MATHEMATICS 300 — SPRING 2001

Introduction to Mathematical Reasoning (640 - 300, Section 03)

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HANDOUT NO. 8 *(March 29, 2001)*

HOMEWORK ASSIGNMENT DUE ON MONDAY, APRIL 9

1. Section 11 of the book, Problems 1(c),(d),(e); 2(a),(b),(c),(e); 6; 7; 8; 10; 11.

2. Give a formal proof of

$$(\forall x, (\mathcal{A}(x) \longrightarrow \mathcal{B}(x))) \longrightarrow ((\neg\exists x, \mathcal{B}(x)) \longrightarrow (\neg\exists x, \mathcal{A}(x)))\,.$$

3. Let
$$E = \{x : x \in \mathbb{Z} \wedge (\exists y, (y \in \mathbb{Z} \wedge x = 2 \cdot y))\}\,.$$
(That is, E is the set of even integers.) Let
$$M = \{(x, y) : x \in \mathbb{Z} \wedge y \in \mathbb{Z} \wedge x = 2 \cdot y\}\,.$$
Prove that M is a matching between E and Z. (The definition of "matching" is given below, in Page 2 of this handout.)

4. Prove that the empty set is finite. (The precise mathematical definition of "finite set" is given below, in Page 2 of this handout.)

THE SECOND MIDTERM TEST

The test will be on **Thursday, April 19.**

MATCHINGS AND FINITE SETS

DEFINITION. Let A, B be sets. A *matching* between A and B is a subset M of $A \times B$ such that

1. $\forall x, (x \in A \longrightarrow (\exists y, (y \in B \wedge (x, y) \in M)))$;
2. $\forall x, \forall y, \forall z, ((x \in A \wedge y \in B \wedge z \in B \wedge (x, y) \in M \wedge (x, z) \in M) \longrightarrow y = z)$
3. $\forall x, (x \in B \longrightarrow (\exists y, (y \in A \wedge (y, x) \in M)))$;
4. $\forall x, \forall y, \forall z, ((x \in B \wedge y \in A \wedge z \in A \wedge (y, x) \in M \wedge (z, x) \in M) \longrightarrow y = z)$

EXAMPLE. Let A be the set of all students who are in a classroom, and let B be the set of all chairs in that classroom. A *matching* between A and B (that is, a matching of the students with the chairs) consists of having students occupy chairs, in such as way that

1. every student occupies a chair,
2. no student occupies two different chairs,
3. every chair is occupied by a student,
4. no chair is occupied by two different students.

NOTATION. Let n be a natural number. We use the expression I_n to denote the set $\{m : m \in \mathbb{N} \wedge 1 \leq m \leq n\}$.

THEOREM. Let n, m be a natural number such that $n \neq m$. Then there is no matching between the sets I_n and I_m.

NOTE. We are not yet in a position to prove this. We will prove this theorem after we discuss mathematical induction.

DEFINITION. Let A be a set. We say that A is *finite* if there is a matching between A and I_n for some $n \in \mathbb{N}$.

DEFINITION. Let A, B be sets. We say that *A and B have the same number of elements* if there is a matching between A and B.

EXAMPLE. Let us continue with our students-and-chairs example. If you see that every student is seated on a chair and every chair is occupied by a student then you know, **without having to count the students or the chairs**, that there are as many students as there are chairs.