MATHEMATICS 300
Introduction to Mathematical Reasoning

H. J. Sussmann — SPRING 2000

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Proof of (19) in Table 4.2, page 102

We want to prove: $\exists x(P \to Q) \leftrightarrow (\forall x\, P \to \exists x\, Q)$

Here P and Q are sentences having no x-quantifiers. If k is a constant, $P(k)$ and $Q(k)$ are the sentences obtained from P, Q by substituting k for x.

PROOF:

(1)[1]	Assume $\exists x(P \to Q)$	[Preparation to apply Rule 3]
(2)[2]	Assume $\forall x\, P$	[Preparation to apply Rule 3]
(3)[3]	Let a be such that $P(a) \to Q(a)$	[Rule 6, from (1)]
(4)[3]	$P(a)$	[Rule 4, specializing x to a in (2)]
(4)[3]	$Q(a)$	[Rule 2, from (3) and (4)]
(5)[3]	$\exists x\, Q$	[Rule 7, from (4)]
(6)[2]	$\exists x\, Q$	[Rule 6, from (3) and (5)]
(7)[1]	$\forall x\, P \to \exists x\, Q$	[Rule 3, from (2) and (6)]
(8)[0]	$\exists x(P \to Q) \to (\forall x\, P \to \exists x\, Q)$	[Rule 3, from (1) and (7)]
(9)[1]	Assume $\forall x\, P \to \exists x\, Q$	[Preparation to apply Rule 3]
(10)[2]	Assume $\sim \exists x(P \to Q)$	[Preparation to apply Rule 3]
(11)[3]	Let a be arbitrary.	[Preparation to apply Rule 5]
(12)[4]	Assume $\sim P(a)$	[Preparation to apply Rule 3]
(13)[4]	$(\sim P(a)) \to (P(a) \to Q(a))$	[Tautology $(\sim \alpha) \to (\alpha \to \beta)$]
(14)[4]	$P(a) \to Q(a)$	[Rule 2, from (12) and (13)]
(15)[4]	$\exists x(P \to Q)$	[Rule 7, from (14)]
(16)[4]	$(\sim \exists x(P \to Q)) \to (\exists x(P \to Q) \to P(a))$	[Tautology $(\sim \alpha) \to (\alpha \to \beta)$]
(17)[4]	$\exists x(P \to Q) \to P(a)$	[Rule 2, from (10) and (16)]
(18)[4]	$P(a)$	[Rule 2, from (15) and (17)]
(19)[3]	$(\sim P(a)) \to P(a)$	[Rule 3, from (12) and (19)]
(20)[3]	$((\sim P(a)) \to P(a)) \to P(a)$	[Tautology $((\sim \alpha) \to \alpha) \to \alpha$]
(21)[3]	$P(a)$	[Rule 2, from (19) and (20)]
(22)[2]	$\forall x\, P$	[Rule 5, from (11) and (21)]
(23)[2]	$\exists x\, Q$	[Rule 2, from (9) and (22)]
(24)[3]	Let b be such that $Q(b)$	[Rule 6, from (23)]
(25)[3]	$Q(b) \to (P(b) \to Q(b))$	[Tautology $\alpha \to (\beta \to \alpha)$]
(26)[3]	$P(b) \to Q(b)$	[Rule 2, from (24) and (25)]
(27)[3]	$\exists x(P \to Q)$	[Rule 7, from (26)]
(28)[2]	$\exists x(P \to Q)$	[Rule 6, from (24) and (27)]
(29)[1]	$(\sim \exists x(P \to Q)) \to \exists x(P \to Q)$	[Rule 2, from (10) and (28)]
(30)[1]	$((\sim \exists x(P \to Q)) \to \exists x(P \to Q)) \to \exists x(P \to Q)$	[Tautology $((\sim \alpha) \to \alpha) \to \alpha$]
(31)[1]	$\exists x(P \to Q)$	[Rule 2, from (29) and (30)]
(32)[0]	$(\forall x\, P \to \exists x\, Q) \to (\exists x(P \to Q))$	[Rule 3, from (9) and (31)]
(33)[0]	$(\exists x(P \to Q) \to (\forall x\, P \to \exists x\, Q)) \to (((\forall x\, P \to \exists x\, Q) \to (\exists x(P \to Q))) \to$ $(\exists x(P \to Q) \leftrightarrow (\forall x\, P \to \exists x\, Q)))$	[Tautology $(\alpha \to \beta) \to ((\beta \to \alpha) \to (\alpha \leftrightarrow \beta))$]
(34)[0]	$((\forall x\, P \to \exists x\, Q) \to (\exists x(P \to Q))) \to (\exists x(P \to Q) \leftrightarrow (\forall x\, P \to \exists x\, Q))$	[Rule 2, from (8) and (33)]
(35)[0]	$\exists x(P \to Q) \leftrightarrow (\forall x\, P \to \exists x\, Q)$	[Rule 2, from (32) and (34)] $\diamondsuit$