MATHEMATICS 300 — SPRING 2001

Introduction to Mathematical Reasoning

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HANDOUT NO. 5—February 26, 2001

1 HOMEWORK ASSIGNMENT NO. 7

This assignment is due on **Thursday, March 1**. It consists of the following three problems.

PROBLEM 1. Let $\mathcal{A}$ be a two-variable predicate letter. Give formal proofs of each of the following:

$$(\exists y, \forall x, \mathcal{A}(x,y)) \longrightarrow (\forall x, \exists y, \mathcal{A}(x,y)) \tag{1}$$

$$(\forall x, \exists y, \mathcal{A}(x,y)) \longrightarrow (\exists y, \forall x, \mathcal{A}(x,y)) \tag{2}$$

PROBLEM 2. Let A, B, C be sets. Assume that $A \subseteq B$ and $B \subseteq C$. Give a formal proof of the statement $A \subseteq C$.

PROBLEM 3. Let $\mathcal{A}$ and $\mathcal{B}$ be one-variable predicate letters. Give formal proofs of each of the following:

$$(\forall x, (\mathcal{A}(x) \wedge \mathcal{B}(x))) \longleftrightarrow ((\forall x, \mathcal{A}(x)) \wedge (\forall x, \mathcal{B}(x))) \tag{3}$$

$$(\forall x, (\mathcal{A}(x) \vee \mathcal{B}(x))) \longleftrightarrow ((\forall x, \mathcal{A}(x)) \vee (\forall x, \mathcal{B}(x))) \tag{4}$$

2 REMINDERS ON THE FIRST MIDTERM EXAM

The first midterm exam will be on **Monday, March 5**.

The exam will consist of problems similar to the ones in the homework, plus questions on the material of the book and the handouts. (Examples of possible questions: define "even number," explain what a tautology is, explain what a prime number is, explain what the Peano paradox is.)

Use of the book, the instructor's handouts, calculators, notes of any kind, will **not** be permitted.

Remember that the **new grading policy**, explained in detail in Handout No. 4, will be used to grade this exam.

3 SOME EXAMPLES OF FORMAL PROOFS

PROBLEM. Let $\mathcal{A}$ and $\mathcal{B}$ be one-variable predicate letters. Give formal proofs of each of the following:

$$(\exists x, (\mathcal{A}(x) \wedge \mathcal{B}(x))) \longleftrightarrow ((\exists x, \mathcal{A}(x)) \wedge (\exists x, \mathcal{B}(x))) \tag{$*$}$$

$$(\exists x, (\mathcal{A}(x) \vee \mathcal{B}(x))) \longleftrightarrow ((\exists x, \mathcal{A}(x)) \vee (\exists x, \mathcal{B}(x))) \tag{@}$$

SOLUTION. Formula (*) cannot be proved because it is not logically valid. To see this, let us give a particular example of predicates $\mathcal{A}(x)$, $\mathcal{B}(x)$ for which (*) is false. Let $\mathcal{A}(x)$ stand for "x is a Democratic Senator" and $\mathcal{B}(x)$ for "x is a Republican Senator." Then $(\exists x, \mathcal{A}(x)) \wedge (\exists x, \mathcal{B}(x))$ says that "there are Democratic Senators and there are Republican Senators," which is true. On the other hand, $\exists x, (\mathcal{A}(x) \wedge \mathcal{B}(x))$ says that "some people are both Democratic Senators and Republican Senators," which is false. So (*) is not true for this particular interpretation of our predicates.

We now give a formal proof of Formula (@).

