MATHEMATICS 300
Introduction to Mathematical Reasoning

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HANDOUT NO. 8 — April 6, 2000

Homework due on Monday, April 10

1. Give an informal proof and a completely formal proof of the following fact:
$$2.2 = 4\,,$$
using the rules of inference, the equality axioms and the real number axioms, and the definitions: $2 = 1 + 1$, $3 = 2 + 1$, $4 = 3 + 1$.

2. Give an informal proof and a completely formal proof of the following fact:
$$0 \neq 2\,,$$
using the rules of inference, the equality axioms and the real number axioms, and the definition: $2 = 1 + 1$.

3. Is it possible to prove that $0 \neq 2$ using only the field axioms?

Some examples of proofs

THEOREM 1: $\forall x(0 + x = x)$

PROOF:

(1)[1] Let a be arbitrary	[Preparation to apply Rule 5]
(2)[1] $\forall x(x + 0 = x)$	[Axiom A8]
(3)[1] $a + 0 = a$	[Rule 4, specializing x to a in (2)]
(4)[1] $\forall x\forall y(x + y = y + x)$	[Axiom A5]
(5)[1] $\forall y(0 + y = y + 0)$	[Rule 4, specializing x to 0 in (4)]
(6)[1] $0 + a = a + 0$	[Rule 4, specializing y to a in (5)]
(7)[1] $\forall x\forall y\forall z((x = y \wedge y = z) \rightarrow x = z)$ [Axiom Eq3]	
(8)[1] $\forall y\forall z((0 + a = y \wedge y = z) \rightarrow 0 + a = z)$	[Rule 4, specializing x to $0 + a$ in (7)]
(9)[1] $\forall z((0 + a = a + 0 \wedge a + 0 = z) \rightarrow 0 + a = z)$	[Rule 4, specializing y to $a + 0$ in (8)]
(10)[1] $(0 + a = a + 0 \wedge a + 0 = a) \rightarrow 0 + a = a$	[Rule 4, specializing z to a in (9)]
(11)[1] $0 + a = a + 0 \rightarrow (a + 0 = a \rightarrow (0 + a = a + 0 \wedge a + 0 = a))$	[Tautology $\alpha \rightarrow (\beta \rightarrow (\alpha \wedge \beta))$]
(12)[1] $a + 0 = a \rightarrow (0 + a = a + 0 \wedge a + 0 = a)$	[Rule 2, from (6) and (11)]
(13)[1] $0 + a = a + 0 \wedge a + 0 = a$	[Rule 2, from (3) and (12)]
(14)[1] $0 + a = a$	[Rule 2, from (10) and (13)]
(15)[0] $\forall x(0 + x = x)$	[Rule 3, from (1) and (14)] $\diamondsuit$

THEOREM 2: $\forall x \forall y \forall z(x = y \to x + z = y + z)$

PROOF:

(1)[1] Let a be arbitrary [Preparation to apply Rule 5]

(2)[2] Let b be arbitrary [Preparation to apply Rule 5]

(3)[3] Let c be arbitrary [Preparation to apply Rule 5]

(4)[4] Assume $a = b$ [Preparation to apply Rule 3]

(5)[4] $\forall x \forall y(x = y \to (x + c = x + c \leftrightarrow x + c = y + c))$ [Axiom Eq4]

(6)[4] $\forall y(a = y \to (a + c = a + c \leftrightarrow a + c = y + c)$ [Rule 4, specializing x to a in (5)]

(7)[4] $a = b \to (a + c = a + c \leftrightarrow a + c = b + c$ [Rule 4, specializing y to b in (6)]

(8)[4] $a + c = a + c \leftrightarrow a + c = b + c$ [Rule 2, from (4) and (7)]

(9)[4] $\forall x(x = x)$ [Axiom Eq1]

(10)[4] $a + c = a + c$ [Rule 4, specializing x to $a + c$ in (9)]

(11)[4] $(a + c = a + c \leftrightarrow a + c = b + c) \to (a + c = a + c \to a + c = b + c)$ [Tautology $(\alpha \leftrightarrow \beta) \to (\alpha \to \beta)$]

(12)[4] $a + c = a + c \to a + c = b + c$ [Rule 2, from (8) and (11)]

(13)[4] $a + c = b + c$ [Rule 2, from (10) and (12)]

(14)[3] $a = b \to a + c = b + c$ [Rule 3, from (4) and (13)]

(15)[2] $\forall z(a = b \to a + z = b + z)$ [Rule 5, from (3) and (14)]

(16)[1] $\forall y \forall z(a = y \to a + z = y + z)$ [Rule 5, from (2) and (15)]

(17)[0] $\forall x \forall y \forall z(x = y \to x + z = y + z)$ [Rule 5, from (1) and (16)] $\diamondsuit$

THEOREM 3: $\forall x \forall y \forall z(x = y \to z + x = z + y)$

PROOF:

