MATHEMATICS 300 — Introduction to Mathematical Reasoning

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(With thanks to several students who have pointed out mistakes in the original March 9 version.)

NOTE: In the March 9 version this handout was called "Handout No. 6." The correct number should have been 5, not 6.

HOMEWORK DUE ON THURSDAY MARCH 30: Problems 1 through 5 in the last page of this handout.

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1. AN EXAMPLE OF A FORMAL PROOF

Let us work in a predicate calculus used as a language for *kinship*. This calculus has two binary predicates, namely, P and $=$, and two 1-variable predicates m and f. The predicate P is written in **prefix** form (that is, we write "$P(x,y)$") and the predicate $=$ is written in **infix** form (that is, we write "$x = y$ rather than "$=(x,y)$"). "$P(x,y)$" should be thought of as meaning "x is y's parent," "$m(x)$" as "x is male," and "$f(x)$" as "x is female.")

There are two axioms:

(K1) $\quad \forall x(m(x) \vee f(x) \wedge (\sim (m(x) \wedge f(x))))$

(K2) $\quad \forall x \exists y \exists z(P(y,x) \wedge P(z,x) \wedge m(y) \wedge f(z) \wedge \forall u(P(u,x) \rightarrow (u = y \vee u = z)))$

We define a new "grandmother" binary predicate g:

(DEF) $\quad \forall x \forall y(g(x,y) \leftrightarrow \exists z(P(x,z) \wedge P(z,y) \wedge f(x)))$

THEOREM G. Everybody has a grandmother.

INFORMAL PROOF. Let a be arbitrary. We are going to prove that a has a grandmother. By Axiom K2, a has two parents, one of whom is male and one of whom is female. Let b be a male parent of a, and let c be a female grandparent of a. By Axiom K2, b has two parents, one of whom is male and one of whom is female. Let d be a male parent of b, and let e be a female parent of b. Then e is a female parent of b and b is a parent of a. So there is a person—namely, b—who is a parent of a and is such that e is a mother of that person. The definition of "grandmother" says that e is a grandmother of a if and only if there is a person who is a parent of a and is such that e is a mother of that person. So e is a grandmother of a. Therefore a has a grandmother. Since a was arbitrary, we have proved that a has a grandmother. $\diamondsuit$

AN EVEN MORE INFORMAL PROOF. Let a be arbitrary. By Axiom K2, a has a father and a mother. Let b be a father of a. By Axiom K2, b has a father and a mother. Let e be a mother of b. Then e is a mother of b and b is a parent of a. So by the definition of "grandmother" e is a grandmother of a. Therefore a has a grandmother. $\diamondsuit$

A COMMENT ON INFORMAL VS. FORMAL PROOFS. A *proof* (a.k.a "formal proof") in a first-order mathematical theory (that is, in a formal language which is a first-order predicate calculus with its usual rules of inference, in which some axioms and definitions are specified) is a sequence of *steps*, each of which consists of an *assertion* together with a *justification*.

The **assertion** must be either

(a) a sentence in the language, which

- is an axiom

or

- is a definition,

or

- follows from the assertions of previous steps by one of the rules of inference

or

(b) is an introduction of a new assumption or a new name.

(Examples of Type (a) steps: all the ones in the formal proof of Theorem G below, except for Steps (1), (4), (9), (10), and all the ones in the formal proof of Formula (I), except for Steps (1), (2), (13), (17), (18). Examples of Type (b) steps: Steps (1), (4), (9), (10) in the formal proof of Theorem G below, and Steps (1), (2), (13), (17), (18) in the formal proof of (I).)

The **justification** must be an explanation of why the assertion or introduction is permitted. So the justification must tell us whether the assertion is an axiom (and, in that case, which one), or a definition (and, in that case, which one), or something that follows from the assertions of previous steps by one of the rules of inference (and, if this is the case, it must tell us *which* previous steps it follows from, and by means of *which rule of inference*), or an introduction of a new assumption or a new name (and in that case it must tell us why this is permitted; if it is an introduction of a new assumption, or of an arbitrary object, then no such explanation is needed, because we can always introduce new assumptions and new arbitrary objects; if we want to introduce a new object characterized by a condition—for example, by saying "Let a be such that $p(a)$"—then this is permitted by Rule 6, provided that we have an earlier step saying "$\exists x\, p(x)$.")

