MATHEMATICS 300 — FALL 1998

Introduction to Mathematical Reasoning

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HANDOUT NO. 6

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READING FOR SEPTEMBER 28-30

Book, Sections 1.4, 1.5 and 1.6.

HOMEWORK ASSIGNMENT NO. 3
DUE ON WEDNESDAY, SEPTEMBER 30

a. Problems 4, 5, 6, 7, 9, 10 in pages 38, 39 (nonstarred parts only).
b. Problems 5 and 9 in pages 46-47-48-49 (nonstarred parts only).

REVIEW QUESTIONS

59. What is an "open sentence"?
60. What is the "truth set" of a sentence?
61. Explain what is wrong with the following answer to the question "what is the 'truth set' of a sentence?": **truth set** is the collection of objects that may be substituted to make an open sentence a true proposition.
62. What does it mean for two open sentences to be "equivalent"?
63. Give an example of a pair of sentences with one open variable that are equivalent when the universe of discourse is the set of natural numbers but are not equivalent when the universe of discourse is the set of real numbers.
64. Give an example of a pair of sentences with one open variable that are equivalent when the universe of discourse is the set of real numbers but are not equivalent when the universe of discourse is the set of natural numbers.
65. What does it mean for two quantified sentences to be "equivalent"?

Two quantified sentences without open variables are *logically equivalent* if they are equivalent for every possible choice of the universe of discourse and every choice of interpretation for the predicates and function sysmbols involved. For example:

a. $\sim(\forall x)p(x)$ and $(\exists x)(\sim p(x))$ are logically equivalent, because, no matter what "$p(x)$" means, the first one says that "it is not true that all x's satisfy $p(x)$," and this is equivalent to saying that "there is one x that does not satisfy $p(x)$," which is precisely the second sentence.

b. $(\forall x)p(x)$ and $(\exists x)p(x)$ are not logically equivalent, because one can give an example of a way to choose the universe of discourse and the interpretation of the predicate p that makes the two sentences not equivalent. It suffices to take the universe of discourse to be the set of all people, and have $p(x)$ stand for "x is a Rutgers student." Then "$(\forall x)p(x)$" says "everybody is a Rutgers student," which is clearly false, but $(\exists x)p(x)$ says "somebody is a Rutgers student," which is true.

66. In a predicate calculus in which there is a 2-variable predicate $p(x, y)$, is it true or false that the sentences $(\forall x)(\exists y)p(x, y)$ and $(\exists y)(\forall x)p(x, y)$ are logically equivalent?
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68. In a predicate calculus in which there are two 1-variable predicates $p(x)$, $q(x)$, is it true or false that the sentences $(\forall x)(p(x) \wedge q(x))$ and $(\forall x)p(x) \wedge (\forall x)q(x)$ are logically equivalent?
69. In a predicate calculus in which there are two 1-variable predicates $p(x)$, $q(x)$, is it true or false that the sentences $(\forall x)(p(x) \vee q(x))$ and $(\forall x)p(x) \vee (\forall x)q(x)$ are logically equivalent?
70. In a predicate calculus in which there are two 1-variable predicates $p(x)$, $q(x)$, is it true or false that the sentences $(\exists x)(p(x) \wedge q(x))$ and $(\exists x)p(x) \wedge (\exists x)q(x)$ are logically equivalent?
71. In a predicate calculus in which there are two 1-variable predicates $p(x)$, $q(x)$, is it true or false that the sentences $(\exists x)(p(x) \vee q(x))$ and $(\exists x)p(x) \vee (\exists x)q(x)$ are logically equivalent?
72. In a predicate calculus in which there are two 1-variable predicates $p(x)$, $q(x)$, is it true or false that the sentences $(\forall x)(p(x) \Longrightarrow q(x))$ and $(\forall x)p(x) \Longrightarrow (\forall x)q(x)$ are logically equivalent?
73. In a predicate calculus in which there are two 1-variable predicates $p(x)$, $q(x)$, is it true or false that the sentences $(\exists x)(p(x) \Longrightarrow q(x))$ and $(\exists x)p(x) \Longrightarrow (\exists x)q(x)$ are logically equivalent?
74. Explain what "logical equivalence" means.
75. What is the "modus ponens" rule?
76. Prove that the set of primes is infinite.
77. What is an irrational number?
78. Prove that $\sqrt{2}$ is irrational.
79. Is $\sqrt{2} + \sqrt{3}$ rational or irrational? Justify your answer.
80. Prove or disprove: the sum of two rational numbers is rational.
81. Prove or disprove: the sum of two irrational numbers is irrational.
82. Prove or disprove: the sum of a rational number and an irrational number is rational.
83. Prove or disprove: the sum of a rational number and an irrational number is irrational.
84. Define "absolute value."
85. Prove that $-|x| \leq x \leq |x|$ for every real number x.
86. Prove that $|x + y| \leq |x| + |y|$ for all real numbers x, y.
87. For a predicate calculus with a two-variable symbol $p(x, y)$, take the universe of discourse to be the set of all people, and interpret $p(x, y)$ to mean "x is y's father." Translate each of the following sentences into symbolic language:

    a. Everybody has a father.
    b. Everybody has one and only one father.
    c. Nobody is his own father.
    d. The father of the father of a person is not the father of that person.
    e. Some people are fathers and some people are not.