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Introduction to Mathematical Reasoning

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1. WHY DO WE NEED RULES FOR PROOFS?

A **proof** of a statement is a **convincing argument** that establishes **beyond the shadow of a doubt** that the statement is true.

You should think of proving statements as a **game** that you are playing against an opponent who does not believe that the statement you are proving is true. You make statements, your opponent may find them convincing or not, and it is up to you to clear up all your opponent's doubts by arguing your case persuasively every time your opponent isn't convinced. (Maybe you should think that I, the instructor, am the "opponent." **You** have to convince **me**. If you convince me, then your proof has worked. Otherwise it has not.)

The trouble with all this is that it is not very clear what will count as a "convincing argument." Suppose your opponent says, after you have given what looks to you like a very convincing proof: "I am not convinced; I don't see why what you just said proves anything." How would you deal with something like that?

Moreover, any argument you give will have to *start* somewhere. (You cannot prove anything starting from nothing.) What can you do if your opponent refuses to accept your starting point?

EXAMPLE: suppose you want to prove that,

> *If a real number is greater than or equal to 4, then the square root of the number is less than or equal to half the number.*

Equivalently —and more concisely— you want to show that

> *If x is any real number such that $x \geq 4$, then $\sqrt{x} \leq \frac{x}{2}$.*

Or —even more concisely— you want to prove that

$$(\forall x)\Big(x \geq 4 \Longrightarrow \sqrt{x} \leq \frac{x}{2}\Big)\,.$$

To prove this, you can proceed as follows:
To prove $(\forall x)\Big(x \geq 4 \Longrightarrow \sqrt{x} \leq \frac{x}{2}\Big)$, we pick an arbitrary x and try to prove that $x \geq 4 \Longrightarrow \sqrt{x} \leq \frac{x}{2}$.
So, let x be arbitrary. We want to prove $x \geq 4 \Longrightarrow \sqrt{x} \leq \frac{x}{2}$. To prove this, we assume $x \geq 4$ and try to prove that $\sqrt{x} \leq \frac{x}{2}$.
Now we are assuming that $x \geq 4$. From this it follows —multiplying both sides by x— that $x^2 \geq 4x$. Then —dividing both sides by 4— we get that $\frac{x^2}{4} \geq x$. Then —taking the square root of both sides— we get $\sqrt{\frac{x^2}{4}} \geq \sqrt{x}$. Since $\sqrt{\frac{x^2}{4}} = \frac{x}{2}$, we have shown that $\frac{x}{2} \geq \sqrt{x}$, which is "the same as" $\sqrt{x} \leq \frac{x}{2}$. **END OF PROOF**

Is this a truly convincing argument?

The answer is "yes" if all the steps are convincing. But **if one step is not convincing, then the whole argument fails.** (This is why you should **not** expect to get partial credit for a wrong proof if most steps are O.K. but just a few of them are wrong. A proof is like a chain. And **a chain is no stronger than its weakest link.** If a 16-Ton weight is hanging right above your head, held by a chain, and the chain breaks because of one weak link, and the weight falls on your head and kills you, do you think the chain manufacturer should be entitled to "partial credit" because "most links were O.K.; only one broke"? If the driver of Lady Diana's car didn't drink every night, but only drank on the particular night of the accident, should he be entitled to partial credit because "after all, most of the time he didn't drink"?)

So, are **ALL** the steps in the above proof truly convincing?

In the proof, we have given justifications for each step. For example, at one point we went from

$$x \geq 4$$

to

$$x^2 \geq 4x\,,$$

and we said that this was done "multiplying both sides by x." Now, why is this step permissible?

Suppose I tell you that "I don't believe that you can do this"? How do you persuade me? Well, let us give it a try. You can say: "because there is a general rule that says if you take an inequality and multiply both sides by a number, then you still get a valid inequality." (Or, in more symbolic notation:

$$(\forall a)(\forall b)(\forall c)(a \geq b \Longrightarrow ac \geq bc)\,,$$

using the language of a predicate calculus that we will discuss later.)

This, however, will not do. This "rule" is NOT VALID. (For example, if $a = 5$, $b = 2$ and $c = -2$, then $a \geq b$, but it's not true that $ac \geq bc$, since $ac = -10$ and $bc = -4$, so $ac < bc$.) So WE CANNOT USE IT.

Now, here is a second try. You can say: "because there is a general rule that says if you take an inequality and multiply both sides by a nonnegative number, then you still get a valid inequality, that is, using symbolic notation:

$$(\forall a)(\forall b)(\forall c)\Big((a \geq b \wedge c \geq 0) \Longrightarrow ac \geq bc\Big) .$$

It turns out that this time our rule IS VALID. But notice that, in order to apply this rule, we must be sure that $c \geq 0$. In our case, the number that plays the role of c is x, and it is indeed true that $x \geq 0$, because $x \geq 4$. (Since this is a crucial fact for us to be able to apply the above "rule," it would have been better to include in our proof a little remark highlighting this. For example, instead of "multiplying both sides by x," it would have been better to say "multiplying both sides by x, which is possible because $x \geq 0$, since we are assuming that $x \geq 4$.")

But we haven't solved our problem. Remember that your proof is really a two-person game that you are playing against me, your opponent, in which you have to convince me. What if I say "How do we know that $(\forall a)(\forall b)(\forall c)\Big((a \geq b \wedge c \geq 0) \Longrightarrow ac \geq bc\Big)$? I am not really convinced that this is true."?

You could try to prove for me that $(\forall a)(\forall b)(\forall c)\Big((a \geq b \wedge c \geq 0) \Longrightarrow ac \geq bc\Big)$ is indeed true. But I am sure that in our proof you will use other "rules," and I, your opponent, could pick any of these and claim, once again, that I am not convinced of its validity.

