MATHEMATICS 300 — SPRING 2001

Introduction to Mathematical Reasoning (640 - 300, Section 03)
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HANDOUT NO. 3 *(February 15, 2001)*

1 THE FIRST MIDTERM EXAM

The *first midterm exam* is going to be on **Thursday, March 1**. No books, notes of any kind, or calculators will be allowed.

2 HOMEWORK ASSIGNMENTS NOS. 4, 5, AND 6

Assignment No. 4 was due today, February 15. Assignment No. 5—which is very short—is due on Monday, February 19. Assignment No. 6 is due on Thursday, February 22.

2.1 ASSIGNMENT NO. 4:

- Section 33 of the book, Problems 1, 2, 3, 4, 5, 6, 7, 8, 10.

2.2 ASSIGNMENT NO. 5:

1. Here are five sentences about rabbits, unicorns, elephants, Peter, Minnie, and the binary relation "taller than."

(1) Unicorns are taller than elephants. (That is, "every unicorn is taller than every elephant," or, if you prefer, "for every x and every y, if x is a unicorn and y is an elephant, then x is taller than y.")

(2) Rabbits are taller than unicorns.

(3) If something is taller than something else which is in turn taller than a third thing, then the first one is taller than the third one. (That is "for every x, every y and every z, if x taller than y and y is taller than z, then x is taller than z.")

(4) Peter is a rabbit.

(5) Minnie is an elephant.

Using sentences (1), (2), (3), (4), (5), *and no other fact about rabbits, unicorns, elephants, Peter, Minnie, and the binary relation "taller than,"* prove that Peter is taller than Minnie.

2. (a) Are statements (1), (2), (3) of the previous problem true? (Please do not just answer "yes" or "no." Give *reasons* for your answers.)

(b) (Assume that statements (4) and (5) are true, and Peter is a real rabbit and Minnie a real elephant.) Is the conclusion (that is, the statement that "Peter is taller than Minnie") true?

3. *(Optional.)* Try to guess why I am assigning this particular problem. Compare this problem with the one on giraffes, cows and sheep in Homework Assignment No. 3. Reexamine your solution of that problem, and see if you can say something new about it after thinking about the "rabbits, unicorns and elephants" version.

2.3 ASSIGNMENT NO. 6:

- Section 6 of the book, Problems 8, 10, 14.
- Section 7 of the book, Problems 1, 5, 7, 8.
- Section 8 of the book, Problems 1, 2, 3, 6, 8, 9.
- Translate the sentence

$$\forall x, \forall y, ((x \in \mathbb{Z} \wedge y \in \mathbb{Z}) \longrightarrow \exists z, (z \in \mathbb{Z} \wedge z + y^2 = x))$$

  into an English statement consisting entirely of words, without any letter variables whatsoever.

3 A REMARK ON THE USE OF LETTERS

Students have been using letter symbols in ways that clearly violate the rules. **Please be careful and try not to do this again!**

Actually, the rules are quite simple, so you should not have any difficulty with them.

The first two rules are:

1. A letter symbol has to be either a constant or a variable and it cannot be both.

2. If a letter symbol is a constant, then it has to be introduced by declaring its value[1]. The value has to be an object that will remain fixed from then on. The object need not be known, but it has to be a definite object.

For example, here are some things you could say making a perfectly legitimate use of the letter symbols a, b:

$$\boxed{\text{Let } a = 5. \text{ Then } a^2 = 25.} \tag{1}$$

$$\boxed{\text{Let } a = 5. \text{ Then } a^2 = 20.} \tag{2}$$

$$\boxed{\text{Let } a \text{ be an arbitrary real number. Then } (a+1)^2 = a^2 + 2a + 1.} \tag{3}$$

$$\boxed{\text{Let } a \text{ be an arbitrary real number. Then } (a+1)^2 = a^2 + 1.} \tag{4}$$

$$\boxed{\text{Let } a \text{ be a rational number such that } a^2 = 2. \text{ Then } a^4 = 4.} \tag{5}$$

$$\boxed{\text{Let } a \text{ be a rational number such that } a^2 = 2. \text{ Then } a^4 = 3.} \tag{6}$$

$$\boxed{\text{Let } a \text{ be a real number such that } a^2 = 2. \text{ Then } a^4 = 4.} \tag{7}$$

$$\boxed{\text{Let } a \text{ be a real number such that } a^2 = 2. \text{ Then } a^4 = 3.} \tag{8}$$

$$\boxed{\text{Let } a \text{ and } b \text{ be arbitrary real numbers. Then } (a+b)^2 = a^2 + 2ab + b^2.} \tag{9}$$

$$\boxed{\text{Let } a \text{ and } b \text{ be arbitrary real numbers. Then } (a+b)^2 = a^2 + b^2.} \tag{10}$$

(Here we are only discussing *grammatical correctness*, that is, agreement with the rules of mathematical grammar. The above examples are "right" in the same way as the sentence "Clinton was the fifth president of the United States" is "right," that is, in the sense of being grammatically right, even though it is clearly false. This sentence should be contrasted with something like "Clinton president fifth was," which is grammatically unacceptable. All we are saying here is

[1]The "value" of a constant is the object for which it stands.

that our seven examples are written with correct mathematical grammar. Whether what they say is true is a different problem altogether. In (1), (3), (5), (6), (7), and (9), the second sentence is true under the assumption made in the first sentence. On the other hand, in (2), (4), (8) and (10), the second sentence can be false[2] under the assumption made in the first sentence[3].)