(1)	Assume $\exists x, (\mathcal{A}(x) \vee \mathcal{B}(x))$.	[Preparation to apply Rule 4.P]
(2)	Pick a such that $\mathcal{A}(a) \vee \mathcal{B}(a)$.	[Rule 7.U, from (1)]
(3)	Assume $\mathcal{A}(a)$.	[Preparation to apply Rule 3.U]
(4)	$\exists x, (\mathcal{A}(x))$.	[Rule 7.P, from (3)]
(5)	$(\exists x, (\mathcal{A}(x))) \longrightarrow ((\exists x, (\mathcal{A}(x))) \vee (\exists x, (\mathcal{B}(x))))$	[Rule 0]
(6)	$(\exists x, (\mathcal{A}(x))) \vee (\exists x, (\mathcal{B}(x)))$	[Rule 4.U, from (4) and (5)]
(7)	$\mathcal{A}(a) \longrightarrow ((\exists x, (\mathcal{A}(x))) \vee (\exists x, (\mathcal{B}(x))))$.	[Rule 4.P, from (3) and (6)]
(8)	Assume $\mathcal{B}(a)$.	[Preparation to apply Rule 3.U]
(9)	$\exists x, (\mathcal{B}(x))$.	[Rule 7.P, from (8)]
(10)	$(\exists x, (\mathcal{B}(x))) \longrightarrow ((\exists x, (\mathcal{A}(x))) \vee (\exists x, (\mathcal{B}(x))))$	[Rule 0]
(11)	$(\exists x, (\mathcal{A}(x))) \vee (\exists x, (\mathcal{B}(x)))$	[Rule 4.U, from (9) and (10)]
(12)	$\mathcal{B}(a) \longrightarrow ((\exists x, (\mathcal{A}(x))) \vee (\exists x, (\mathcal{B}(x))))$.	[Rule 4.P, from (3) and (6)]
(13)	$(\exists x, (\mathcal{A}(x))) \vee (\exists x, (\mathcal{B}(x)))$.	[Rule 3.U, from (2), (7) and (12)]
(14)	$(\exists x, (\mathcal{A}(x))) \vee (\exists x, (\mathcal{B}(x)))$.	[Rule 7.U, from (1), (2) and (13)]
(15)	$(\exists x, (\mathcal{A}(x) \vee \mathcal{B}(x))) \longrightarrow ((\exists x, (\mathcal{A}(x))) \vee (\exists x, (\mathcal{B}(x))))$.	[Rule 4.P, from (1) and (14)]
(16)	Assume $(\exists x, (\mathcal{A}(x))) \vee (\exists x, (\mathcal{B}(x))))$.	[Preparation to apply Rule 4.P]
(17)	Assume $\exists x, (\mathcal{A}(x))$.	[Preparation to apply Rule 3.U]
(18)	Pick a such that $\mathcal{A}(a)$.	[Rule 7.U, from (17)]
(19)	$\mathcal{A}(a) \longrightarrow \mathcal{A}(a) \vee \mathcal{B}(a)$.	[Rule 0]
(20)	$\mathcal{A}(a) \vee \mathcal{B}(a)$.	[Rule 4.U, from (18) and (19)]
(21)	$\exists x, ((\mathcal{A}(x)) \vee (\mathcal{B}(x)))$.	[Rule 7.P, from (20)]
(22)	$\exists x, ((\mathcal{A}(x)) \vee (\mathcal{B}(x)))$.	[Rule 7.U, from (17), (18) and (21)]
(23)	$(\exists x, (\mathcal{A}(x))) \longrightarrow (\exists x, ((\mathcal{A}(x)) \vee (\mathcal{B}(x))))$.	[Rule 4.P, from (17) and (22)]
(24)	Assume $\exists x, (\mathcal{B}(x))$.	[Preparation to apply Rule 3.U]
(25)	Pick a such that $\mathcal{B}(a)$.	[Rule 7.U, from (24)]
(26)	$\mathcal{B}(a) \longrightarrow \mathcal{A}(a) \vee \mathcal{B}(a)$.	[Rule 0]
(27)	$\mathcal{A}(a) \vee \mathcal{B}(a)$.	[Rule 4.U, from (25) and (26)]
(28)	$\exists x, ((\mathcal{A}(x)) \vee (\mathcal{B}(x)))$.	[Rule 7.P, from (27)]
(29)	$\exists x, ((\mathcal{A}(x)) \vee (\mathcal{B}(x)))$.	[Rule 7.U, from (24), (25) and (28)]
(30)	$(\exists x, (\mathcal{B}(x))) \longrightarrow (\exists x, ((\mathcal{A}(x)) \vee (\mathcal{B}(x))))$.	[Rule 4.P, from (24) and (29)]
(31)	$((\exists x, (\mathcal{A}(x))) \vee (\exists x, (\mathcal{B}(x)))) \longrightarrow (\exists x, (\mathcal{A}(x) \vee \mathcal{B}(x)))$.	[Rule 4.P, from (16) and (30)]
(32)	$((\exists x, (\mathcal{A}(x))) \vee (\exists x, (\mathcal{B}(x)))) \longleftrightarrow (\exists x, (\mathcal{A}(x) \vee \mathcal{B}(x)))$.	[Rule 5.P, from (15) and (31)]

END OF PROOF

PROBLEM. Let $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, be one-variable predicate letters. Let $\mathcal{S}$ be a two-variable predicate letter. Let a, b be constants.