IF I WANTED TO, I COULD GO ON LIKE THIS FOREVER. NO MATTER WHAT JUSTIFICATION YOU GIVE FOR A STEP IN YOUR PROOF, I COULD ALWAYS ASK: "WHY IS THAT VALID?"

In other words, if your opponent is truly stubborn and uncooperative, then there is *nothing* you can do. Maybe you will come up with some reason, but this reason will be based on some other facts that look obviously true to you. But your opponent can still refuse to accept these other facts, and this process can go on for ever.

To take care of this "problem of the opponent that doesn't play fair," mathematicians have come up with the concept of a **formal proof**. We give very precise rules defining what a "formal proof" is, and if you prove something correctly, and strictly following the rules, then you are *guaranteed* that your proof will be accepted as valid.

If you do *not* obey the rules, then you are on uncertain ground: you might be able to convince an instructor or grader, but you cannot be sure, because any time you take a step one of us might ask "how do you justify that step?"

You should think of giving formal proofs as similar to playing a game such as chess: you are given very strict, precise rules, and you are only allowed to make moves following the rules. As long as you obey the rules, your opponent does not have the right to reject your moves. (If your opponent unilaterally decides that a move of yours is not acceptable, because "you are not allowed to move a bishop if in the previous move you moved a knight," then you would say that your

opponent is not playing fair, because this "rule" was not in the list of rules that had been agreed upon before the game started.)

In Mathematics we play fair. We tell you what the rules are, and if you prove something correctly following the rules, then you can be sure that it will be accepted. You may think that having to follow the rules is a restriction to your freedom. But this is not so. You should think of it as **your own protection against unfair play.** If you correctly justify every step according to the rules, then we promise you that we will not change the rules in the middle of the game, kick the board, or come up with other surprises.

REMARK. I said that you should think of giving formal proofs of theorems as a game similar to chess, in which you are given a small number of very strict rules telling you which moves are permitted, and each move has to be justifiable by explaining exactly which rule you are using and how you are using it. However, exactly as in the case of chess, if all you know is the rules, and you always apply the rules carefully and scrupulously, you may still lose all the time. If you want to **win** (that is, in the case of a proof, if you want to get to the desired conclusion) then you have to **think**. You have to plan a strategy, make a move now because it will be useful five or six steps later, etc.

The one big difference between chess and theorem-proving is that **chess is a zero-sum game**, but **theorem-proving is a cooperative game.** What this means is that in chess your opponent wants you to lose, whereas in theorem-proving you and your opponent are working together, trying to find out whether something is true, and to understand why.

That's why in chess you don't explain openly why you make a move. You do *not* say to your opponent things like: "I am making this move now so that then in the next move I can do this other thing, so that three moves later I can do that, and then I will checkmate you." *You don't want your opponent to be able to figure out your strategy.*

In theorem-proving you definitely **do** want your opponent to understand your strategy. Your opponent is going to be ruthless in criticizing you if you don't obey the rules, but as long as you obey them you will get full coopertaion. In fact, usually you don't have to go through the whole business of explaining every detail of every application of every rule. *As long as you persuade your opponent that something can be done using the rules, there is no need to go through all the steps and actually do it.* And, since your opponent wants to be persuaded, the more you do to explain why you are doing something, the better. That's why *in theorem-proving it's always nice to explain your strategy, so that your opponent can understand where you are headed.*

Because theorem-proving is a cooperative game, when writing proofs it is always nice to use the first person plural —the so-called "editorial we"— as we did in our first example of a proof ("we pick," "we are assuming," "we have shown"). This is a good way to convey the idea that the writer and the reader are *working together*, that **the writer is leading the reader gently along a path of discovery.** □

2. VALID INFERENCES

Proofs involve **steps** in which you move from some statements (the *premisses*) to a new statement (the conclusion). We also say that you *infer*, or *derive*, or *deduce* the conclusion from the premisses, and the steps where you do that are called *inferences*.

So the first thing we need to discuss is:

WHAT IS A VALID INFERENCE?

An *inference* is a way of passing from certain statements (the *premisses*) to another statement (the *conclusion*).

> A *valid inference* is one for which it is guaranteed that, if the premisses are true, then the conclusion is also true.

For example:

$$\frac{\begin{array}{r}\text{All men are mortal}\\ \text{Socrates is a man}\end{array}}{\text{Socrates is mortal}} \qquad (1)$$

is an example of a valid inference known as a *syllogism*. The premisses are "All men are mortal" and "Socrates is a man," and the conclusion is "Socrates is mortal."

Roughly speaking, we would like to say that

> An inference is **logically valid** if the conclusion follows from the premisses **by virtue of the "form" of the statements involved**.

Eventually, we will say exactly that. But we are not ready to say it yet, because we still do not know precisely what we mean by the "form" of a statement.

Before we talk about "form," let us go on with examples to illustrate what we are trying to say. We would like to call the syllogism of (1) logically valid **because in (1) it does not matter what "Socrates" or "man" or "mortal" mean.** It would be equally valid to say

$$\frac{\begin{array}{r}\text{All men are immortal}\\ \text{Socrates is a man}\end{array}}{\text{Socrates is immortal}} \qquad (2)$$

or

$$\frac{\begin{array}{r}\text{All pigs are immortal}\\ \text{Socrates is a pig}\end{array}}{\text{Socrates is immortal}} \qquad (3)$$

or

$$\frac{\begin{array}{r}\text{All men are mortal}\\ \text{Donald Duck is a man}\end{array}}{\text{Donald Duck is mortal}} \qquad (4)$$

or

$$\frac{\begin{array}{r}\text{All borogoves are mimsy}\\ \text{The Mad Hatter is a borogove}\end{array}}{\text{The Mad Hatter is mimsy}} \qquad (5)$$

It does not matter whether the premisses are true (in some of the above examples they are true, in others they are not) or whether the conclusion is true. It doesn't even matter whether you know what a "borogove" is or what "mimsy" means. The point is:

The four examples of syllogism given above all are of the same **logical form**, *namely,*

$$\frac{\begin{array}{r}\text{All } X \text{ are } Y\\ Z \text{ is an } X\end{array}}{Z \text{ is a } Y} \qquad (6)$$

To be able to talk about an inference being "logically valid" we have to be able to talk about the **logical form** (or, even better, "a" logical form, rather than "the" logical form) of an inference.

