Homework assignment No. 8, due on Tuesday, November 14:

NOTE: This assignment is due on Tuesday, Nov. 14 rather than on Thursday, November 9, because of the November 9 game. Homework No. 9 will be due on Thursday, November 16.

1. Book, pages 106 to 110, problem 8(c), (e), (h), (i), (m).

2. In pages 2 and 3 of this handout, you have two completely formal proofs of two simple arithmetical statements. In each case, *write detailed justifications for all the steps*. There is no need for you to actually copy the proofs. All you have to do is list the step numbers and, for each step, write down a complete justification of each step. For example, one line of your list of justfications for the second proof (corresponding to Step 10) should read more or less like this

   10. From 6, by the special case rule, plugging in the constant term $a \cdot 0$ for x.

3. Write completely formal proofs of Statements I and II below. You are allowed to use the 17 rules of logic, the axioms of arithmetic, the Equality axiom $(\forall x)x = x$. and the following four definitions

$$\begin{aligned} 2 &= 1+1\,, \\ 3 &= 2+1\,, \\ 4 &= 3+1\,, \end{aligned}$$

   and $(\forall x)(-x = 0 - x)$. (This last one is the definition of "minus" as a one-input term symbol, as in -3, or -7, or $-(-4)$, as opposed to the two-input term also written "$-$", as in $5-2$.) You are also allowed to use as a known fact the theorem that $(\forall x \in \mathbb{R})x.0 = 0$.

   I. $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(-x) \cdot (-y) = x \cdot y\,.$

   (Informal proof: We first show that $(\forall x)x + (-x) = 0$. To prove this, we observe that, if $a \in \mathbb{R}$ is arbitrary, then $0 - a = -a$ by the definition of $-a$, and $0 - a = -a \Leftrightarrow 0 = a + (-a)$ by Axiom Sub2. Hence $0 = a + (-a)$, so $a + (-a) = 0$, completing the proof that $(\forall x)x + (-x) = 0$. We now pick arbitrary real numbers a, b, and compute $a \cdot b + a \cdot (-b) = a \cdot (b + (-b)) = a \cdot 0 = 0$. Also, $(-a) \cdot (-b) + a \cdot (-b) = ((-a) + a) \cdot (-b) = 0 \cdot (-b) = (-b) \cdot 0 = 0$. Since $a \cdot b + a \cdot (-b) = 0$, Axiom Sub2 implies $a \cdot b = 0 - a \cdot (-b)$. Since $(-a) \cdot (-b) + a \cdot (-b) = 0$. Axiom Sub2 implies the equality $(-a) \cdot (-b) = 0 - a \cdot (-b)$. Hence $a \cdot b = (-a) \cdot (-b)$.)

   II. $2 \cdot 2 = 4$.

Theorem.
$(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(\forall z \in \mathbb{R})(\forall w \in \mathbb{R})((x < y \wedge z < w) \Rightarrow x + y < z + y)$

Informal proof: Suppose a, b, c, d are real numbers such that $a < b$ and $c < d$. It follows from Axiom Or7 that $a+c < b+c$, and also that $c+b < c+d$. Axiom Add2 implies $b + c = c + b$. Hence $b + c < c + d$. Since $a + c < b + c$ and $b + c < c + d$. Axiom Or6 implies $a + b = c + d$. ♢

Formal proof:

1. Let $a \in \mathbb{R}$ be arbitrary.
2. Let $b \in \mathbb{R}$ be arbitrary.
3. Let $c \in \mathbb{R}$ be arbitrary.
4. Let $d \in \mathbb{R}$ be arbitrary.
5. Assume $a < b \wedge c < d$.
6. $a < b$.
7. $c < d$.
8. $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(\forall z \in \mathbb{R})(x < y \Rightarrow x + z < y + z)$.
9. $(\forall y \in \mathbb{R})(\forall z \in \mathbb{R})(a < y \Rightarrow a + z < y + z)$.
10. $(\forall z \in \mathbb{R})(a < b \Rightarrow a + z < b + z)$.
11. $a < b \Rightarrow a + c < b + c$.
12. $a + c < b + c$.
13. $(\forall y \in \mathbb{R})(\forall z \in \mathbb{R})(c < y \Rightarrow c + z < y + z)$.
14. $(\forall z \in \mathbb{R})(c < d \Rightarrow c + z < d + z)$.
15. $c < d \Rightarrow c + b < d + b$.
16. $c + b < d + b$.
17. $c + b < d + b$.
18. $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})x + y = y + x$.
19. $(\forall y \in \mathbb{R})c + y = y + c$.
20. $c + b = b + c$.
21. $b + c < d + b$.
22. $(\forall y \in \mathbb{R})d + y = y + d$.
23. $d + b = b + d$.
24. $b + c < b + d$.
25. $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(\forall z \in \mathbb{R})((x < y \wedge y < z) \Rightarrow x < z)$
26. $(\forall y \in \mathbb{R})(\forall z \in \mathbb{R})((a + c < y \wedge y < z) \Rightarrow a + c < z)$
27. $(\forall z \in \mathbb{R})((a + c < b + c \wedge b + c < z) \Rightarrow a + c < z)$
28. $(a + c < b + c \wedge b + c < b + d) \Rightarrow a + c < b + d$
29. $a + c < b + c \wedge b + c < b + d$
30. $a + c < b + d$
31. $(a < b \wedge c < d) \Rightarrow a + c < b + d$.
32. $(a < b \wedge c < d) \Rightarrow a + c < b + d$.
33. $(\forall w \in \mathbb{R})((a < b \wedge c < w) \Rightarrow a + c < b + w)$.
34. $(\forall z \in \mathbb{R})(\forall w \in \mathbb{R})((a < b \wedge z < w) \Rightarrow a + c < z + w)$.
35. $(\forall y \in \mathbb{R})(\forall z \in \mathbb{R})(\forall w \in \mathbb{R})((a < y \wedge z < w) \Rightarrow a + y < z + w)$.
36. $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(\forall z \in \mathbb{R})(\forall w \in \mathbb{R})((x < y \wedge z < w) \Rightarrow x + y < z + w)$.

END OF PROOF

Theorem.
$(\forall x \in \mathbb{R}) x \cdot 0 = 0$

Informal proof: Let a be an arbitrary real number. Axiom ZO1 implies that $0+0=0$, so $a\cdot(0+0) = a\cdot 0$. Then Axiom DIS tells us that $a\cdot(0+0) = a\cdot 0 + a\cdot 0$, so $a \cdot 0 = a \cdot 0 + a \cdot 0$. Therefore $a \cdot 0 - a \cdot 0 = a \cdot 0$. But $a \cdot 0 - a \cdot 0 = 0$. So $a \cdot 0 = 0$.

Formal proof:

1. $(\forall x \in \mathbb{R}) x + 0 = x.$
2. $0 \in \mathbb{Z}.$
3. $(\forall x \in \mathbb{Z}) x \in \mathbb{R}.$
4. $0 \in \mathbb{R}.$
5. $0 + 0 = 0.$
6. $(\forall x) x = x.$
7. $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(\forall z \in \mathbb{R}) x \cdot (y + z) = x \cdot y + x \cdot z.$
8. $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(\forall z \in \mathbb{R})(x - y = z \Leftrightarrow x = y + z)$
9. Let $a \in \mathbb{R}$ be arbitrary.
10. $a \cdot 0 = a \cdot 0.$
11. $a \cdot 0 = a \cdot (0 + 0).$
12. $(\forall y \in \mathbb{R})(\forall z \in \mathbb{R}) a \cdot (y + z) = a \cdot y + a \cdot z.$
13. $(\forall z \in \mathbb{R}) a \cdot (0 + z) = a \cdot 0 + a \cdot z.$
14. $a \cdot (0 + 0) = a \cdot 0 + a \cdot 0.$
15. $a \cdot 0 = a \cdot 0 + a \cdot 0.$
16. $(\forall y \in \mathbb{R})(\forall z \in \mathbb{R})(a \cdot 0 - y = z \Leftrightarrow a \cdot 0 = y + z)$
17. $(\forall z \in \mathbb{R})(a \cdot 0 - a \cdot 0 = z \Leftrightarrow a \cdot 0 = a \cdot 0 + z)$
18. $a \cdot 0 - a \cdot 0 = a \cdot 0 \Leftrightarrow a \cdot 0 = a \cdot 0 + a \cdot 0.$
19. $a \cdot 0 - a \cdot 0 = a \cdot 0.$
20. $a \cdot 0 - a \cdot 0 = 0 \Leftrightarrow a \cdot 0 = a \cdot 0 + 0.$
21. $a \cdot 0 + 0 = a \cdot 0.$
22. $a \cdot 0 - a \cdot 0 = 0 \Leftrightarrow a \cdot 0 = a \cdot 0.$
23. $a \cdot 0 = a \cdot 0.$
24. $a \cdot 0 - a \cdot 0 = 0.$
25. $a \cdot 0 = 0.$
26. $(\forall x \in \mathbb{R}) x \cdot 0 = 0$

END OF PROOF