I. Using the following six hypotheses

(H1) $$\forall x, \forall y, ((\mathcal{P}(x) \wedge \mathcal{Q}(y)) \longrightarrow \mathcal{S}(x,y))\,,$$

(H2) $$\forall x, \forall y, ((\mathcal{Q}(x) \wedge \mathcal{R}(y)) \longrightarrow \mathcal{S}(x,y))\,,$$

(H3) $$\forall u, \forall v, \forall w, ((\mathcal{S}(u,v) \wedge \mathcal{S}(v,w)) \longrightarrow \mathcal{S}(u,w))\,,$$

(H4) $$\mathcal{P}(a)\,,$$

(H5) $$\mathcal{R}(b)\,,$$

(H6) $$\exists x, (\mathcal{Q}(x))\,,$$

give a formal proof of

(T) $$\mathcal{S}(a,b)\,.$$

II. Give a formal proof of (T) from the hypotheses (H1), (H2), (H3), (H4), (H5) alone, without using (H6).

SOLUTION. Formula (T) cannot be proved from (H1), (H2), (H3), (H4), (H5) alone, without using (H6), because (T) does not follow logically from (H1), (H2), (H3), (H4), (H5). To see this, let us give a particular example of predicates $\mathcal{P}(x)$, $\mathcal{Q}(x)$, $\mathcal{R}(x)$, $\mathcal{S}(x,y)$, for which (T) is false. Let $\mathcal{P}(x)$, $\mathcal{Q}(x)$, $\mathcal{R}(x)$, $\mathcal{S}(x,y)$, stand for "x is a rabbit," "x is a unicorn," "x is an elephant," "x is taller than y," respectively. (The crucial point about unicorns is that there aren't any. If you believe that unicorns exist, then use something else instead, for example pigs with wings, six-legged ostriches, or round squares.) Let a be a particular rabbit, and b a particular elephant. (You are free to choose any rabbit and any elephant you want.) Then (H1), (H2), (H3), (H4), (H5) are true, but (T) is false. So (T) cannot be proved from (H1), (H2), (H3), (H4), (H5), as asserted.

We now give a formal proof of (T) using (H1), (H2), (H3), (H4), (H5), and (H6).

(1)	$\exists x, (\mathcal{Q}(x))$.	[Hypothesis (H6)]
(2)	Pick c such that $\mathcal{Q}(c)$.	[Rule 7.U, from (1)]
(3)	$\forall x, \forall y, ((\mathcal{P}(x) \wedge \mathcal{Q}(y)) \longrightarrow \mathcal{S}(x,y))$.	[Hypothesis (H1)]
(4)	$\forall x, \forall y, ((\mathcal{Q}(x) \wedge \mathcal{R}(y)) \longrightarrow \mathcal{S}(x,y))$.	[Hypothesis (H2)]
(5)	$\forall y, ((\mathcal{P}(a) \wedge \mathcal{Q}(y)) \longrightarrow \mathcal{S}(a,y))$.	[Rule 6.U. from (3), plugging a for x]
(6)	$(\mathcal{P}(a) \wedge \mathcal{Q}(c)) \longrightarrow \mathcal{S}(a,c))$.	[Rule 6.U. from (5), plugging c for y]
(7)	$\mathcal{P}(a)$.	[Hypothesis (H4)]
(8)	$\mathcal{P}(a) \wedge \mathcal{Q}(c)$.	[Rule 2.P, from (2) and (7)]
(9)	$\mathcal{S}(a,c)$.	[Rule 4.U. from (6) and (8)]
(10)	$\forall y, ((\mathcal{Q}(c) \wedge \mathcal{R}(y)) \longrightarrow \mathcal{S}(c,y))$.	[Rule 6.U. from (4), plugging c for x]
(11)	$(\mathcal{Q}(c) \wedge \mathcal{R}(b)) \longrightarrow \mathcal{S}(c,b))$.	[Rule 6.U. from (10), plugging b for y]
(12)	$\mathcal{R}(b)$.	[Hypothesis (H5)]
(13)	$\mathcal{Q}(c) \wedge \mathcal{R}(b)$.	[Rule 2.P, from (2) and (12)]
(14)	$\mathcal{S}(c,b)$.	[Rule 4.U. from (11) and (13)]
(15)	$\forall u, \forall v, \forall w, ((\mathcal{S}(u,v) \wedge \mathcal{S}(v,w)) \longrightarrow \mathcal{S}(u,w))$.	[Hypothesis (H3)]
(16)	$\forall v, \forall w, ((\mathcal{S}(a,v) \wedge \mathcal{S}(v,w)) \longrightarrow \mathcal{S}(a,w))$.	[Rule 6.U. from (15), plugging a for u]
(17)	$\forall w, ((\mathcal{S}(a,c) \wedge \mathcal{S}(c,w)) \longrightarrow \mathcal{S}(a,w))$.	[Rule 6.U. from (16), plugging c for v]
(18)	$(\mathcal{S}(a,c) \wedge \mathcal{S}(c,b)) \longrightarrow \mathcal{S}(a,b))$.	[Rule 6.U. from (17), plugging b for w]
(19)	$\mathcal{S}(a,c) \wedge \mathcal{S}(c,b)$.	[Rule 2.P, from (9) and (14)]
(20)	$\mathcal{S}(a,b)$.	[Rule 2.U, from (18) and (19)]
(21)	$\mathcal{S}(a,b)$.	[Rule 7.U, from (1), (2), and (20)]