For example, we could say that all five inferences (1), (2), (3), (4), (5) are of the form (6), and any reasoning of this form is valid.

Another example of a valid inference form would be:

$$\frac{\begin{array}{r}X \text{ is } Z\\ Y \text{ is } Z\end{array}}{X \text{ and } Y \text{ are } Z} \qquad (7)$$

Examples of inferences of the form (7) are:

$$\frac{\begin{array}{r}\text{John is happy}\\ \text{Mary is happy}\end{array}}{\text{John and Mary are happy}} \qquad (8)$$

and

$$\frac{\begin{array}{r}\text{Al Gore is smart}\\ \text{Hillary Clinton is smart}\end{array}}{\text{Al Gore and Hillary Clinton are smart}} \qquad (9)$$

So it would seem that we have two examples of "valid forms", namely, (6) and (7). Consider, however, the following examples:

All men are equal
Socrates is a man

Socrates is equal (10)

and

Al Gore is married
Hillary Clinton is married

Al Gore and Hillary Clinton are married (11)

Superficially, (10) is "of the form (6)," and (11) is "of the form (7)." Something is clearly wrong here, because in (110) the conclusion is clearly false, even though the premisses are true, and in (10) the conclusion is meaningless.

What is wrong here is that "statement forms" such as "All X are Y" or "X and Y are Z" are not good enough to capture logical distinctions that matter. For example, the statements "Al Gore and Hillary Clinton are married" and "Al Gore and Hillary Clinton are smart" both appear to be of the same "form," namely, "X and Y are Z," but they really are not. "Smart" is a **one-variable predicate**. (If x is a person, you can say that "x is smart.") But "married" is *both* a *one-variable predicate* (in "Al Gore is married") and a 2-variable predicate (in "Al Gore and Hillary Clinton are married"). In normal English discourse, when we say that "x and y are married" we do not mean that each one, separately, is married. We mean that "x and y are married to each other." (If you don't believe that, ask yourself the following: when someone asks a couple "are you two married?", isn't it clear that this means "to each other"?)

So, if we really want to talk about **forms** of statements and inferences, then we need a more sophisticated language. We have to be able to distinguish, for example, between one-variable predicates and two-variable predicates. The languages that do that are the **first-order predicate calculi**. ("Calculi" is the plural of "calculus.") Before we discuss these rather complicated languages, we will say a few words about **propositional calculi**, which are simpler. Both propositional calculi and first-order predicate calculi are examples of **formal languages**, so we will start by talking about formal languages in general.

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3. FORMAL LANGUAGES

Formal languages resemble "natural languages" such as English, in that in both cases there is a collection of *symbols* (the letters of the English alphabet, including the blank space, for written English, or phonemes—i. e. units of sound—for spoken English) and certain strings of symbols are distinguishd as "well formed" (that is, acceptable, or meaningful) "words" or "sentences" or "formulas." Formal languages **differ** from natural languages in that in a formal language there are *very precise rules* telling us which strings of symbols are well-formed formulas. These rules should be so precise that one ought to be able to write a computer program to check whether a string of symbols is a well-formed formula or not.

A simple example. Let us describe a very simple formal language. We use two symbols, "a" and "b," and give the following rules:

RULE 1: The one-letter words a, b are well formed formulas.

RULE 2: If A and B are well-formed formulas, then ABA is a well-formed formula.

RULE 3: Only those strings obtained by repeatedly applying Rules 1 and 2 are well-formed formulas.

Then, for example, *bbbbabababababbbb* is well formed, because we can produce it as follows:

STEP 1: a and b are well formed, by Rule 1.

STEP 2: Applying Rule 2 with a in the role of A and b in the role of B, we see that *aba* is well formed.

STEP 3: Applying Rule 2 with b in the role of A and a in the role of B, we see that *bab* is well formed.

STEP 4: Applying Rule 2 with *bab* in the role of A and *aba* in the role of B, we see that *bababababab* is well formed.

STEP 5: Applying Rule 2 with b in the role of A and b in the role of B, we see that *bbb* is well formed.

STEP 6: Applying Rule 2 with *bbb* in the role of A and *bababababab* in the role of B, we see that *bbbbabababababbbb* is well formed.

On the other hand, I claim that *baba* is *not* well formed. It is a good exercise to try to understand *why*. First of all, you can try to produce *baba* using the three rules, and no matter how hard you try you will not be able to. But, *just because you have not been able to do something it does not follow that it cannot be done.* So, how do we know that *baba* **cannot** be obtained using the rules? We need an *argument*. We need *reasoning*. We need a **proof.**

Here is a proof: if *baba* could be produced by applying the rules, it would have to be by means of Rule 2, because Rule 1 alone will only give us one-letter strings. Now, in any application of Rule 2, if A has at least two letters, then the resulting string will have at least five letters, so it cannot be *baba*. Therefore, if there is a way to get *baba* by Rule 2, it would have to be such that A is a one-letter string. If A is the string a, then the resulting word would start with an a, so it cannot be *baba*. If A is the string b, then the resulting word would end with a b, so it cannot be *baba*. END OF PROOF.