(1)[1] Let a be arbitrary [Preparation to apply Rule 5]

(2)[2] Let b be arbitrary [Preparation to apply Rule 5]

(3)[3] Let c be arbitrary [Preparation to apply Rule 5]

(4)[4] Assume $a = b$ [Preparation to apply Rule 3]

(5)[4] $\forall x \forall y(x = y \to (c + x = c + x \leftrightarrow x + c = y + c))$ [Axiom Eq4]

(6)[4] $\forall y(a = y \to (c + a = c + a \leftrightarrow c + a = c + y)$ [Rule 4, specializing x to a in (5)]

(7)[4] $a = b \to (c + a = c + a \leftrightarrow c + a = c + a$ [Rule 4, specializing y to b in (6)]

(8)[4] $c + a = c + a \leftrightarrow c + a = c + b$ [Rule 2, from (4) and (7)]

(9)[4] $\forall x(x = x)$ [Axiom Eq1]

(10)[4] $c + a = c + a$ [Rule 4, specializing x to $c + a$ in (9)]

(11)[4] $(c + a = c + a \leftrightarrow c + a = c + b) \to (c + a = c + b \to c + a = c + b)$ [Tautology $(\alpha \leftrightarrow \beta) \to (\alpha \to \beta)$]

(12)[4] $c + a = c + a \to c + a = c + b$ [Rule 2, from (8) and (11)]

(13)[4] $c + a = c + b$ [Rule 2, from (10) and (12)]

(14)[3] $a = b \to c + a = c + b$ [Rule 3, from (4) and (13)]

(15)[2] $\forall z(a = b \to z + a = z + b)$ [Rule 5, from (3) and (14)]

(16)[1] $\forall y \forall z(a = y \to z + a = z + y)$ [Rule 5, from (2) and (15)]

(17)[0] $\forall x \forall y \forall z(x = y \to z + x = z + y)$ [Rule 5, from (1) and (16)] $\diamondsuit$

THEOREM 4: $\forall x \forall y \forall z(x = y \wedge z = y \to x = z)$

PROOF:

(1)[1] Let a be arbitrary [Preparation to apply Rule 5]

(2)[2] Let b be arbitrary [Preparation to apply Rule 5]

(3)[3] Let c be arbitrary [Preparation to apply Rule 5]

(4)[4] Assume $a = b \wedge c = b$ [Preparation to apply Rule 3]

(5)[4] $\forall x \forall y \forall z((x = y \wedge y = z) \to x = z)$ [Axiom Eq3]

(6)[4] $\forall y \forall z((a = y \wedge y = z) \to x = z)$ [Rule 4, specializing x to a in (5)]

(7)[4] $\forall z((a = b \wedge b = z) \to a = z)$ [Rule 4, specializing y to b in (6)]

(8)[4] $(a = b \wedge b = c) \to a = c$ [Rule 4, specializing z to c in (7)]
(9)[4] $\forall x \forall y(x = y \to y = x)$ [Axiom Eq2]
(10)[4] $\forall y(c = y \to y = c)$ [Rule 4, specializing x to c in (9)]
(11)[4] $c = b \to b = c$ [Rule 4, specializing y to b in (10)]
(12)[4] $(a = b \wedge c = b) \to ((c = b \to b = c) \to (a = b \wedge b = c))$
[Tautology $(\alpha \wedge \beta) \to ((\beta \to \gamma) \to (\alpha \wedge \gamma))$]
(13)[4] $(c = b \to b = c) \to (a = b \wedge b = c)$ [Rule 2, from (4) and (12)]
(14)[4] $a = b \wedge b = c$ [Rule 2, from (11) and (13)]
(15)[4] $a = c$ [Rule 2, from (8) and (14)]
(16)[3] $(a = b \wedge c = b) \to a = c$ [Rule 3, from (4) and (15)]
(17)[2] $\forall z(a = b \wedge z = b) \to a = z$ [Rule 5, from (3) and (16)]
(18)[1] $\forall y \forall z(a = y \wedge z = y) \to a = z$ [Rule 5, from (2) and (17)]
(19)[0] $\forall x \forall y \forall z\ (x = y \wedge z = y) \to x = z$ [Rule 5, from (1) and (18)] $\diamondsuit$

THEOREM 5: $\forall x \forall y \forall z(x + z = y + z \to x = y)$

PROOF:

(1)[1] Let a be arbitrary [Preparation to apply Rule 5]
(2)[2] Let b be arbitrary [Preparation to apply Rule 5]
(3)[3] Let c be arbitrary [Preparation to apply Rule 5]
(4)[4] Assume $a + c = b + c$ [Preparation to apply Rule 3]
(5)[4] $\forall x(x + (-x) = 0)$ [Axiom A8]
(6)[4] $c + (-c) = 0$ [Rule 4, specializing x to c in (5)]
(7)[4] $\forall x \forall y \forall z(x = y \to x + z = y + z)$ [Theorem 2]
(8)[4] $\forall y \forall z(a + c = y \to (a + c) + z = y + z)$ [Rule 4, specializing x to $a + c$ in (7)]
(9)[4] $\forall z(a + c = b + c \to (a + c) + z = (b + c) + z)$ [Rule 4, specializing y to $b + c$ in (8)]
(10)[4] $a + c = b + c \to (a + c) + (-c) = (b + c) + (-c)$ [Rule 4, specializing z to $-c$ in (9)]
(11)[4] $(a + c) + (-c) = (b + c) + (-c)$ [Rule 2, from (4) and (10)]
(12)[4] $\forall x \forall y \forall z(x + (y + z) = (x + y) + z)$ [Axiom A3]
(13)[4] $\forall y \forall z(a + (y + z) = (a + y) + z)$ [Rule 4, specializing x to a in (12)]
(14)[4] $\forall z(a + (c + z) = (a + c) + z)$ [Rule 4, specializing y to c in (13)]
(15)[4] $a + (c + (-c)) = (a + c) + (-c)$ [Rule 4, specializing z to $-c$ in (14)]
(16)[4] $\forall y \forall z(b + (y + z) = (b + y) + z)$ [Rule 4, specializing x to b in (12)]
(17)[1] $\forall z(b + (c + z) = (b + c) + z)$ [Rule 4, specializing y to c in (16)]
(18)[1] $b + (c + (-c)) = (b + c) + (-c)$ [Rule 4, specializing z to $-c$ in (17)]
(19)[4] $\forall x \forall y \forall z((x = y \wedge y = z) \to x = z)$ [Axiom Eq3]
(20)[4] $\forall y \forall z((a + (c + (-c)) = y \wedge y = z) \to a + (c + (-c)) = z)$
[Rule 4, specializing x to $a + (c + (-c))$ in (19)]
(21)[4] $\forall z((a + (c + (-c)) = (a + c) + (-c) \wedge (a + c) + (-c) = z) \to a + (c + (-c)) = z)$
[Rule 4, specializing y to $(a + c) + (-c)$ in (20)]
(22)[4] $(a+(c+(-c)) = (a+c)+(-c) \wedge (a+c)+(-c) = (b+c)+(-c)) \to a+(c+(-c)) = (b+c)+(-c)$
[Rule 4, specializing z to $(b + c) + (-c)$ in (21)]
(23)[4] $a + (c + (-c)) = (a + c) + (-c) \to ((a + c) + (-c) = (b + c) + (-c) \to$
$(a + (c + (-c)) = (a + c) + (-c) \wedge (a + c) + (-c) = (b + c) + (-c)))$
[Tautology $\alpha \to (\beta \to (\alpha \wedge \beta))$]
(24)[4] $(a + c) + (-c) = (b + c) + (-c) \to$
$(a + (c + (-c)) = (a + c) + (-c) \wedge (a + c) + (-c) = (b + c) + (-c))$
[Rule 2, from (15) and (23)]
(25)[4] $a + (c + (-c)) = (a + c) + (-c) \wedge (a + c) + (-c) = (b + c) + (-c)$
[Rule 2, from (18) and (24)]
(26)[4] $a + (c + (-c)) = (b + c) + (-c)$ [Rule 2, from (22) and (25)]
(27)[4] $\forall x \forall y \forall z(x = y \wedge z = y) \to x = z)$ [Theorem 4]
(28)[4] $\forall y \forall z((a + (c + (-c)) = y \wedge z = y) \to a + (c + (-c)) = z)$
[Rule 4, specializing x to $a + (c + (-c))$ in (27)]

