MATHEMATICS 300 — SPRING 2001

Introduction to Mathematical Reasoning

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HANDOUT NO. 6—March 1, 2001

1 REMINDERS ON THE FIRST MIDTERM EXAM

The first midterm exam will be on **Monday, March 5**.

The exam will consist of problems similar to the ones in the homework, plus questions on the material of the book and the handouts. (Examples of possible questions: define "even number," explain what a tautology is, explain what a prime number is, explain what the Peano paradox is.)

The topics covered will be: Sections 1 through 9 of the book, plus Section 33, plus the material in the handouts. NOTE: **I am not including Section 10,** because this section has not been covered in class.

Use of the book, the instructor's handouts, calculators, notes of any kind, will **not** be permitted.

Remember that the **new grading policy**, explained in detail in Handout No. 4, will be used to grade this exam.

2 SOLUTIONS THE THE PROBLEMS OF HOMEWORK ASSIGNMENT NO. 7

PROBLEM 1. Let $\mathcal{A}$ be a two-variable predicate letter. Give formal proofs of each of the following:

$$(\exists y, \forall x, \mathcal{A}(x,y)) \longrightarrow (\forall x, \exists y, \mathcal{A}(x,y)) \tag{1}$$

$$(\forall x, \exists y, \mathcal{A}(x,y)) \longrightarrow (\exists y, \forall x, \mathcal{A}(x,y)) \tag{2}$$

SOLUTION. Formula (2) cannot be proved because it is not logically valid. To see this, let us give a particular example of a predicate $\mathcal{A}(x,y)$ for which (2) is false. Suppose the range of the variable x is the set of all people. Let $\mathcal{A}(x,y)$ stand for "y is x's mother." Then $\forall x, \exists y, \mathcal{A}(x,y)$ says that "everybody has a mother," which is true. On the other hand, $\exists y, \forall x, \mathcal{A}(x,y)$ says that "there is a person who is everybody's mother," which is false. So (2) is not true for this particular interpretation of our predicates. Therefore (2) is not logically valid. Hence (2) cannot be proved.

We now give a formal proof of Formula (1).

(1)	Assume $\exists y, \forall x, \mathcal{A}(x,y)$.	[Preparation to apply Rule 4.P]
(2)	Let a be arbitrary.	[Preparation to apply Rule 6.P]
(3)	Pick b such that $\forall x, \mathcal{A}(x,b)$.	[Rule 7.U, from (1)]
(4)	$\mathcal{A}(a,b)$.	[Rule 6.U, from (3), plugging in a for x]
(5)	$\exists y, \mathcal{A}(a,y)$.	[Rule 7.P, from (4)]
(6)	$\exists y, \mathcal{A}(a,y)$.	[Rule 7.U, from (1), (3) and (5)]
(7)	$\forall x, \exists y, \mathcal{A}(x,y)$.	[Rule 6.P, from (2) and (6)]
(8)	$(\exists y, \forall x, \mathcal{A}(x,y)) \longrightarrow (\forall x, \exists y, \mathcal{A}(x,y))$.	[Rule 4.P, from (1) and (7)]

END OF PROOF

PROBLEM 2. Let A, B, C be sets. Assume that $A \subseteq B$ and $B \subseteq C$. Give a formal proof of the statement $A \subseteq C$.

SOLUTION. Here is a proof:

(1)	$A \subseteq B \longleftrightarrow (\forall x, (x \in A \longrightarrow x \in B))$.	[Definition of $\subseteq$]
(2)	$B \subseteq C \longleftrightarrow (\forall x, (x \in B \longrightarrow x \in C))$.	[Definition of $\subseteq$]
(3)	$A \subseteq C \longleftrightarrow (\forall x, (x \in A \longrightarrow x \in C))$.	[Definition of $\subseteq$]
(4)	$A \subseteq B \longrightarrow (\forall x, (x \in A \longrightarrow x \in B))$.	[Rule 5.U, from (1)]
(5)	$B \subseteq C \longrightarrow (\forall x, (x \in B \longrightarrow x \in C))$.	[Rule 5.U, from (2)]
(6)	$(\forall x, (x \in A \longrightarrow x \in C)) \longrightarrow A \subseteq C$.	[Rule 5.U, from (3)]
(7)	$A \subseteq B$.	[Hypothesis]
(8)	$B \subseteq C$.	[Hypothesis]
(9)	$\forall x, (x \in A \longrightarrow x \in B)$.	[Rule 4.U, from (4) and (7)]
(10)	$\forall x, (x \in B \longrightarrow x \in C)$.	[Rule 4.U, from (5) and (8)]
(11)	Let a be arbitrary.	[Preparation to apply Rule 6.P]
(12)	Assume $a \in A$.	[Preparation to apply Rule 4.P]
(13)	$a \in A \longrightarrow a \in B$.	[Rule 6.U, from (9), plugging in a for x]
(14)	$a \in B$.	[Rule 4.U, from (12) and (13)]
(15)	$a \in B \longrightarrow a \in C$.	[Rule 6.U, from (10), plugging in a for x]
(16)	$a \in C$.	[Rule 4.U, from (14) and (15)]
(17)	$a \in A \longrightarrow a \in C$.	[Rule 4.P, from (12) and (16)]
(18)	$\forall x (x \in A \longrightarrow x \in C)$.	[Rule 6.P, from (11) and (17)]
(19)	$A \subseteq C$.	[Rule 4.U, from (6) and (18)]