If we are given a collection $\mathcal{H}$ of sentences, a *proof from hypotheses* $\mathcal{H}$ is the same as a "proof," except that now a step can also be one of the hypotheses in $\mathcal{H}$.

According to the above definition of "proof," the "informal proofs" we have just given are *not* proofs. A real proof of Theorem G, involving 28 steps, is given below.

What is the point of giving informal proofs? The answer is this. If you want to convince me that that something is true, you have to *persuade me that there is a proof.* For this purpose, it usually suffices for you to give me enough information to show to me how the proof would go. You don't really have to give me the full proof. What you really have to do is tell me, in ordinary English, how you would go about doing a real (i.e., formal) proof.

PROOF OF THEOREM G. In symbolic language, the statement that we have to prove says $\quad \forall x \exists y\, g(y,x)$.

Here is a formal proof of Theorem G:

(1) $\forall x \exists y \exists z (P(y,x) \land P(z,x) \land m(y) \land f(z) \land \forall u (P(u,x) \to (u = y \lor u = z)))$ [Axiom K2]

(2) Let a be arbitrary [Preparation to apply Rule 3]

(3) $\exists y \exists z (P(y,a) \land P(z,a) \land m(y) \land f(z) \land \forall u (P(u,a) \to (u = y \lor u = z)))$ [R. 4, from (1)]

(4) Pick b such that
$\exists z (P(b,a) \land P(z,a) \land m(b) \land f(z) \land \forall u (P(u,a) \to (u = b \lor u = z)))$ [R. 6, from (3)]

(5) Pick c such that
$P(b,a) \land P(c,a) \land m(b) \land f(c) \land \forall u (P(u,a) \to (u = b \lor u = c))$[R. 6, from (4)]

(6) $(P(b,a) \land P(c,a) \land m(b) \land f(c) \land \forall u (P(u,a) \to (u = b \lor u = c))) \to P(b,a)$
(Tautology $(A \land B \land C \land D \land E) \to A$.)

(7) $P(b,a)$ [Rule 2 from (5), (6)]

(8) $\exists y \exists z (P(y,b) \land P(z,b) \land m(y) \land f(z) \land \forall u (P(u,b) \to (u = y \lor u = z)))$
[Rule 4, from (1)]

(9) Pick d such that
$\exists z (P(d,b) \land P(z,b) \land m(d) \land f(z) \land \forall u (P(u,b) \to (u = d \lor u = z)))$
[Rule 4, from (8)]

(10) Pick e such that
$P(d,b) \land P(e,b) \land m(d) \land f(e) \land \forall u (P(u,b) \to (u = d \lor u = e))$
[R. 4, from (9)]

(11) $(P(d,b) \land P(e,b) \land m(d) \land f(e) \land \forall u (P(u,b) \to (u = d \lor u = e)))$
$\to (P(e,a) \land f(e))$ [Tautology $(A \land B \land C \land D \land E) \to (B \land D)$]

(12) $P(e,b) \land f(e)$ [Rule 2 from (10), (11)]

(13) $\forall x \forall y (g(x,y) \leftrightarrow \exists z (P(x,z) \land P(z,y) \land f(x)))$. [Definition of g]

(14) $\forall y (g(e,y) \leftrightarrow \exists z (P(e,z) \land P(z,y) \land f(e)))$. [R.4, spec. x to e in (13)]

(15) $g(e,a) \leftrightarrow \exists z (P(e,z) \land P(z,a) \land f(e)))$. [Rule 5, spec. y to a in (14)]

(16) $P(b,a) \to ((P(e,b) \land f(e)) \to (P(e,b) \land P(b,a) \land f(e)))$
[Tautology $A \to ((B \land C) \to (B \land A \land C))$]

(17) $(P(e,b) \land f(e)) \to (P(e,b) \land P(b,a) \land f(e))$ [R. 2 fr. (7) & (16)]

(18) $P(e,b) \land P(b,a) \land f(e)$ [Rule 2 from (12) and (17)]