Now that we know how to use constants, we have to explain how to use *variables*. Once again, the rules are quite simple:

3. Letter variables can be used in a statement only if they are "dummy variables."

We will discuss dummy variables in great detail later, but the basic meaning of the concept is this: a *dummy variable* within a statement is a letter used as a label (or "tag") to keep track of an object that occurs several times. The distinctive feature of a dummy variable is that you could have used any other symbol instead, and you would be saying exactly the same thing.

Here are seven examples of statements involving dummy variables.

For every x, if x is a natural number then $x \geq 0$.	(11)
For some natural number x, $x^2 = 4$.	(12)
For some natural number x, $x^2 = 2$.	(13)
If you take all the natural numbers n between 1 and 25, compute $(n+1)^2 - n^2$ for each such n, and add all the results, what you get is 676.	(14)
If you take all the natural numbers n between 1 and 5, compute n^2 for each such n, and multiply all the results, what you get is $14,400$.	(15)
If you take the function that to each x assigns x^2, and integrate it from 0 to 1, the result is $\frac{1}{3}$.	(16)
The set of all the natural numbers n such that n is prime and $n > 2$ consists entirely of odd numbers.	(17)

The above statements can be rewritten in a more formal way as follows:

$$\boxed{\forall x, (x \in \mathbb{N} \longrightarrow x \geq 0).} \tag{11.\#}$$

$$\boxed{\exists x, (x \in \mathbb{N} \land x^2 = 4).} \tag{12.\#}$$

$$\boxed{\exists x, (x \in \mathbb{N} \land x^2 = 2).} \tag{13.\#}$$

$$\boxed{\textstyle\sum_{n=1}^{25} \Big((n+1)^2 - n^2\Big) = 676\,.} \tag{14.\#}$$

$$\boxed{\textstyle\prod_{n=1}^{5} n^2 = 14,400\,.} \tag{15.\#}$$

$$\boxed{\textstyle\int_0^1 x^2\,dx = \frac{1}{3}\,.} \tag{16.\#}$$

$$\boxed{\{n : n \in \mathbb{N} \land n \text{ is prime} \land n > 2\} \subseteq \{n : n \in \mathbb{N} \land n \text{ is odd}\}\,.} \tag{17.\#}$$

In addition, they could also be written using other letter variables. For example, we could say:

[2]Why did I write "can be false" rather than "is false"?

[3]Make sure this is clear to you. You may find it strange that (6) is correct. The reason for this, however, is quite simple: *there are no rational numbers whose square is equal to* 2.

$$\boxed{\forall m, (m \in \mathbb{N} \longrightarrow m \geq 0).} \tag{11.*}$$

$$\boxed{\exists q, (q \in \mathbb{N} \wedge q^2 = 4).} \tag{12.*}$$

$$\boxed{\exists n, (n \in \mathbb{N} \wedge n^2 = 2).} \tag{13.*}$$

$$\boxed{\textstyle\sum_{k=1}^{25} \Big((k+1)^2 - k^2\Big) = 676\,.} \tag{14.*}$$

$$\boxed{\textstyle\prod_{j=1}^{5} j^2 = 14,400\,.} \tag{15.*}$$

$$\boxed{\textstyle\int_0^1 z^2\,dz = \frac{1}{3}\,.} \tag{16.*}$$

$$\boxed{\{r : r \in \mathbb{N} \wedge r \text{ is prime} \wedge r > 2\} \subseteq \{u : u \in \mathbb{N} \wedge u \text{ is odd}\}\,.} \tag{17.*}$$

Or we could go to the other extreme and say all these things without using letter symbols at all:

The square of every natural number is nonnegative.	(11.&)
There exists a natural number whose square is equal to 4.	(12.&)
There exists a natural number whose square is equal to 2.	(13.&)
If you take all the natural numbers from 1 to 25, then compute for each such number the sum of the number plus 1, square this sum and subtract from the resulting square the square of the number itself, and, finally, add all the differences of squares obtained in this way, the result is 676.	(14.&)
The product of the squares of all the natural numbers from 1 to 5 is equal to $14,400$.	(15.&)
The integral from 0 to 1 of the function that assigns to each real number its square is equal to $\frac{1}{3}$.	(16.&)
The set of all natural numbers that are prime and greater than 2 is a subset of the set of all odd natural numbers.	(17.&)

We have now finished listing all the ways in which letter symbols can be used. Nothing else is permitted.

So now you know. If you write something like "let x =cows," then you know for sure that it is not acceptable. (Why? Well, because when you say "let x . . .," then the only possibility is that the letter x will be a constant. And if it is a constant then it has to stand for a definite object. And "cows" is not a definite object. On the other hand, "Let x be a cow" would be acceptable, and "Let x be the set of all cows" would be acceptable.)

Let us look at a second example. A student begins his solution of the "giraffes" problem by writing: "C: cow, S: sheep." This is completely unacceptable, and I am sure the student does not quite mean it, really. Does the student mean to say that he is going to use the letter C as an abbreviation for "cow"'? If so, then the student, to be consistent, would have to write things like "x is a C" instead of "x is a cow." But that is not what the student did. What the student did was write expressions like "Cx". Now, if we take seriously the student's assertion that "C means "cow," then when the student wrote "Cx" what he is actually saying is "cow x." But I doubt very much that the student was trying to say that. I bet that what he was trying to say was "x is a cow." But, if this is the case, then why did he write "Cx" rather than "x is a C"?