PROBLEM 1. Is *bbbbbaaaaa* well formed? Justify your answer.

PROBLEM 2. Is *bbbaaabbb* well formed? Justify your answer.

PROBLEM 3. Is *bbbaabbb* well formed? Justify your answer.

The above example of a formal language is very rudimentary, and not useful to us. We need some more sophisticated languages, that can be used to represent **forms of sentences**. In this course, we will be working with two such languages: the **propositional calculus** and the **first-order predicate calculus.**

4. PROPOSITIONAL CALCULI

In a propositional calculus, the **symbols** are:

1. the *propositional variables* (usually letters such as P, Q, R, S),
2. the *logical connectives* $\vee$, $\wedge$, $\Longrightarrow$, $\sim$,
3. the *parentheses* (and) .

The well-formed formulas are also called **propositional forms.** The **rules** determining which strings are well-formed formulas (i. e. propositional forms) are:

RULE 1: One-letter strings consisting of a propositional variable are well-formed formulas.

RULE 2: If A is a well-formed formula then $(\sim A)$ is a well-formed formula.

RULE 3: If A and B are well-formed formulas then $(A \wedge B)$, $(A \vee B)$, and $(A \Longrightarrow B)$ are well-formed formulas.

RULE 4: Only those strings obtained by applying Rules 1, 2 and 3 are well-formed formulas.

For example:

$(((P \vee Q) \wedge (\sim Q)) \Longrightarrow (P \vee Q))$ is well formed, because one can arrive at it by applying the rules as follows:

STEP 1: P and Q are well formed, by Rule 1.

STEP 2: Applying Rule 2 with Q in the role of A, we see that $(\sim Q)$ is well formed.

STEP 3: Applying Rule 3 with P in the role of A and Q in the role of B, we see that $(P \vee Q)$ is well formed.

STEP 4: Applying Rule 3 with $(P \vee Q)$ in the role of A and $(\sim Q)$ in the role of B, we see that $((P \vee Q) \wedge (\sim Q))$ is well formed.

STEP 5: Applying Rule 3 with $((P \vee Q) \wedge (\sim Q))$ in the role of A and $(P \vee Q)$ in the role of B, we we see that $(((P \vee Q) \wedge (\sim Q)) \Longrightarrow (P \vee Q))$ is well formed.

On the other hand, I claim that $((P \wedge Q) \Longrightarrow Q$ is *not* well formed. Once again, as in the previous section, it is a good exercise to try to understand *why*. First of all, you can try to produce $((P \wedge Q) \Longrightarrow Q$ using the three rules, and no matter how hard you try you will not be able to. But, *just because you have not been able to do something it does not follow that it cannot be done.* So, how do we know that $((P \wedge Q) \Longrightarrow Q$ **cannot** be obtained using the rules? We need an *argument.* We need *reasoning.* We need a **proof.**

Here is a proof. First, we prove a small theorem.

THEOREM 1. *In every well-formed formula the number of left parentheses is equal to the number of right parentheses.*

PROOF. Call a string "balanced" if it has the same number of left and right parentheses. Any string obtained using Rule 1 is balanced, because it has zero left parentheses and zero right parentheses. Any application of Rule 2 will add an extra left parenthesis and an extra right parenthesis, so when you apply Rule 2 to a balanced string you get a balanced string. In any application of Rule 3, the number of left parentheses in the result is that of A, plus that of B, plus one, and the same is true for the number of right parentheses. So when you apply Rule 3 to two balanced strings you get a balanced string. So the application of Rules 1, 2, 3 can never create unbalanced strings. END OF PROOF.

Remark. Theorem 1 gives an example of a **necessary condition** for a string to be well formed. (We say that A is a *necessary condition* for B if A happens whenever B does, that is, if B implies A. For example: "he took the final exam" is a necessary condition for "he passed the course," because if he passed the course then he must have taken the final exam. But is of course not a "sufficient condition." Just taking the final exam is not enough to pass the course.) The theorem says that if a string is well formed then it has to have the same number of left and right parentheses. This condition is not sufficient: there are many strings that have the same number of left and right parentheses but are not well formed. (For example, the string "$P)($".) How can we tell if a string is well formed? Even more precisely, can one give a **computer algorithm** to check? The answer is "yes," but this is a hard thing to do, and if you are interested you should look it up in a mathematical logic book. □

PROBLEM 4. Is $(((P \vee Q) \wedge \sim Q) \Longrightarrow (P \vee Q))$ well formed? Justify your answer.

PROBLEM 5. Is $(((P \vee Q) \wedge (\sim Q))(\Longrightarrow (P \vee Q)))$ well formed? Justify your answer.

When we work with a propositional calculus, we usually do not want to have to write too many parentheses. So we add the

PARENTHESIS DELETION OPTION: If there is only one way to put the parentheses back in to produce a well-formed formula, then you are allowed to omit them.

For example, you can write $P \wedge Q$ rather than $(P \wedge Q)$, or $\sim P$ rather than $(\sim P)$.

Remark. The book, in the bottom half of page 6, gives you extra leeway to omit parentheses. For example, the book gives you **extra rules** so that, if you write $\sim P \wedge Q$, it is understood that this stands for $((\sim P) \wedge Q))$. (In principle, it could also have stood for $(\sim (P \wedge Q))$. The rules given in the book exclude this second possibility, because "$\sim$ applies to the smallest proposition following it.")

Notice that **if you want to make the formulas simpler by allowing deletion of lots of parentheses, as the book does, then the rules become more complicated.** In my own personal view, it is better to keep the rules simple, even at the cost of writing formulas with more parentheses. **You are free to do it my way, or the book's way, as you prefer, long as you do it correctly.** □

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5. WHAT ARE PROPOSITIONAL CALCULI GOOD FOR?