(19) $\exists z (P(e,z) \land P(z,a) \land f(e))$ [Rule 7 from (18)]

(20) $(g(e,a)(\exists z (P(e,z) \land P(z,a) \land f(e)) \leftrightarrow \exists z (P(e,z) \land P(z,a) \land f(e)))$
$\to g(e,a))$ [Tautology $(A \leftrightarrow B) \to (B \to A)$]

(21) $\exists z (P(e,z) \land P(z,a) \land f(e)) \to g(e,a)$ [Rule 2, from (15) and (20)]

(22) $g(e,a)$ [Rule 2, from (19) and (21)]

(23) $\exists y\, g(y,a)$ [Rule 7, from (22)]

(24) $\exists y\, g(y,a)$ [Rule 7, from (10) and (23)]

(25) $\exists y\, g(y,a)$ [Rule 7, from (9) and (24)]

(26) $\exists y\, g(y,a)$ [Rule 7, from (5) and (25)]

(27) $\exists y\, g(y,a)$ [Rule 7, from (4) and (26)]

(28) $\forall x \exists y\, g(y,x)$ [Rule 5, from (1) and (23)] $\diamondsuit$

COMMENT: In the above proof, we used *indentations* to indicate "levels of proof." That is, every time we introduced a new assumption (for example, in Steps (2), (4), (5), (9), (10)) we added an extra indentation. That way, every time a step is proved "under" certain assumptions, the step is literally, physically, *under* those assumptions. For example, Step (22) was proved under *five assumptions*, namely, those made in (2), (4), (5), (9), and (10), so Step (22) is in Level 5. Once we get to Step (23), also in Level 5, we observe that the way we got to Level 5 was by picking e in Step (10). But the assertion of Step (23) does not involve e, so Rule 7 enables us to go up one level with the same assertion, as we do in Step (24). But then we are in Level 4, and the way we got there was by picking d in Step (9). Since the assertion of Step (24) does not involve d, Rule 7 enables us to go up one level with the same assertion, as we do in Step (25). Then we land in Level 3, and we see that we got there by picking c in Step (5). Once again, the assertion of Step (25) does not involve d, so Rule 7 enables us to go up one level with the same assertion, and we do that in Step (26), landing in Level 2. Next, we see that we got to Level 2 by picking b in Step (4), and the assertion of Step (26) does not involve b. So Rule 7 enables us to go up one level with the same assertion, and we do that in Step (27), landing in Level 1. Now, the way we got to Level 1 was by picking an arbitrary a, and in Step (27) we proved $\exists y\, g(y,a)$ in the same level. So Rule 5 enables us to go to $\forall x \exists y\, g(y,x)$ in Level 0.

Another way to do the same thing is to use numbers, instead of indentation, to indicate the various levels. Here is the same proof as above, written in this way.