(29)[4] $\forall z((a+(c+(-c)) = (b+c)+(-c) \wedge z = (b+c)+(-c)) \to a+(c+(-c)) = z)$
[Rule 4, specializing y to $(b+c)+(-c)$ in (20)]
(30)[4] $(a+(c+(-c)) = (b+c)+(-c) \wedge b+(c+(-c)) = (b+c)+(-c)) \to a+(c+(-c)) = b+(c+(-c))$
[Rule 4, specializing z to $b+(c+(-c))$ in (29)]
(31)[4] $a+(c+(-c)) = (b+c)+(-c) \to (b+(c+(-c)) = (b+c)+(-c) \to$
$(a+(c+(-c)) = b+(c+(-c)) \wedge b+(c+(-c)) = (b+c)+(-c)))$
[Tautology $\alpha \to (\beta \to (\alpha \wedge \beta))$]
(32)[4] $b+(c+(-c)) = (b+c)+(-c) \to$
$(a+(c+(-c)) = b+(c+(-c)) \wedge b+(c+(-c)) = (b+c)+(-c))$
[Rule 2, from (26) and (31)]
(33)[4] $a+(c+(-c)) = b+(c+(-c)) \wedge b+(c+(-c)) = (b+c)+(-c)$
[Rule 2, from (18) and (32)]
(34)[4] $a+(c+(-c)) = b+(c+(-c))$ [Rule 2, from (22) and (33)]
(35)[4] $\forall x \forall y(x = y \to (a+x = b+x \leftrightarrow a+y = b+y))$ [Axiom Eq4]
(36)[4] $\forall y(c+(-c) = y \to (a+(c+(-c) = b+(c+(-c)) \leftrightarrow a+y = b+y))$
[Rule 4, specializing x to $c+(-c)$ in (35)]
(37)[4] $c+(-c) = 0 \to (a+(c+(-c)) = b+(c+(-c)) \leftrightarrow a+0 = b+0)$
[Rule 4, specializing z to 0 in (36)]
(38)[4] $a+(c+(-c) = b+(c+(-c)) \leftrightarrow a+0 = b+0$ [Rule 2, from (6) and (37)]
(39)[4] $(a+(c+(-c) = b+(c+(-c)) \leftrightarrow a+0 = b+0) \to (a+(c+(-c) = b+(c+(-c)) \to a+0 = b+0)$
[Tautology$(\alpha \leftrightarrow \beta) \to (\alpha \to \beta)$]
(40)[4] $a+(c+(-c) = b+(c+(-c)) \to a+0 = b+0$ [Rule 2, from (38) and (39)]
(41)[4] $a+0 = b+0$ [Rule 2, from (34) and (40)]
(42)[4] $\forall x(x+0 = x)$ [Axiom A8]
(43)[4] $\forall x(0+x = x)$ [Theorem 1]
(44)[4] $a = a+0$ [Rule 4, specializing x to a in (43)]
(45)[4] $b+0 = b$ [Rule 4, specializing x to b in (42)]
(46)[4] $\forall y \forall z((a+0 = y \wedge y = z) \to a+0 = z)$ [Rule 2, specializing x to $a+0$ in (19)]
(47)[4] $\forall z((a+0 = b+0 \wedge b+0 = z) \to a+0 = z)$ [Rule 2, specializing y to $b+0$ in (46)]
(48)[4] $(a+0 = b+0 \wedge b+0 = b) \to a+0 = b$ [Rule 2, specializing z to b in (47)]
(49)[4] $a+0 = b+0 \to (b+0 = b \to (a+0 = b+0 \wedge b+0 = b))$ [Tautology $\alpha \to (\beta \to (\alpha \wedge \beta))$]
(50)[4] $b+0 = b \to (a+0 = b+0 \wedge b+0 = b)$ [Rule 2, from (41) and (49)]
(51)[4] $a+0 = b+0 \wedge b+0 = b$ [Rule 2, from (45) and (50)]
(52)[4] $a+0 = b$ [Rule 2, from (48) and (51)]
(53)[4] $\forall y \forall z((a = y \wedge y = z) \to a = z)$ [Rule 2, specializing x to a in (19)]
(54)[4] $\forall z((a = a+0 \wedge a+0 = z) \to a = z)$ [Rule 2, specializing y to $a+0$ in (53)]
(55)[4] $(a = a+0 \wedge a+0 = b) \to a = b$ [Rule 2, specializing z to b in (54)]
(56)[4] $a = a+0 \to (a+0 = b \to (a = a+0 \wedge a+0 = b))$ [Tautology $\alpha \to (\beta \to (\alpha \wedge \beta))$]
(57)[4] $a+0 = b \to (a = a+0 \wedge a+0 = b)$ [Rule 2, from (44) and (56)]
(58)[4] $a = a+0 \wedge a+0 = b$ [Rule 2, from (53) and (57)]
(59)[4] $a = b$ [Rule 2, from (55) and (58)]
(60)[3] $a+c = b+c \to a = b$ [Rule 3, from (4) and (59)]
(61)[2] $\forall z(a+z = b+z \to a = b)$ [Rule 5, from (3) and (60)]
(62)[1] $\forall y \forall z(a+z = y+z \to a = y)$ [Rule 5, from (2) and (61)]
(63)[1] $\forall x \forall y \forall z(x+z = y+z \to x = y)$ [Rule 5, from (1) and (62)] $\diamondsuit$

THEOREM 6: $\forall x \forall y \forall z(z+x = z+y \to x = y)$

PROOF:

(1)[1] Let a be arbitrary [Preparation to apply Rule 5]
(2)[2] Let b be arbitrary [Preparation to apply Rule 5]
(3)[3] Let c be arbitrary [Preparation to apply Rule 5]
(4)[4] Assume $c+a = c+b$ [Preparation to apply Rule 3]
(5)[4] $\forall x \forall y(x+y = y+x)$ [Axiom A5]