So far, we have just given some rules for writing certain weird strings of symbols and calling them "propositional forms." What do we want to with all that?

The answer is: **we use propositional forms to represent forms of propositions or statements.**

For example, the statement

(S1) Abraham Lincoln was a great president and all the serious historians who have studied his career think very highly of him.

could be represented by the propositional form $P \wedge Q$, where the propositional variable P stands for "Abraham Lincoln was a great president" and the propositional variable Q stands for "all the serious historians who have studied Abraham Lincoln's career think very highly of him."

Notice that (S1) could also be represented in the form P, where the propositional variable P stands for "Abraham Lincoln was a great president and all the serious historians who have studied his career think very highly of him."

So **the representation of a statement by means of a propositional form is not unique.** There may be more than one way to represent the statement, depending on how deeply you want to go into the details its structure. In our example, "P" is a **coarser** propositional-form representation of (S1) than "$P \wedge Q$" (and "$P \wedge Q$" is **finer** than "P").

As explained in the book, a propositional form has a **truth table**. The truth table tells you how to assing a "truth value" (True or False) to a propositional form if you assign truth values to the propositional variables.

A **tautology** is a propositional form that has the value True for all possible ways of assigning truth values to its propositional variables.

A **contradiction** is a propositional form that has the value False for all possible ways of assigning truth values to its propositional variables.

Often, we also apply the words "tautology" and "contradiction" to statements in ordinary English. In that case, what we mean by a "tautology" is "a statement that has a propositional form which is a tautology." Also, what we mean by a "contradiction" is "a statement that has a propositional form which is a contradiction."

For example: the statement

(S2) Hector Sussmann is a great teacher.

is true (of course!), but it is of the form P, which is not a tautology, and it has no other, finer propositional form. So (S2) is not a tautology.

On the other hand, the statement

(S3) Either Hector Sussmann is a great teacher or he is not.

has the propositional form P, which is not a tautology, but is also has the form $(P \vee (\sim P))$, which is a tautology. So (S3) **is** a tautology.

Remark. The example of (S3) shows that, for something to be a tautology, it suffices that **one** of its propositional forms be a tautology. In the case of statement (S3), we can find at least three different propositional form for it, namely, P, $(P \vee Q)$, and $(P \vee (\sim P))$, which are progressively finer, and only the third one is a tautology. This, however, suffices to make (S3) a tautology. □

Let us look at another example of a formal representation. Consider the statement (in which we assume that f is a fixed function described earlier, and x_0 is some given number)

(S4) If f has a relative minimum at x_0 and f is differentiable at x_0 then $f'(x_0) = 0$.

This is of the form P (taking P to stand for the whole statement), or of the form $P \Longrightarrow Q$ (taking P to stand for "f has a relative minimum at x_0 and f is differentiable at x_0" and Q to stand for "$f'(x_0) = 0$"), or of the form $(P \wedge Q) \Longrightarrow R$ (taking P to stand for "f has a relative minimum at x_0," Q to stand for "f is differentiable at x_0," and R to stand for "$f'(x_0) = 0$"). Once again, notice that the logical form is not unique.

6. SOME TYPES OF VALID INFERENCES

Now that we know how to represent statements, we can start giving examples of valid inferences of the propositional calculus.

Assume that f is a given function, and n is a given natural number. Then the inferences

$$\begin{array}{r} \text{If } f \text{ is differentiable then } f \text{ is continuous.} \\ f \text{ is differentiable.} \\ \hline f \text{ is continuous.} \end{array} \qquad (12)$$

and

$$\begin{array}{r} \text{If } n \text{ is not prime then } n \text{ is divisible by 3.} \\ n \text{ is not a prime number.} \\ \hline n \text{ is divisible by 3.} \end{array} \qquad (13)$$

are both of the form

$$\begin{array}{r} P \Longrightarrow Q \\ P \\ \hline Q \end{array} \qquad (14)$$

This is a **valid inference form.** Why is it valid? Because *if the premisses are true then the conclusion must be true.* Indeed: if $P \Longrightarrow Q$ is true and P is true then Q is true, because if Q was false then $P \Longrightarrow Q$ would be false (since P is true) because the truth table for $\Longrightarrow$ gives an F to $P \Longrightarrow Q$ if P is T and Q is F.

Remark. Notice that in (13) the first premiss is usually not true. (For example, it is not true if $n = 25$, because 25 is not a prime number and it's not divisible by 3. So, if $n = 25$, the first premiss says "If 25 is not a prime number then 25 is divisible by 3." This is of the form $P \Longrightarrow Q$, where P stands for "25 is not a prime number" and Q for "25 is divisible by 3." Then P is true and Q is false, so $P \Longrightarrow Q$ is false.)

The inference form (14) is one of the most basic ones of all logic. It is called **modus ponens.**

What do we mean, in general, by a "valid inference form"?

First of all, an **inference form** is a list of propositional forms (the premisses), followed by a horizontal line and then a propositional form (the conclusion).

A **valid inference form** is an inference form such that the conclusion is true for all assignments of truth values to the propositional variables that render the premisses true.