(1)[0] $\forall x \exists y \exists z (P(y,x) \wedge P(z,x) \wedge m(y) \wedge f(z) \wedge \forall u (P(u,x) \to (u = y \vee u = z)))$ [Axiom K2]
(2)[1] Let a be arbitrary [Preparation to apply Rule 3]
(3)[1] $\exists y \exists z (P(y,a) \wedge P(z,a) \wedge m(y) \wedge f(z) \wedge \forall u (P(u,a) \to (u = y \vee u = z)))$ [Rule 4, from (1)]
(4)[2] Pick b such that
$\exists z (P(b,a) \wedge P(z,a) \wedge m(b) \wedge f(z) \wedge \forall u (P(u,a) \to (u = b \vee u = z)))$ [Rule 6, from (3)]
(5)[3] Pick c such that
$P(b,a) \wedge P(c,a) \wedge m(b) \wedge f(c) \wedge \forall u (P(u,a) \to (u = b \vee u = c))$ [Rule 6, from (4)]
(6)[3] $(P(b,a) \wedge P(c,a) \wedge m(b) \wedge f(c) \wedge \forall u (P(u,a) \to (u = b \vee u = c))) \to P(b,a)$
(Tautology $(A \wedge B \wedge C \wedge D \wedge E) \to A$.)
(7)[3] $P(b,a)$ [Rule 2 from (5), (6)]
(8)[3] $\exists y \exists z (P(y,b) \wedge P(z,b) \wedge m(y) \wedge f(z) \wedge \forall u (P(u,b) \to (u = y \vee u = z)))$ [Rule 4, from (1)]
(9)[4] Pick d such that $\exists z (P(d,b) \wedge P(z,b) \wedge m(d) \wedge f(z) \wedge \forall u (P(u,b) \to (u = d \vee u = z)))$
[Rule 4, from (8)]
(10)[5] Pick e such that
$P(d,b) \wedge P(e,b) \wedge m(d) \wedge f(e) \wedge \forall u (P(u,b) \to (u = d \vee u = e))$ [Rule 4, from (9)]
(11)[5] $(P(d,b) \wedge P(e,b) \wedge m(d) \wedge f(e) \wedge \forall u (P(u,b) \to (u = d \vee u = e)))$
$\to (P(e,a) \wedge f(e))$ [Tautology $(A \wedge B \wedge C \wedge D \wedge E) \to (B \wedge D)$]
(12)[5] $P(e,b) \wedge f(e)$ [Rule 2 from (10), (11)]
(13)[5] $\forall x \forall y (g(x,y) \leftrightarrow \exists z (P(x,z) \wedge P(z,y) \wedge f(x)))$. [Definition of g]
(14)[5] $\forall y (g(e,y) \leftrightarrow \exists z (P(e,z) \wedge P(z,y) \wedge f(e)))$. [Rule 4, specializing x to e in (13)]
(15)[5] $g(e,a) \leftrightarrow \exists z (P(e,z) \wedge P(z,a) \wedge f(e)))$. [Rule 5, specializing y to a in (14)]
(16)[5] $P(b,a) \to ((P(e,b) \wedge f(e)) \to (P(e,b) \wedge P(b,a) \wedge f(e)))$
[Tautology $A \to ((B \wedge C) \to (B \wedge A \wedge C))$]
(17)[5] $(P(e,b) \wedge f(e)) \to (P(e,b) \wedge P(b,a) \wedge f(e))$ [Rule 2, from (7) and (16)]
(18)[5] $P(e,b) \wedge P(b,a) \wedge f(e)$ [Rule 2, from (12) and (17)]
(19)[5] $\exists z (P(e,z) \wedge P(z,a) \wedge f(e))$ [Rule 7, from (18)]
(20)[5] $(g(e,a)(\exists z (P(e,z) \wedge P(z,a) \wedge f(e)) \leftrightarrow \exists z (P(e,z) \wedge P(z,a) \wedge f(e))) \to g(e,a))$
[Tautology $(A \leftrightarrow B) \to (B \to A)$]
(21)[5] $\exists z (P(e,z) \wedge P(z,a) \wedge f(e)) \to g(e,a)$ [Rule 2, from (15) and (20)]
(22)[5] $g(e,a)$ [Rule 2, from (19) and (21)]
(23)[5] $\exists y\, g(y,a)$ [Rule 7, from (22)]
(24)[4] $\exists y\, g(y,a)$ [Rule 7, from (10) and (23)]
(25)[3] $\exists y\, g(y,a)$ [Rule 7, from (9) and (24)]
(26)[2] $\exists y\, g(y,a)$ [Rule 7, from (5) and (25)]
(27)[1] $\exists y\, g(y,a)$ [Rule 7, from (4) and (26)]
(28)[0] $\forall x \exists y\, g(y,x)$ [Rule 5, from (1) and (23)] $\diamondsuit$

An informal proof is an abbreviated outline of how a real proof would go. The goal of giving an informal proof is to make it clear to the listener or reader that a proof exists and that the speaker or writer knows how to produce such a proof if asked, without having to exhibit one.

You should think of proof-giving or proof-writing as a "two-person game." When you write or present a proof, you have to convince somebody else (the "challenger," whose role will usually be played in Math 300 by the teacher) that you really know how to write a true formal proof. As long as you convince the challenger, you are doing O.K. But the challenger is not easily convinced. If you make a wrong move, or a move that you don't know how to justify by the rules, the challenger is going to challenge you to provide a more detailed justification. If you cannot do it, you lose.

