MATHEMATICS 300 — FALL 1999

Introduction to Mathematical Reasoning

H. J. Sussmann

HANDOUT NO. 4 *(September 27, 1999)*

READING FOR THE FIRST EIGHT DAYS OF CLASS

(that is, for Sept. 2-9-13-16-20-23-27-30)

- the book's "Preface for the Student,"
- the book's Chapter 1 (all of it);
- the book's Chapter 2, first two sections ("Two theorems, about equations," and "Quantifiers and logical statements");
- the instructor's Handouts Nos. 1, 2, 3 and 4.

REMARKS AND CORRECTIONS TO THE BOOK'S SECTION ON "THE REAL NUMBER SYSTEM" (PAGES 16-17-18)

1. In the list of axioms given in pages 17-18. the authors forgot to include one important axiom, namely:

   $$\boxed{\text{AXIOM N:} \qquad 0 \neq 1.}$$

   This axiom is necessary, because without it we cannot prove that the real number system contains more than one member, as shown below. (See Homework Assignment No. 4 in page 2 of this handout.)

2. After they discuss the arithmetical axioms (A0 to A4, M0 to M4, and DL) the authors tell us that "the additive inverse of x is the **negative** of x, written as $-x$." That's very nice, but *how do we know that there is such a thing as* **the** *additive inverse of* x*?* In English, when we use the word *the*, we mean that there is only one such thing. (For example: we say that Clinton is *the* president of the United States, but we do not say that Robert Torricelli is *the* U.S. senator. That's because there is only one president, but there are many senators.)

   So, how do we know that a real number x has only one additive inverse? The answer is: **because we can prove it:**

   THEOREM A: Let $x \in \mathbb{R}$. Then x has only one additive inverse.

PROOF: Suppose u, v are two additive inverses of x. We want to prove that $u = v$. By definition (according to the book, Axiom A4, although it would have nicer if they had said this more clearly) an *additive inverse* of x is a number y such that $x + y = 0$. So, since u and v are supposed to be additive inverses of x, we have $x + u = 0$ and $x + v = 0$.

Now, Axiom A3 tells us that $w + 0 = w$ for all $w \in \mathbb{R}$. (Question for you: why do you think I used w rather than x when I quoted Axiom A3?) Let us apply this with v in the role of w. Then $v + 0 = v$. But we know that $x + u = 0$. So $v + (x + u) = v$. On the other hand, Axiom A1 implies that $(v + x) + u = v + (x + u)$. So $(v + x) + u = v$. But $v + x = x + v$ by Axiom A2. So $(x + v) + u = v$. Now we use the fact that $x + v = 0$. Then $0 + u = v$. Next, Axiom A2 implies $0 + u = u + 0$. So Then $u + 0 = v$. Finally, Axiom A3 implies $u + 0 = u$. So $\boxed{u = v}$, as desired. $\diamondsuit$

3. Right after Definition 1.40, the authors tell us that "we define $x < y$ to mean $y - x \in P$." That's fine, but **what is the meaning of "$y - x$"?** *The authors never tell us.* It's true that they define what is meant by "$-x$" earlier. But that does not tell us what "$y - x$" means.

   When we use "$-$" as in "-3", the symbol $-$ stands for a **one-argument operation** (that is, a **unary operation**. You take **one** number x and you produce **its negative**, called $-x$.

   On the other hand, when we use "$-$" as in "$8 - 3$", the symbol $-$ stands for a **two-argument operation** (that is, a **binary operation**): you take **two** numbers x and y and you produce **their difference**, called $x - y$. This is quite different!

4. The authors should have done the following:

   a. First, they should have stated and proved Theorem A.

   b. Then they could have said, as they do in page 17, that "the additive inverse of x is the **negative** of x, written as $-x$." (This is now justified thanks to Theorem A.)

   c. Then should have said "if x, y are real numbers, the **difference** of x and y, written as $x - y$, is the number $x + (-y)$."

5. Exactly the same problem discussed above for the additive inverse arises with the multiplicative inverse. The authors talk about about *the* reciprocal of a nonzero number x. *How do we know that there is only one reciprocal?*

HOMEWORK ASSIGNMENT NO. 4

This assignment is due on Monday October 4.

- Consider a new (and, of course, rather silly) "number system" in which the only number is 0. Addition is defined by letting $0 + 0 = 0$. Multiplication is defined by letting $0.0 = 0$. The set P is the empty set. 1 is defined to be 0.

  Verify that all the axioms of page 17 of the book are verified in this "silly number system." (One example of such a verification is given below. You are asked to verify all the other axioms.)

  Conclude that, with the axioms given in the book, we **cannot** prove that $0 < 1$.

EXAMPLE: Here is the verification of Axiom A4 for the silly number system. We have to show that "given x, there is a y such that $x + y = 0$. Let x be given. Then $x = 0$, because in our silly number system 0 is the only number. By definition, $0 + 0 = 0$. So, if we take $y = 0$, we see that $x + y = 0$. So for our given x there is indeed a y such that $x + y = 0$. So Axiom A4 holds. $\diamondsuit$