MATHEMATICS 300 — FALL 1998

Introduction to Mathematical Reasoning

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HANDOUT NO. 5

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Here are three answers—written by three students in our class—to the problem in Homework assignment No. 1 having to do with statement 3 of Handout No. 1. One of the answers is perfect (and would get a grade of 10 on a scale from 0 to 10), another one is fairly good but not quite complete (in the sense that I can see that the author got the main point, but some of the things it says are confusing; I would give it a grade of 7 or 7.5), and one of them is completely unacceptable, because it is not clear what the author means, and to the extent that something is being said at all it is not true. (This one would get 2 out of 10, because I am giving two points for saying correctly whether the answer is "true" or "false," and eight points for giving a convincing reason.) Can you see by yourself which is which?
Let me refresh your memory by reminding you what the question was:

STATEMENT 3: There exists exactly one positive integer n such that $n > 1$ and the numbers n, $n+2$ and $n+4$ are prime. (Note: If we take $n = 3$ then $n > 1$, and the three numbers n, $n+2$ and $n+4$ —that is, 3, 5 and 7— are prime. So a number n with the desired properties definitely does exist. The question here is whether 3 is the only number with this property. Are there others? Yes? No? Why?)

And here are the three answers:

ANSWER NO. 1: This is a true statement. $n = 3$ is the only case that works. If n is even, then n, $n+2$ and $n+4$ are divisible by 2. If n is odd, we have three consecutive odd numbers, one of which must be odd. Every third number is divisible by 3, alternating between even and odd, so every third odd number is divisible by 3. 3, 5 and 7 has one number divisible by 3, but that number is 3 and it is prime.

ANSWER NO. 2: Statement 3 is true. All positive integers that are greater than 1 can be written in one of the following forms: $3k$, $3k-1$, $3k+1$, where k is a natural number. If $n = 3k$, it is divisible by 3. If $n = 3k-1$, then $n+4 = 3k+3 = 3(k+1)$, which is divisible by 3. If $n = 3k+1$ then $n+2 = 3k+3 = 3(k+1)$, which is divisible by 3. The only prime number divisible by 3 is 3 itself. 3 only appears when $n = 3$. Therefore this is the only case when n, $n+2$ and $n+4$ are all prime.

ANSWER NO. 3: This statement is **true**, because for any n prime number to satisfy $n+2$ and $n+4$ giving prime numbers, the next two prime numbers to that n would have to be 2 integeres (that is: n, $n+1$, $n+2$, $n+3$, $n+4$) away from each other. After 3, 5 and 7 the distance between prime numbers increases so that there are no three prime numbers, within two digits of each other.

Here are three answers to the problem in Homework assignment No. 1 having to do with statement 5 of Handout No. 1. One of the answers is perfect (and would get a grade of 10 on a scale from 0 to 10), another one is good but not quite O.K. (in the sense that I can see that the author got the main point, but some of the things actually said are inaccurate and/or confusing; I would give it a grade of 6), and one of them is completely unacceptable, because it seems to have been written by someone who followed the example of "how **not** to write" in Handout 2 and made the mistake of taking this as a model of how one **should** write. (This one would get 3 out of 10, because I am giving two points for saying correctly whether the answer is "true" or "false," and eight points for giving a convincing reason, and the author seems to have some idea of what might be the reason why the statement is true.) Can you see by yourself which is which? Also, can you tell in how many ways the answer that gets 3 out of 10 fits exactly the pattern of my "how not to write" example? Let me refresh your memory by reminding you what the question was:

STATEMENT 5: If x and y are positive real numbers then $\sqrt{xy} \leq \frac{x+y}{2}$.

And here are the three answers:

ANSWER NO. 1: If $\sqrt{xy} \leq \frac{x+y}{2}$, then $xy \leq \frac{x^2+y^2+2xy}{4}$ by squaring both sides. Rearranging sides, $xy \leq x^2+y^2+2xy$ or $o \leq x^2+y^2-2xy$. This means that $(x-y) \geq 0$. Since statement 5 is equivalent to the tautology $(x-y)^2 \geq 0$. Statement 5 is therefore, also a tautology. Statement 5 is true.

ANSWER NO. 2: True

$$\sqrt{xy} \leq \tfrac{x+y}{2} \text{ for } x, y \text{ positive real}$$

$$2\sqrt{xy} \leq x+y$$

$$4xy \leq (x+y)^2$$

$$(x+y)^2 - 4xy \geq 0$$

$$(x-y)^2 > 0$$

Since the square of any real number is positive So the statement is true.

ANSWER NO. 3: The statement is true.
PROOF: Let x, y be real numbers such that $x \geq 0$, $y \geq 0$. We want to prove that $\sqrt{xy} \leq \frac{x+y}{2}$. Now, $\sqrt{xy} \leq \frac{x+y}{2}$ is equivalent to $2\sqrt{xy} \leq x+y$, and $2\sqrt{xy} \leq x+y$ is equivalent to $4xy \leq (x+y)^2$, because if a, b are real numbers, $a \geq 0$ and $b \geq 0$, then $a \leq b$ is equivalent to $a^2 \leq b^2$.
Moreover, $4xy \leq (x+y)^2$ is equivalent to $(x+y)^2 - 4xy \geq 0$. But

$$(x+y)^2 - 4xy = x^2 + y^2 + 2xy - 4xy = x^2 + y^2 - 2xy = (x-y)^2 .$$

So $(x+y)^2 - 4xy \geq 0$ is equivalent to $(x-y)^2 \geq 0$. But the square of any real number is nonnegative. So $(x-y)^2 \geq 0$ is true. But then all the other formulas that are equivalent to it are also true. In particular, $\sqrt{xy} \leq \frac{x+y}{2}$ is true, and our proof is complete.