This raises another point. **How can we guarantee that the challenger will play fair?** *For example, the challenger could question everything you say, and always ask "why?" or "how do you know that?" If you use an axiom, the challenger could ask "how do you justify that axiom?" And if you answer "because it is an axiom, and axioms do not need justification," then the challenger could ask "why do axioms not need justification?" A truly mean-spirited challenger could go on challenging you forever, and you would never be able to win. That is why we have* **precise rules**. *When we give you rules for proofs, we are making a commitment to accept as valid—without any further challenges—anything you do that follows the rules.*

The rules are for your protection. *When I give you rules, I am telling you in advance: "as long as you can justify what you are doing according to these rules, I guarantee that I will accept it and will not challenge it."*

The challenger is tough but fair. *If you play by the rules, the challenger will not demand extra justification. But when you do not play by the rules then the challenger may—and often will—ask you to justify what you did according to the rules.*

Is it safe to give informal arguments that fail to obey the rules? Is it safe to cut corners? *The answer is "yes if you know what you are doing, not otherwise." Keep in mind that* **in Math 300 the challenger is someone much more experienced than you are, someone who can smell a rat a mile away.** *If you give an informal proof involving 200 steps, and 199 of these steps are O.K. but one is not, then the challenger almost certainly will realize right away which step is the "weak link in the chain" and challenge that step. And if this one step cannot be justifed then your whole proof is wrong!*

MORE EXAMPLES OF FORMAL PROOFS

We will show that, if $q(x)$ is a 1-variable predicate and r is a proposition[1], then

$$\exists x \Big(r \wedge q(x)\Big) \leftrightarrow (r \wedge \exists x\, q(x)) \qquad (I)$$

and

$$\forall x \Big(r \vee q(x)\Big) \leftrightarrow (r \vee \forall x\, q(x)) \qquad (II)$$

are theorems of the predicate calculus. Let us prove these two things using our rules.

PROOF OF (I).

(1)[1]	Assume $\exists x(r \wedge q(x))$.	[Preparation to apply Rule 3]
(2)[2]	Pick a such that $r \wedge q(a)$.	[Rule 6, from (1)]
(3)[2]	$(r \wedge q(a)) \rightarrow r$	[Tautology $(\alpha \wedge \beta) \rightarrow \alpha$]
(4)[2]	$(r \wedge q(a)) \rightarrow q(a)$	[Tautology $(\alpha \wedge \beta) \rightarrow \beta$]
(5)[2]	r	[Rule 2, from (2) and (3)]
(6)[2]	$q(a)$	[Rule 2, from (2) and (4)]
(7)[2]	$\exists x\, q(x)$	[Rule 7, from (6)]
(8)[2]	$r \rightarrow (\exists x\, q(x) \rightarrow (r \wedge \exists x\, q(x)))$	[Tautology $\alpha \rightarrow (\beta \rightarrow (\alpha \wedge \beta))$]
(9)[2]	$\exists x\, q(x) \rightarrow (r \wedge \exists x\, q(x))$	[Rule 2, from (5) and (8)]
(10)[2]	$r \wedge \exists x\, q(x)$	[Rule 2, from (7) and (9)]
(11)[1]	$r \wedge \exists x\, q(x)$	[Rule 6, from (2) and (10)]
(12)[0]	$\exists x(r \wedge q(x)) \rightarrow (r \wedge \exists x\, q(x))$	[Rule 3, from (1) and (11)]
(13)[1]	Assume $r \wedge \exists x\, q(x)$.	[Preparation to apply Rule 3]
(14)[1]	$(r \wedge \exists x\, q(x)) \rightarrow r$	[Tautology $(\alpha \wedge \beta) \rightarrow \alpha$])
(15)[1]	$(r \wedge \exists x\, q(x)) \rightarrow \exists x\, q(x)$	[Tautology $(\alpha \wedge \beta) \rightarrow \beta$]
(16)[1]	r	[Rule 2, from (12) and (13)]
(17)[1]	$\exists x\, q(x)$	[Rule 2, from (13) and (15)]
(18)[2]	Pick a such that $q(a)$.	[Rule 6, from (17)]
(19)[2]	$r \rightarrow (q(a) \rightarrow (r \wedge q(a)))$	[Tautology $\alpha \rightarrow (\beta \rightarrow (\alpha \wedge \beta))$]
(20)[2]	$q(a) \rightarrow (r \wedge q(a))$	[Rule 2, from (16) and (19)]
(21)[2]	$r \wedge q(a)$	[Rule 2, from (18) and (20)]
(22)[2]	$\exists x(r \wedge q(x))$	[Rule 7, from (21)]
(23)[1]	$\exists x(r \wedge q(x))$	[Rule 6, from (18) and (22)]
(24)[0]	$(r \wedge \exists x\, q(x)) \rightarrow \exists x(r \wedge q(x))$	[Rule 3, from (13) and (23)]