Here is another example: the form

$$\begin{array}{r} P \Longrightarrow Q \\ \sim Q \\ \hline \sim P \end{array} \qquad (15)$$

is valid. Examples of inferences of this form are:

$$\begin{array}{r} \text{If I was rich I would own a Mercedes.} \\ \text{I don't own a Mercedes.} \\ \hline \text{I am not rich.} \end{array} \qquad (15)$$

and

$$\begin{array}{r} \text{If } x > 3 \text{ then } x > 2\,. \\ \sim x > 2\,. \\ \hline \sim x > 3\,. \end{array} \qquad (15)$$

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7. FIRST-ORDER PREDICATE CALCULI

We now turn to the formal languages that are really important for this course, namely, first-order predicate calculi. Inferences such as (1) above cannot be justified by just looking at their propositional form. If we look at propositional form, then (1) is of the form

$$\begin{array}{r} P \\ Q \\ \hline R \end{array} \qquad (15)$$

which is not a valid inference form of the propositional calculus. In a first-order predicate calculus, (1) will have associated to it a much **finer** form, namely

$$\begin{array}{r} (\forall x)(p(x) \Longrightarrow q(x)) \\ p(a) \\ \hline q(a) \end{array} \qquad (16)$$

(In (1), "$p(x)$" stands for "x is a man," "$q(x)$" for "x is mortal," and "a" for "Socrates.")

On the other hand, *(10) is not if the form (16).* In fact, to represent (10) correctly we need a binary predicate $E(x, y)$ (meaning "x and y are equal to each other"), and then *(10) cannot even be represented in the language.* Using a 1-variable predicate $M(x)$ for "x is a man", and the symbol a for "Socrates", we can represent the premisses as $(\forall x)(\forall y)((M(x) \wedge M(y)) \Longrightarrow E(x, y))$ and $M(a)$. But *there is no way to represent the conclusion*, which is fine because the conclusion is meaningless. In other words: **by trying to represent an inference such as (10) in a first-order predicate calculus, we discover that it cannot be done, and this explains why it is mistaken to think that (10) is valid.**

Similarly, (8) and (9) are of the form

$$\begin{array}{r} p(x) \\ p(y) \\ \hline p(x) \wedge p(y) \end{array} \qquad (17)$$

which is valid, but (11) is of the form

$$\frac{\begin{array}{r} p(x) \\ p(y) \end{array}}{q(x,y)} \qquad (18)$$

where q is a *different predicate* from p. (In fact, q is a two-variable predicate, and p is a one-variable predicate. Naturally, p and q are related, because $p(x)$ —i.e. "x is married"— is a way of saying $(\exists y)q(x,y)$, i.e. "x is married to somebody," or "there exists a person that x is married to," or "there exists a y such that x is married to y.")

Once again, **we looked at an inference that appeared to be valid but really should not have been. We tried to represent it in a first-order predicate calculus, and we discovered why it is not valid. the reason is that a word that appeared in it was being used with two different meanings, namely, the one-variable predicate "married" and the two-variable predicate "married."** The attempt to formalize our inference in a first-order predicate calculus forced us to be precise about what kind of predicate "married" is, and we discovered that it is really two different predicates. This solved our problem.

We now turn to a more precise description of **PREDICATE CALCULI**.

A predicate calculus is a formal language in which there are two kind of well-formed expressions, namely, **terms** and **formulas**.

The basic symbols will be:

1. constants;
2. variables;
3. function symbols;
4. predicate symbols;
5. the logical connectives $\forall$, $\exists$, $\wedge$, $\vee$, $\Rightarrow$, $\Longleftrightarrow$, $\sim$;
6. the left and right parentheses, "(" and ")," and the comma ",".

VARIABLES are letters such as x, y, z, m, n, k, ε, X, Y, etc. In fact, any letter (or even other symbols, such as $\diamond$, or $\aleph$, or even a sequence of letters such as "grrrr" can be used as a variable, provided it is *declared* to be a variable. Often, there is no need to declare a symbol to be a variable, because the context makes it clear that it is a variable. For example, if I write $(\forall x)(x^2 \geq 0)$, then x *has* to be a variable, because only variables can appear in quantifiers such as $(\forall x)(x^2 \geq 0)$.

CONSTANTS are letters such as a, b, c, α, A, B, etc. Actually, any letter (or even other symbols, such as $\diamond$, or $\aleph$, or ε, or x, or y, or π, or e, or 0, or 1, or 2, or 3, or $\emptyset$, or even a sequence of letters such as "grrrr" or "34612) can be used as a constant, provided it is *declared* to be a constant. Often, there is no need to declare a symbol to be a constant, because the context makes it clear that it is a constant. For example, if I write "let a be an arbitrary real number," or "let $a = 3$," or "suppose A and B are sets," then I am declaring that a, b, A, B are going to be treated as constants.

Notice that, if a symbol is a constant, then there are certain things you cannot do with it. For example, after you have declared "a" to be a constant by saying, for example, "let a be an arbitrary real number," *you cannot use quantifiers such as* $(\forall a)$, because a is now fixed, not variable.

Similarly, if a symbol is a variable, then there things you cannot do with it. For example, after you have said in a proof $(\forall x)(\forall y)(x + y = y + x)$, you cannot write in your proof something like $x + y = y + x$, because *this is not a proposition.* (This means that, if x and y are variables, then "$x + y = y + x$" does not have a definite truth value. It doesn't make sense to say that $x + y = y + x$" is true or false unless x and y are given specific values. And, if you give x and y specific, definite values, then they become constants.)

FUNCTION SYMBOLS are letters such as f, g, F, G, or other symbols such as $+$, $\cdot$, $\cap$, $\cup$, $\in$, $\sim$. Each function symbol admits a *fixed* number of arguments. (If admits *one* argument, then it is a *unary*, or *one-variable* symbol. If admits *two* arguments, then it is a *binary*, or *two-variable* symbol. If admits *three* arguments, then it is a *ternary*, or *three-variable* symbol. And so on.) If you write a function symbol, followed by a "(," then as many arguments as the function symbol admits, separated by commas, and then ")," you get a *term*. The strings that can be used as arguments are variables, or constants, or more general terms.