(6)[4] $\forall y(a+y=y+a)$ [Rule 4, specializing x to a in (5)]
(7)[4] $a+c=c+a$ [Rule 4, specializing y to a in (6)]
(8)[4] $\forall y(c+y=y+c)$ [Rule 4, specializing x to c in (5)]
(9)[4] $c+b=b+c$ [Rule 4, specializing y to b in (8)]
(10)[4] $\forall x\forall y\forall z((x=y\wedge y=z)\rightarrow x=z)$ [Axiom Eq3]
(11)[4] $\forall y\forall z((a+c=y\wedge y=z)\rightarrow a+c=z)$ [Rule 4, specializing x to $a+c$ in (10)]
(12)[4] $\forall z((a+c=c+a\wedge c+a=z)\rightarrow a+c=z)$ [Rule 4, specializing y to $c+a$ in (11)]
(13)[4] $(a+c=c+a\wedge c+a=c+b)\rightarrow a+c=c+b$ [Rule 4, specializing z to $c+b$ in (12)]
(14)[4] $a+c=c+a\rightarrow(c+a=c+b\rightarrow(a+c=c+a\wedge c+a=c+b))$
[Tautology $\alpha\rightarrow(\beta\rightarrow(\alpha\wedge\beta))$]
(15)[4] $c+a=c+b\rightarrow(a+c=c+a\wedge c+a=c+b)$ [Rule 2, from (7) and (14)]
(16)[4] $a+c=c+a\wedge c+a=c+b$ [Rule 2, from (4) and (15)]
(17)[4] $a+c=c+b$ [Rule 2, from (13) and (16)]
(18)[4] $\forall z((a+c=c+b\wedge c+b=z)\rightarrow a+c=z)$ [Rule 4, specializing y to $c+b$ in (11)]
(19)[4] $(a+c=c+b\wedge c+b=b+c)\rightarrow a+c=b+c$ [Rule 4, specializing z to $c+b$ in (18)]
(20)[4] $a+c=c+b\rightarrow(c+b=b+c\rightarrow(a+c=c+b\wedge c+b=b+c))$
[Tautology $\alpha\rightarrow(\beta\rightarrow(\alpha\wedge\beta))$]
(21)[4] $c+b=b+c\rightarrow(a+c=c+b\wedge c+b=b+c)$ [Rule 2, from (17) and (20)]
(22)[4] $a+c=c+b\wedge c+b=b+c$ [Rule 2, from (9) and (21)]
(23)[4] $a+c=b+c$ [Rule 2, from (19) and (22)]
(24)[4] $\forall x\forall y\forall z((x+z=y+z\rightarrow x=y)$ [Theorem 5]
(25)[4] $\forall y\forall z((a+z=y+z\rightarrow a=y)$ [Rule 4, specializing x to a in (24)]
(26)[4] $\forall z((a+z=b+z\rightarrow a=b)$ [Rule 4, specializing y to b in (25)]
(27)[4] $a+c=b+c\rightarrow a=b$ [Rule 4, specializing z to c in (26)]
(28)[4] $a=b$ [Rule 2, from (23) and (27)]
(29)[3] $c+a=c+b\rightarrow a=b$ [Rule 3, from (4) and (28)]
(30)[2] $\forall z(z+a=z+b\rightarrow a=b)$ [Rule 5, from (3) and (29)]
(31)[1] $\forall y\forall z(z+a=z+y\rightarrow a=y)$ [Rule 5, from (2) and (30)]
(32)[0] $\forall x\forall y\forall z(z+x=z+y\rightarrow x=y)$ [Rule 5, from (1) and (31)] ♢

THEOREM 7: $\forall x\forall y\forall z(x=y\rightarrow x.z=y.z)$

PROOF:

(1)[1] Let a be arbitrary [Preparation to apply Rule 5]
(2)[2] Let b be arbitrary [Preparation to apply Rule 5]
(3)[3] Let c be arbitrary [Preparation to apply Rule 5]
(4)[4] Assume $a=b$ [Preparation to apply Rule 3]
(5)[4] $\forall x\forall y(x=y\rightarrow(x.c=x.c\leftrightarrow x.c=y.c))$ [Axiom Eq4]
(6)[4] $\forall y(a=y\rightarrow(a.c=a.c\leftrightarrow a.c=y.c)$ [Rule 4, specializing x to a in (5)]
(7)[4] $a=b\rightarrow(a.c=a.c\leftrightarrow a.c=b.c$ [Rule 4, specializing y to b in (6)]
(8)[4] $a.c=a.c\leftrightarrow a.c=b.c$ [Rule 2, from (4) and (7)]
(9)[4] $\forall x(x=x)$ [Axiom Eq1]
(10)[4] $a.c=a.c$ [Rule 4, specializing x to $a.c$ in (9)]
(11)[4] $(a.c=a.c\leftrightarrow a.c=b.c)\rightarrow(a.c=a.c\rightarrow a.c=b.c)$
[Tautology $(\alpha\leftrightarrow\beta)\rightarrow(\alpha\rightarrow\beta)$]
(12)[4] $a.c=a.c\rightarrow a.c=b.c$ [Rule 2, from (8) and (11)]
(13)[4] $a.c=b.c$ [Rule 2, from (10) and (12)
(14)[3] $a=b\rightarrow a.c=b.c$ [Rule 3, from (4) and (13)]
(15)[2] $\forall z(a=b\rightarrow a.z=b.z)$ [Rule 5, from (3) and (14)]
(16)[1] $\forall y\forall z(a=y\rightarrow a.z=y.z)$ [Rule 5, from (2) and (15)]
(17)[0] $\forall x\forall y\forall z(x=y\rightarrow x.z=y.z)$ [Rule 5, from (1) and (16)] ♢