Now let us write A for $\exists x(r \wedge q(x))$, and B for $r \wedge \exists x\, q(x)$.

(25)[0]	$(A \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow (A \leftrightarrow B))$	[Tautology]
(26)[0]	$(B \rightarrow A) \rightarrow (A \leftrightarrow B)$	[Rule 2, from (12) and (26)]
(27)[0]	$A \leftrightarrow B$	[Rule 2, from (24) and (25)] $\diamondsuit$

[1] In first-order predicate calculus, a “proposition” is a closed sentence

AN ABBREVIATED PROOF OF (II):

(1)	Assume $\forall x(r \vee q(x))$	[Preparation to apply Rule 3]
(2)	Assume $\sim r$	[Preparation to apply Rule 3]
(3)	Let a be arbitrary.	[Preparation to apply Rule 5]
(4)	$r \vee q(a)$	[Rule 4, specializing to $x = a$ in (1)]
(5)	$q(a)$	[***]
(6)	$\forall x\, q(x)$	[Rule 5, from (3) and (5)]
(7)	$r \vee \forall x\, q(x)$	[***]
(8)	Assume $r \vee \forall x\, q(x)$.	[Preparation to apply Rule 3]
(9)	Assume $\sim r$.	[Preparation to apply Rule 3]
(10)	$\forall x\, q(x)$	[***]
(11)	Let a be arbitrary.	[Preparation to apply Rule 5]
(12)	$q(a)$	[Rule 4, specializing to $x = a$ in (10)]
(13)	$r \vee q(a)$	[***]
(14)	$\forall x\,(r \vee q(x))$	[Rule 5, because we proved $r \vee q(a)$ for arbitrary a]
(15)	Assume r.	[Preparation to apply Rule 5]
(16)	Let a be arbitrary.	[Preparation to apply Rule 5]
(17)	$r \vee q(a)$	[***]
(18)	$\forall x(r \vee q(x))$	[Rule 5, because we proved $r \vee q(a)$ for arbitrary a]
(19)	$\forall x(r \vee q(x))$	[***]
(20)	$\forall x(r \vee q(x)) \leftrightarrow \vee\forall x\, q(x)$	[***] $\diamondsuit$

AN INFORMAL PROOF OF (II): Assume that $\forall x(r \vee q(x))$. We will prove that $r \vee \forall x\, q(x)$. If r is true then $r \vee \forall x\, q(x)$ is true, so we are done. Next assume $\sim r$. We prove $\forall x\, q(x)$. Let a be arbitrary. Then $r \vee q(a)$, because $\forall x(r \vee q(x))$. So, in particular, $q(a)$. Since a was arbitrary, we have proved $\forall x\, q(x)$. So we have established that $\forall x(r \vee q(x)) \to r \vee \forall x\, q(x)$.

Now let us prove the converse, that is, that $r \vee \forall x\, q(x) \to \forall x(r \vee q(x))$. Assume $r \vee \forall x\, q(x)$. We want to prove $\forall x(r \vee q(x))$. Let a be arbitrary. We want to prove $r \vee q(a)$. From the assumption that $r \vee \forall x\, q(x)$ it follows that $\forall x\, q(x)$, and then that $q(a)$. Therefore $r \vee q(a)$ is true. Since a was arbitrary, we have shown that $\forall x(r \vee q(x))$, as desired. So we have proved that $r \vee \forall x\, q(x) \to \forall x(r \vee q(x))$.