END OF PROOF

PROBLEM 3. Let $\mathcal{A}$ and $\mathcal{B}$ be one-variable predicate letters. Give formal proofs of:

(3) $$(\forall x, (\mathcal{A}(x) \vee \mathcal{B}(x))) \longleftrightarrow ((\forall x, \mathcal{A}(x)) \vee (\forall x, \mathcal{B}(x)))$$

(4) $$(\forall x, (\mathcal{A}(x) \wedge \mathcal{B}(x))) \longleftrightarrow ((\forall x, \mathcal{A}(x)) \wedge (\forall x, \mathcal{B}(x)))$$

SOLUTION. Formula (3) cannot be proved because it is not logically valid. To see this, let us give a particular example of predicates $\mathcal{A}(x)$, $\mathcal{B}(x)$ for which (3) is false. Suppose the range of the variable x is the set of all people. Let $\mathcal{A}(x)$ stand for "x is male" and $\mathcal{B}(x)$ for "x is female." Then $\forall x, (\mathcal{A}(x) \vee \mathcal{B}(x))$ says that "everybody is either male or female," which is true. On the other hand, $(\forall x, \mathcal{A}(x)) \vee (\forall x, \mathcal{B}(x))$ says that "either everybody is male or everybody is female," which is false. So (3) is not true for this particular interpretation of our predicates. Therefore (3) is not logically valid. Hence (3) cannot be proved.

We now give a formal proof of Formula (4).

(1)	Assume $\forall x, (\mathcal{A}(x) \wedge \mathcal{B}(x))$.	[Preparation to apply Rule 4.P]
(2)	Let a be arbitrary.	[Preparation to apply Rule 6.P]
(3)	$\mathcal{A}(a) \wedge \mathcal{B}(a)$.	[Rule 6.U, from (1), plugging a for x]
(4)	$\mathcal{A}(a)$.	[Rule 2.U, from (3)]
(5)	$\forall x, (\mathcal{A}(x))$.	[Rule 6.P, from (2) and (4)]
(6)	Let a be arbitrary.	[Preparation to apply Rule 6.P]
(7)	$\mathcal{A}(a) \wedge \mathcal{B}(a)$.	[Rule 6.U, from (1), plugging a for x]
(8)	$\mathcal{B}(a)$.	[Rule 2.U, from (7)]
(9)	$\forall x, (\mathcal{B}(x))$.	[Rule 6.P, from (6) and (8)]
(10)	$(\forall x, (\mathcal{A}(x))) \wedge (\forall x, (\mathcal{B}(x)))$.	[Rule 2.P, from (5) and (9)]
(11)	$(\forall x, (\mathcal{A}(x) \wedge \mathcal{B}(x))) \longrightarrow ((\forall x, (\mathcal{A}(x))) \wedge (\forall x, (\mathcal{B}(x))))$.	[Rule 4.P, from (1) and (10)]
(12)	Assume $(\forall x, (\mathcal{A}(x))) \wedge (\forall x, (\mathcal{B}(x)))$.	[Preparation to apply Rule 4.P]
(13)	$\forall x, (\mathcal{A}(x))$.	[Rule 2.U, from (12)]
(14)	$\forall x, (\mathcal{B}(x))$.	[Rule 2.U, from (12)]
(15)	Let a be arbitrary.	[Preparation to apply Rule 6.P]
(16)	$\mathcal{A}(a)$.	[Rule 6.U, from (13), plugging a for x]
(17)	$\mathcal{B}(a)$.	[Rule 6.U, from (14)], plugging a for x]
(18)	$\mathcal{A}(a) \wedge \mathcal{B}(a)$.	[Rule 2.P, from (16) and (17)]
(19)	$\forall x, (\mathcal{A}(x) \wedge \mathcal{B}(x))$.	[Rule 6.P, from (15) and (18)]
(20)	$((\forall x, (\mathcal{A}(x))) \wedge (\forall x, (\mathcal{B}(x)))) \longrightarrow (\forall x, (\mathcal{A}(x) \wedge \mathcal{B}(x)))$.	[Rule 4.P, from (12) and (19)]
(21)	$((\forall x, (\mathcal{A}(x))) \wedge (\forall x, (\mathcal{B}(x)))) \longleftrightarrow (\forall x, (\mathcal{A}(x) \wedge \mathcal{B}(x)))$.	[Rule 5.P, from (11) and (20)]

END OF PROOF