EXAMPLE. Suppose you want to have a predicate calculus to talk about kinship and comparing ages of people. You could introduce one-variable function symbols f and g to stand for "father of" and "mother of" and, say, a two-variable function symbol y to stand for "the younger of two people." Then, for example, the term $f(Bill)$ would stand for "Bill's father," the term $m(Bill)$ would stand for "Bill's mother," and the term $y(Bill, John)$ would stand for "the younger of Bill and John" (that is, $y(Bill, John) = Bill$ if Bill is younger than John, and $y(Bill, John) = John$ if John is younger than Bill). As explained before, function letters can themselves admit terms as argument, so for example you could write $f(f(Bill))$, which would stand for "Bill's paternal grandfather," $f(m(Bill))$, which would stand for "Bill's maternal grandfather," or even more complicated expressions such as

$$f(m(y(Bill, y(John, Patricia)))),$$

which stands for "the maternal grandfather of the youngest of Bill, John and Patricia." (QUESTION: could we have written $f(m(y(Bill, John, Patricia)))$. Yes? No? Why?)

ANOTHER EXAMPLE. In mathematics, we use a predicate calculus that has symbols for the basic arithmetic operations "$+$" and "$\cdot$" ("plus" and "times"), which are *binary* (that is, you add and multiply *two* numbers). In this case, it is customary to write things such as "$a + b$," "$a \cdot b$," rather than "$+(a, b)$," "$\cdot(a, b)$." There are, of course, many other function symbols in common use. For example, $|\ldots|$ ("absolute value of") is a *one-variable* function symbol, so we use terms such as $|3|$. As in our previous example, one can use terms as arguments, so for example we can form terms such as $|(|3| + 5).|7 + 8| + 4|$.

PREDICATE SYMBOLS are letters such as p, q, P, Q, or other symbols such as $=$, $>$, $<$, $\leq$, $\geq$, ϵ, $\subseteq$. Each function symbol admits a *fixed* number of arguments. (If admits *one* argument, then it is a *unary*, or *one-variable* predicate. If admits *two* arguments, then it is a *binary*, or *two-variable* predicate. If admits *three* arguments, then it is a *ternary*, or *three-variable* predicate. And so on.) If you write a predicate symbol, followed by a "(," then as many arguments as the predicate symbol admits, separated by commas, and then ")," you get an *atomic formula*. The strings that can be used as arguments are *terms*.

EXAMPLE. Let us go on with the predicate calculus that enables us to talk about kinship and compare ages of people. We could introduce two-variable predicates s, d, c, so that $s(a,b)$ would stand for "a is b's son", $d(a,b)$ would stand for "a is b's daughter", $c(a,b)$ would stand for "a is b's child." (Then $c(a,b) \Longleftrightarrow (s(a,b) \vee d(a,b))$. Make sure you understand why!) We could also have the binary predicate "=" with its usual meaning (that is, "$a = b$" means that a and b are the same person). With these symbols, we could say lots of more complicated things. For example, suppose I want to say "a is b's uncle." I could say this by writing

$$(\exists x)(\exists y)(\exists z)\Big(s(a,x) \wedge s(a,y) \wedge (\sim x = y) \wedge (\sim a = z) \wedge c(z,x) \wedge c(z,y) \wedge c(b,z)\Big) .$$

ANOTHER EXAMPLE. In mathematics, the predicate calculi we use have symbols for the basic predicates "=," ">," "<," "≥," "≤," which are *binary*. In this case, it is customary to write things such as "$a > b$," "$a = b$," rather than "$> (a,b)$," "$= (a,b)$." There are, of course, many other predicates in common use. For example, for integers we could introduce a binary predicate symbol "|" that stands for "divides." (That is, $m|n$ stands for $(\exists k)(k.m = n)$.) These symbols admit terms as arguments, so for example we can write things such as $(3+5).7 = |50+6|$. Here $(3+5).7$ and $|50+6|$ are terms, and $(3+5).7 = |50+6|$ is an atomic formula.)

Atomic formulas can be combined with the logical connectives to yield *formulas*. For example,

$$(\exists x)(\exists y)(\exists z)\Big(s(a,x) \wedge s(a,y) \wedge (\sim x = y) \wedge (\sim a = z) \wedge c(z,x) \wedge c(z,y) \wedge c(b,z)\Big)$$

is *not* an atomic formula, but is obtained from the atomic formulas $s(a,x)$, $s(a,y)$, $x = y$, $a = z$, $c(z,x)$, $c(z,y)$, $c(b,z)$ by combining them via the connectives $\wedge$, $\sim$ and $\exists$.

TERMS VS. FORMULAS

Terms are supposed to be names of "things." **Formulas** are expressions that —when we plug in values for the open variables— become sentences or statements.

For example, the following are terms:

$$\begin{array}{rrrr} 3 & 3+5 & x & 5xyz \\ x+3 & 3+5 & x^2+3y+7 & \{x : x^2 < 3y\} , \end{array}$$

and the following are formulas:

$$\begin{array}{rrrr} 3<5 & 3+5=8 & x+3=7y & 5xyz < x+z \\ (\forall x)(x+3>x) & (\forall x)(x+y>x) & (\exists x)(x+y>xz) & (\forall x)(x>y \Longrightarrow y=z) . \end{array}$$

If you want to see a completely precise definition of "term" and "formula" then you have to consult a mathematical logic textbook. In this course, we will take these concepts for granted.

How can you tell whether something is a term or a formula?