Since we have proved both implications

$$\forall x(r \vee q(x)) \to r \vee \forall x\, q(x)$$

and

$$r \vee \forall x\, q(x) \to \forall x(r \vee q(x))\,,$$

the equivalence $\forall x(r \vee q(x)) \leftrightarrow r \vee \forall x\, q(x)$ follows.

AN EVEN MORE INFORMAL PROOF OF (II). We want to prove that "either x is a q or r is true" is true of all x, if and only if either r is true or all xs are qs. Suppose that "either x is a q or r is true" is true of all x. Let us show that either r is true or all xs are qs. If r is true then we are done. Suppose r is false. Let us show that all xs are qs. Pick an x. Then "either x is a q or r is true" is true. Since r isn't true, we can infer that x is a q. Since x was completely arbitrary, we have shown that all xs are qs.

Now let us prove the converse. Assume that either r is true or all xs are qs. We want to show that the statement "either x is a q or r is true" is true for every x. Pick an x. We have to show that either x is a q or r is true. If r is true then the desired conclusion follows. Suppose r is false. Let us show that x is a q. We are assuming that either r is true or all things are qs, and also that r isn't true. Then all things are qs, so in particular our x is a q. Therefore the desired conclusion —that either x is a q or r is true—follows.

AN INCORRECT INFORMAL PROOF OF (II): Assume that $\forall x(r \vee q(x))$. We will prove that $r \vee \forall x\, q(x)$. Pick x. Then $r \vee q(x)$ is true. Since x was arbitrary, we can conclude that $r \vee \forall x\, q(x)$.

Now let us prove the converse, that is, that $r \vee \forall x\, q(x) \rightarrow \forall x(r \vee q(x))$. Assume $r \vee \forall x\, q(x)$. We want to prove $\forall x(r \vee q(x))$. Let a be arbitrary. We want to prove $r \vee q(a)$. From the assumption that $r \vee \forall x\, q(x)$ it follows that $\forall x\, q(x)$, and then that $q(a)$. Therefore $r \vee q(a)$ is true. Since a was arbitrary, we have shown that $\forall x(r \vee q(x))$, as desired. So we have proved that $r \vee \forall x\, q(x) \rightarrow \forall x(r \vee q(x))$. $\diamondsuit$

PROBLEM 1: give a completely formal proof of (II), without skipping any steps.

PROBLEM 2: write an informal proof of (I), along the lines of the informal proofs of (II) given above.

PROBLEM 3: explain why the "incorrect informal proof of (II)" given above is incorrect. (*Note: the proof contains at least two serious logical errors, each one of which would suffice to make the proof totally invalid.*)

PROBLEM 4: an informal proof of

$$\exists x(p(x) \vee q(x)) \leftrightarrow (\exists x\, p(x) \vee \exists x\, q(x))\,. \tag{III}$$

is given below. Write down a complete formal proof.

INFORMAL PROOF OF (III): Suppose there is a thing which is either a p or a q. Then either that thing is a p, in which case p's exist, or that thing is a q, in which case q's exist. In either case, it is true that either p's exist or q's exist. To prove the converse, suppose that either p's exist or q's exist. If p's exist, pick one and call it a. Then this a is either a p or a q (because it is a p) so there is a thing which is either a p or a q. If p's do not exist, then q's exist, so pick one and call it a. Then this a is either a p or a q (because it is a q) so there is a thing which is either a p or a q. In both cases, there is a thing which is either a p or a q.

PROBLEM 5. This problem deals with a first-order predicate calculus with a two-variable predicate symbol p. For each of the following four statements, either write down a formal proof and an informal proof, or give an argument showing that this cannot be done.

$$(\exists x\, p(x) \wedge \exists x\, q(x)) \rightarrow \exists x(p(x) \wedge q(x)) \tag{IV}$$

$$\exists x(p(x) \wedge q(x)) \rightarrow (\exists x\, p(x) \wedge \exists x\, q(x))\,, \tag{V}$$

$$\exists x \forall y\, p(x,y) \rightarrow \forall y \exists x\, p(x,y)\,, \tag{VI}$$

$$\forall y \exists x\, p(x,y) \rightarrow \exists x \forall\, p(x,y)\,. \tag{VII}$$