END OF PROOF

INFORMAL VERSION OF THE ABOVE PROOF. Since we know that there exist things x for which $\mathcal{Q}(x)$ is true, pick one and call it c. Since $\mathcal{P}(a)$ and $\mathcal{Q}(c)$ are true, Hypothesis (H1) implies that $\mathcal{S}(a,c)$. Since $\mathcal{Q}(c)$ and $\mathcal{R}(b)$ are true, Hypothesis (H2) implies that $\mathcal{S}(c,b)$. Since $\mathcal{S}(a,c)$ and $\mathcal{S}(c,b)$ are true, Hypothesis (H3) implies that $\mathcal{S}(a,b)$. **THE END.**

INFORMAL VERSION OF THE ABOVE PROOF, USING GIRAFFES, COWS AND SHEEP. Think of "$\mathcal{P}(x)$," "$\mathcal{Q}(x)$," "$\mathcal{R}(x)$," as standing for "x is a giraffe," "x is a cow," and "x is a sheep," respectively, and of "$\mathcal{S}(x,y)$," as standing for "x is taller than y." Think of a as Olivia and of b as Dolly. Hypothesis (H6) tells us that there are cows. So pick one cow and give her a name, e.g. Clarabella. Since Olivia is a giraffe and Clarabella is a cow, Hypothesis (H1) tells us that Olivia is taller than Clarabella. Since Clarabella is a cow and Dolly is a sheep, Hypothesis (H2) tells us that Clarabella is taller than Dolly. Since Olivia is taller than Clarabella and Clarabella is taller than Dolly. Hypothesis (H3) tells us that Olivia is taller than Dolly. **THE END.**

LOGICAL VALIDITY (A.K.A. LOGICAL TRUTH

A sentence in a formal language is *logically valid* if it becomes a true sentence no matter which predicates are chosen to stand for the predicate letters.
For example, if P is a one-variable predicate letter, then "$\forall x, (P(x) \vee (\neg P(x)))$" is logically valid.

LOGICAL CONSEQUENCES (A.K.A. SENTENCES THAT FOLLOW LOGICALLY FROM A SET OF HYPOTHESES)

Given a set **H** of hypotheses, a sentence S is a *logical consequence* of the hypotheses (or "logically follows" from the hypotheses) if no matter what meaning is assigned to the predicate letters and constants involved in **H** and S, if the hypotheses **H** are all true then S is true.
We use the notation $\mathbf{H} \vdash S$ to indicate that S logically follows from **H**.
For example, if **H** consists of the sentences $\forall x, (A(x) \longrightarrow B(x))$ and $A(a)$, then $B(a)$ follows logically from **H**. In other words,

$$\{\forall x, (A(x) \longrightarrow B(x))\,,\ A(a)\} \vdash B(a)\,.$$

PREDICATES AND PREDICATE LETTERS

A *one-variable predicate* is a statement containing one variable, so that for each value of the variable the resulting assertion is a sentence, i.e. a statement with a definite truth value. For example: "x is a Rutgers professor" is a one-variable predicate. When you plug in "Héctor Sussmann" for the variable x you get a true sentence, and when you plug in "George W. Bush" for the variable x you get a false sentence.
Here is another example: "n is an integer such that $n = m^2$ for some integer m" is a one-variable predicate. When you plug in 4 for the variable n you get a true sentence, but when you plug in 3 you get a false sentence.

REMARK. In the predicate "n is an integer such that $n = m^2$ for some integer m," you may think that there are two variables, namely, n and m. Notice, however, that m is *quantified*, that is, under the scope of a quantifier ("for some," that is, an *existential* quantifier).
Another way to see this is to ask yourself *which are the variables whose values I need to know in order to decide if the sentence is true or false?* For the predicate "n is an integer such that $n = m^2$ for some integer m," if you want to know if it is true or false, what you need to know is n.
A *two-variable predicate* is a statement containing two variables, so that for each choice of values of these two variables the resulting assertion is a sentence, i.e. a statement with a definite truth value. For example: "x is wealthier than y" is a two-variable predicate. When you plug in "Bill Gates" for the variable x and "Héctor Sussmann" for the variable y you get a true sentence, but if you plug in "George W. Bush" for the variable x and "Bill Gates" for the variable y you get a false sentence.
Here is another example. Assume the variables n and m take integer values. Then "n is divisible by m" is a two-variable predicate. When you plug in 24 for the variable n and 6 for the variable m you get a true sentence, but when you plug in 24 for n and 7 for m you get a false sentence.

REMARK. The predicate "n is divisible by m" is equivalent to "there exists an integer q such that $n = qm$." So you may think that this predicate really involves three variables, namely, n, m and q. Notice, however, that q is quantified, so we only need to ask for the values of n and m if we want to decide if the predicate is true or false.

We use **letters** to represent predicates.