TERMS. A **term** stands for a **definite thing** (it may be a number, or a set, or a line, or a function, or some other entity such a person or an animal), or for a "variable thing" that becomes definite once you plug in values for the variables. (For example, "$3+5$" denotes a definite number, whereas "$3+x$" denotes a "variable number" that becomes definite once you plug in values for x: when you plug in $x = 5$, $3+x$ has the value 8; when you plug in $x = -3$, $3+x$ has the value 0; when you plug in $x = -20$, $3+x$ has the value -17; and so on.) For a more complicated example of a term, consider the expression "$f(x)$". This is a term that contains two letter variables, namely, f and x, standing, respectively, for a function and a number. So, when you plug in a specific function for

f, and a specific number for x, you get a number. (For example, if f is the "squaring" function, i.e. the function such that $f(y) = y^2$ for all y, and $x = 3$, then $f(x)$ has the value 9. If f is the "exponential" function, i.e. the function such that $f(y) = e^y$ for all y, and $x = 6$, then $f(x)$ has the value e^6. If f is the function such that $f(y) = y^2 + 3y + 7$ for all y, and $x = 6$, then $f(x)$ has the value 61.)

FORMULAS. A **formula** makes an **assertion**, which may be true or false. So a formula has a **definite truth value** ("true" or "false," often abbreviated T or F) or a "variable truth value" that becomes definite once you plug in values for the variables. (For example, "$3 + 5 = 8$" has a definite truth value, namely, T, and "$3 + 5 = 6$" has a definite truth value, namely, F. On the other hand, "$3 + x = y$" has a variable truth value, which becomes definite once you plug in values for the variables x and y: when you plug in $x = 5$ and $y = 8$, $3 + x = y$ is T; when you plug in $x = 2$ and $y = 8$, $3 + x = y$ is F; when you plug in $x = 2$ and $y = 5$, $3 + x = y$ is T; and so on.)

VARIABLES AND CONSTANTS

As explained before, the basic constituents of terms are *variables* and *constants*.

Constants are symbols that stand —permanently or temporarily— for one *fixed*, *definite* thing. *Variables* are letters such as x, y that are meant to stand for things that only become definite when you assign a specific *value* to the variable.

In principle, any symbol could be used for a constant, and also for a variable. What matters is that in each case it should be clear which use is intended, e.g. by *declaring* a symbol to be a constant or a variable. Usually, letters such as x, y, z, are used as variables for real numbers, letters such as k, m, n, p are used as variables for natural numbers, and letters such as X, Y, Z are used as variables for sets. So, for example, we would write formulas such as

$$(\forall x)(\exists y)(y > x)\,,$$

$$(\forall m)(\forall n)(\forall k)((k|m \wedge k|n) \Rightarrow k|m+n)\,,$$

$$(\forall X)(\forall Y)(\forall Z)((X \subseteq Y \wedge Y \subseteq Z) \Rightarrow X \subseteq Z)\,.$$

But there is nothing sacred about this particular use. We could, for example, write

$$(\forall a)(\exists b)(b > a)\,,$$

$$(\forall x)(\forall y)(\forall z)((z|x \wedge z|y) \Rightarrow z|x+y)\,,$$

$$(\forall \alpha)(\forall \beta)(\forall \gamma)((\alpha \subseteq \beta \wedge \beta \subseteq \gamma) \Rightarrow \alpha \subseteq \gamma)\,,$$

provided that we specify that "a, b are real variables, x, y, z are natural-number variables, α, β, γ are set variables."

One way to make it clear that something is a variable and to specify its range is to put the range in the quantifier. For example, if we write

$$(\forall a \in \mathbf{R})(\exists b \in \mathbf{R})(b > a)\,,$$

$$(\forall x \in \mathbb{N})(\forall y \in \mathbb{N})(\forall z \in \mathbb{N})((z|x \wedge z|y) \Rightarrow z|x+y)\,,$$

then it is clear that a, b are real variables, and x, y, z are natural-number variables.

For our third sentence, involving sets, the situation is a little bit more complicated. The obvious thing to do would be to write something like

$$(\forall X \in \mathcal{S})(\forall Y \in \mathcal{S})(\forall Z \in \mathcal{S})((X \subseteq Y \wedge Y \subseteq Z) \Rightarrow X \subseteq Z)\,,$$

where $\mathcal{S}$ is "the set of all sets." This, however, is *forbidden*, because the "set of all sets" has to be avoided, due to the Russell Paradox (to be discussed later in the course).

So we can get around this by writing, instead,

$$(\forall X)(\forall Y)(\forall Z)((X \subseteq Y \wedge Y \subseteq Z) \Rightarrow X \subseteq Z)\,, \quad \text{where } X, Y, Z \text{ are set variables,}$$

or

$$\text{If } X, Y, Z \text{ are sets, then } (X \subseteq Y \wedge Y \subseteq Z) \Rightarrow X \subseteq Z\,.$$

Naturally, we could even use more informal expressions such as

$$\text{If } X, Y, Z \text{ are sets, then if } X \subseteq Y \text{ and } Y \subseteq Z \text{ it follows that } \Rightarrow X \subseteq Z\,,$$

or

$$\text{If } X, Y, Z \text{ are sets, and } X \subseteq Y \text{ and } Y \subseteq Z, \text{ then } \Rightarrow X \subseteq Z\,.$$

More complicated terms can be constructed from the basic constants and variables by means of (i) function symbols, and (ii) the "curly bracket" set notation. For example, the expression $x + 3$ is a term, obtained by combining the variable x and the constant 3 by means of the binary (i.e. 2-argument) function symbol $+$. The expression $\{x \in \mathbf{R} : x^2 > 2\}$ is a term, that stands for a certain set of numbers. **A more detailed description of how function symbols work, and how to build more complicated terms and work with them, will be given later. For the time being, let us forget about function symbols until further notice, and concentrate on predicates, constants, variables, and the logical connectives.**

(to be continued)