THEOREM 8: $\forall x \forall y \forall z(x = y \rightarrow z.x = z.y)$

PROOF:

(1)[1] Let a be arbitrary [Preparation to apply Rule 5]
(2)[2] Let b be arbitrary [Preparation to apply Rule 5]
(3)[3] Let c be arbitrary [Preparation to apply Rule 5]
(4)[4] Assume $a = b$ [Preparation to apply Rule 3]
(5)[4] $\forall x \forall y(x = y \rightarrow (c.x = c.x \leftrightarrow x.c = y.c))$ [Axiom Eq4]
(6)[4] $\forall y(a = y \rightarrow (c.a = c.a \leftrightarrow c.a = c.y)$ [Rule 4, specializing x to a in (5)]
(7)[4] $a = b \rightarrow (c.a = c.a \leftrightarrow c.a = c.a$ [Rule 4, specializing y to b in (6)]
(8)[4] $c.a = c.a \leftrightarrow c.a = c.b$ [Rule 2, from (4) and (7)]
(9)[4] $\forall x(x = x)$ [Axiom Eq1]
(10)[4] $c.a = c.a$ [Rule 4, specializing x to $c.a$ in (9)]
(11)[4] $(c.a = c.a \leftrightarrow c.a = c.b) \rightarrow (c.a = c.b \rightarrow c.a = c.b)$
[Tautology $(\alpha \leftrightarrow \beta) \rightarrow (\alpha \rightarrow \beta)$]
(12)[4] $c.a = c.a \rightarrow c.a = c.b$ [Rule 2, from (8) and (11)]
(13)[4] $c.a = c.b$ [Rule 2, from (10) and (12)
(14)[3] $a = b \rightarrow c.a = c.b$ [Rule 3, from (4) and (13)]
(15)[2] $\forall z(a = b \rightarrow z.a = z.b)$ [Rule 5, from (3) and (14)]
(16)[1] $\forall y \forall z(a = y \rightarrow z.a = z.y)$ [Rule 5, from (2) and (15)]
(17)[0] $\forall x \forall y \forall z(x = y \rightarrow z.x = z.y)$ [Rule 5, from (1) and (16)] $\diamondsuit$

THEOREM 9: $\forall x(x.0 = 0)$.

PROOF:

(1)[0] $\forall x(x + 0 = 0)$ [Axiom A8]
(2)[0] $\forall x(x = x)$ [Axiom Eq1]
(3)[0] $0 = 0 + 0$ [Rule 4, specializing x to 0 in (1)]
(4)[1] Let a be arbitrary [Preparation to apply Rule 5]
(5)[1] $\forall x \forall y \forall z(x = y \rightarrow z.x = z.y)$ [Theorem 7]
(6)[1] $\forall y \forall z(0 = y \rightarrow z.0 = z.y)$ [Rule 4, specializing x to 0 in (5)]
(7)[1] $\forall z(0 = 0 + 0 \rightarrow z.0 = z.(0 + 0))$ [Rule 4, specializing y to $0 + 0$ in (6)]
(8)[1] $0 = 0 + 0 \rightarrow a.0 = a.(0 + 0)$ [Rule 4, specializing z to a in (7)]
(9)[1] $a.0 = a.(0 + 0)$ [Rule 2, from (3) and (8)]
(10)[1] $\forall x \forall y \forall z(x.(y + z) = x.y + x.z)$ [Axiom A7]
(11)[1] $\forall y \forall z(a.(y + z) = a.y + a.z)$ [Rule 4, specializing x to a in (10)]
(12)[1] $\forall z(a.(0 + z) = a.0 + a.z)$ [Rule 4, specializing y to 0 in (11)]
(13)[1] $a.(0 + 0) = a.0 + a.0$ [Rule 4, specializing z to 0 in (12)]
(14)[1] $\forall x \forall y \forall z((x = y \wedge y = z) \rightarrow (x = z))$ [Axiom Eq3]
(15)[1] $\forall y \forall z((a.0 = y \wedge y = z) \rightarrow 4a.0 = z)$ [Rule 4, specializing x to $a.0$ in (14)]
(16)[1] $\forall z((a.0 = a.(0 + 0) \wedge a.(0 + 0) = z) \rightarrow a.0 = z)$[Rule 4, specializing y to $a.(0 + 0)$ in (15)]
(17)[1] $(a.0 = a.(0 + 0) \wedge a.(0 + 0) = a.0 + a.0) \rightarrow a.0 = a.0 + a.0)$
[Rule 4, specializing z to $a.0 + a.0$ in (16)]
(18)[1] $a.0 = a.(0 + 0) \rightarrow (a.(0 + 0) = a.0 + a.0 \rightarrow (a.0 = a.(0 + 0) \wedge a.(0 + 0) = a.0 + a.0))$
[Tautology $\alpha \rightarrow (\beta \rightarrow (\alpha \wedge \beta))$]
(19)[1] $a.(0+0) = a.0+a.0 \rightarrow (a.0 = a.(0+0) \wedge a.(0+0) = a.0+a.0)$[Rule 2, from (9) and (18)]
(20)[1] $a.0 = a.(0 + 0) \wedge a.(0 + 0) = a.0 + a.0$ [Rule 2, from (13) and (19)]
(21)[1] $a.0 = a.0 + a.0$ [Rule 2, from (17) and (20)]
(22)[1] $a.0 + 0 = a.0$ [Rule 4, specializing x to $a.0$ in (1)]
(23)[1] $\forall x \forall y \forall z((x = y \wedge y = z) \rightarrow x = z)$ [Axiom Eq3]
(24)[1] $\forall y \forall z((a.0 + 0 = y \wedge y = z) \rightarrow a.0 + 0 = z)$ [Rule 4, specializing x to $0 + a.0$ in (23)]
(25)[1] $\forall z((a.0 + 0 = a.0 \wedge a.0 = z) \rightarrow a.0 + 0 = z)$ [Rule 4, specializing y to $a.0$ in (24)]
(26)[1] $(a.0 + 0 = a.0 \wedge a.0 = a.0 + a.0) \rightarrow a.0 + 0 = a.0 + a.0$
[Rule 4, specializing z to $a.0 + a.0$ in (25)]

(28)[1] $a.0 + 0 = a.0 \to (a.0 = a.0 + a.0 \to (a.0 + 0 = a.0 \wedge a.0 = a.0 + a.0))$ [Tautology $\alpha \to (\beta \to (\alpha \wedge \beta))$]
(29)[1] $a.0 = a.0 + a.0 \to (a.0 + 0 = a.0 \wedge a.0 = a.0 + a.0)$ [Rule 2, from (22) and (28)]
(30)[1] $a.0 + 0 = a.0 \wedge a.0 = a.0 + a.0$ [Rule 2, from (21) and (29)]
(31)[1] $a.0 + 0 = a.0 + a.0$ [Rule 2, from (26) and (30)]
(32)[1] $\forall x \forall y \forall z(z + x = z + y \to x = y)$ [Theorem 6]
(33)[1] $\forall y \forall z(z + 0 = z + y \to 0 = y)$ [Rule 4, specializing x to 0 in (32)]
(34)[1] $\forall z(z + 0 = z + a.0 \to 0 = a.0)$ [Rule 4, specializing y to $a.0$ in (33)]
(35)[1] $a.0 + 0 = a.0 + a.0 \to 0 = a.0$ [Rule 4, specializing z to $a.0$ in (34)]
(36)[1] $0 = a.0$ [Rule 2, from (31) and (35)]
(37)[1] $\forall x \forall y(x = y \to y = x)$ [Axiom Eq2]
(38)[1] $\forall y(0 = y \to y = 0)$ [Rule 4, specializing x to 0 in (37)]
(39)[1] $0 = a.0 \to a.0 = 0$ [Rule 4, specializing y to $a.0$ in (38)]
(40)[1] $a.0 = 0$ [Rule 2, from (36) and (39)]
(41)[0] $\forall x(x.0 = 0)$ [Rule 5, from (4) and (40)] $\diamondsuit$

THEOREM 10: $\forall x \forall y(x + y = 0 \to y = -x)$

PROOF:

(1)[1] Let a be arbitrary [Preparation to apply Rule 5]
(2)[2] Let b be arbitrary [Preparation to apply Rule 5]
(3)[3] Assume $a + b = 0$ [Preparation to apply Rule 3]
(4)[3] $\forall x(x + (-x) = 0)$ [Axiom A8]
(5)[3] $a + (-a) = 0$ [Rule 4, specializing x to a in (4)]
(6)[3] $\forall x \forall y \forall z((x = y \wedge z = y) \to x = z)$ [Theorem 4]
(7)[3] $\forall y \forall z((a + b = y \wedge z = y) \to a + b = z)$ [Rule 4, specializing x to $a + b$ in (6)]
(8)[3] $\forall z((a + b = 0 \wedge z = 0) \to a + b = z)$ [Rule 4, specializing y to 0 in (7)]
(9)[3] $(a + b = 0 \wedge a + (-a) = 0) \to a + b = a + (-a)$ [Rule 4, specializing z to $a + (-a)$ in (8)]
(10)[3] $a + b = 0 \to (a + (-a) = 0 \to (a + b = 0 \wedge a + (-a) = 0))$ [Tautology $\alpha \to (\beta \to (\alpha \wedge \beta))$]
(11)[3] $a + (-a) = 0 \to (a + b = 0 \wedge a + (-a) = 0)$ [Rule 2, from (3) and (10)]
(12)[3] $a + b = 0 \wedge a + (-a) = 0$ [Rule 2, from (5) and (11)]
(13)[3] $a + b = a + (-a)$ [Rule 2, from (9) and (12)]
(14)[3] $\forall x \forall y \forall z(z + x = z + y \to x = y)$ [Theorem 6]
(15)[3] $\forall y \forall z(z + b = z + y \to b = y)$ [Rule 4, specializing x to b in (14)]
(16)[3] $\forall z(z + b = z + (-a) \to b = -a)$ [Rule 4, specializing y to $-a$ in (15)]
(17)[3] $a + b = a + (-a) \to b = -a$ [Rule 4, specializing z to a in (16)]
(18)[3] $b = -a$ [Rule 2, from (13) and (17)]
(19)[2] $a + b = 0 \to b = -a$ [Rule 3, from (3) and (18)]
(20)[1] $\forall y(a + y = 0 \to y = -a)$ [Rule 5, from (2) and (19)]
(21)[0] $\forall x \forall y(x + y = 0 \to y = -x)$ [Rule 5, from (1) and (20)] $\diamondsuit$

THEOREM 11: $\forall x(-(-x) = x)$

PROOF:

(1)[0] $\forall x \forall y(x + y = 0 \to y = -x)$ [Theorem 10]
(2)[0] $\forall x(x + (-x) = 0$ [Axiom A8]
(3)[1] Let a be arbitrary [Preparation to apply Rule 5]
(4)[1] $a + (-a) = 0$ [Rule 4, specializing x to a in (2)]
(5)[1] $\forall x \forall y(x + y = y + x)$ [Axiom A5]
(6)[1] $\forall y((-a)) + y = y + (-a))$ [Rule 4, specializing x to $-a$ in (5)]
(7)[1] $(-a) + a = a + (-a)$ [Rule 4, specializing y to a in (6)]
(8)[1] $\forall x \forall y \forall z((x = y \wedge y = z) \to x = z)$ [Axiom Eq3]
(9)[1] $\forall y \forall z(((-a) + a = y \wedge y = z) \to (-a) + a = z)$ [Rule 4, specializing x to $(-a) + a$ in (8)]

(10)[1] $\forall z(((-a) + a = a + (-a) \wedge a + (-a) = z) \to (-a) + a = z)$
[Rule 4, specializing y to $a + (-a)$ in (9)]

(11)[1] $((-a) + a = a + (-a) \wedge a + (-a) = 0) \to (-a) + a = 0$
[Rule 4, specializing z to 0 in (11)]

(12)[1] $(-a) + a = a + (-a) \to (a + (-a) = 0 \to ((-a) + a = a + (-a) \wedge a + (-a) = 0))$
[Tautology $\alpha \to (\beta \to (\alpha \wedge \beta))$]

(13)[1] $a + (-a) = 0 \to ((-a) + a = a + (-a) \wedge a + (-a) = 0)$ [Rule 2, from (7) and (12)]

(14)[1] $(-a) + a = a + (-a) \wedge a + (-a) = 0$ [Rule 2, from (4) and (13)]

(15)[1] $(-a) + a = 0$ [Rule 2, from (11) and (14)]

(16)[1] $\forall x \forall y(x + y = 0 \to y = -x)$ [Theorem 10]

(17)[1] $\forall y((-a) + y = 0 \to y = -(-a))$ [Rule 4, specializing x to $-a$ in (16)]

(18)[1] $(-a) + a = 0 \to a = -(-a))$ [Rule 4, specializing y to a in (17)]

(19)[1] $a = -(-a)$ [Rule 2, from (15) and (18)]

(20)[1] $\forall x \forall y(x = y \to y = x)$ [Axiom Eq2]

(21)[1] $\forall y(a = y \to y = a)$ [Rule 4, specializing x to a in (20)]

(22)[1] $a = -(-a) \to -(-a) = a$ [Rule 4, specializing y to $-(-a)$ in (21)]

(23)[1] $-(-a) = a$ [Rule 2, from (19) and (22)]

(24)[0] $\forall x(-(-x) = x)$ [Rule 5, from (3) and (23)] $\diamondsuit$