MATHEMATICS 300
INTRODUCTION TO MATHEMATICAL REASONING
SPRING 2000

INSTRUCTOR: Héctor J. Sussmann

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I. ANSWERS TO THE QUESTIONS IN THE FIRST MIDTERM EXAM

1. *In this problem, we use a formal language which is a predicate calculus with the usual logical connectives —that is $\vee$, $\wedge$, $\sim$, $\rightarrow$, $\leftrightarrow$, $\exists$ and $\forall$— as well as the left and right parentheses —"(" and ")"— and binary predicate symbols s, d for the predicates "son of," "daughter of," and "is equal to" (so $s(x,y)$ stands for "x is y's son," $d(x,y)$ stands for "x is y's daughter," $x=y$ stands for "x and y are the same person"), individual symbols (that is, the letters x, y, z, u, v, n, m, p, q, etc,),* **and nothing else**. *We take the universe of discourse U to be the set of all human beings.*

I. Give formal definitions of the following four predicates: "is male," "is female," "is the mother of," "is a brother of." (Note: first you have to decide how many arguments —i.e. how many variables— are involved in each case. EXAMPLE: if I ask you to give a formal definition of "has no children," then you would first remark that this is a one-variable predicate, so the thing that must be translated into the formal language is "x has no children." Then a correct translation is $\sim \exists y(s(y,x) \vee d(y,x))$.)

ANSWERS:
"x is male" means "$\exists y\, s(x,y)$";
"x is female" means "$\exists y\, d(x,y)$";
"x is the mother of y" means "$\exists z\,(d(x,z) \wedge (s(y,x) \vee d(y,x)))$";
"x is a brother of y" means
"$\exists z \exists w(\sim z = w \wedge s(x,z) \wedge s(x,w) \wedge (s(y,z) \vee d(y,z)) \wedge (s(y,w) \vee d(y,w))) \wedge \sim x = y$."

II. Translate each of the following three English sentences into a formula of the formal language:

i. Some people have a son and a daughter.
ANSWER: $\exists x \exists y \exists z(s(y,x) \wedge d(z,x))$.

ii. x has exactly two children.
ANSWER:
$\exists y \exists z\,((s(y,x) \vee d(y,x)) \wedge (s(z,x) \vee d(z,x)) \wedge \sim y = z \wedge \forall w((s(w,x) \vee d(w,x)) \rightarrow (w = y \vee w = z)))$.

iii. x is y's uncle.
ANSWER: $\exists z \exists w(s(x,z) \wedge (s(w,z) \vee d(w,z)) \wedge \sim x = w \wedge (s(y,w) \vee d(y,w)))$.

2. *In this problem, we use the same formal language as in Problem 1, with the same interpretation.* Translate each of the following five formulas into plain English, using normal English sentences (*without* letter variables).

i. $\forall x \exists y\, s(x,y)$. ANSWER: everybody is somebody's son.

ii. $\forall x \exists y\, s(y,x)$. ANSWER: everybody has a son.

iii. $\forall x \exists y\Big(s(y,x) \vee d(y,x)\Big)$. ANSWER: everybody has a son or a daughter. (Even better: "everybody has children.")

iv. $\forall x \forall y \forall z \forall w\left(\Big(s(x,z) \wedge d(y,w)\Big) \rightarrow \sim x = y\right)$. ANSWER: a son and a daugther are not the same person.

v. $(\forall x)(\forall y)\left(\left((\exists z)\Big(s(x,z) \wedge s(y,z)\Big)\right) \rightarrow \left((\exists z)(\exists w)\Big(s(x,z) \wedge s(y,z) \wedge s(x,w) \wedge s(y,w) \wedge \sim z = w\Big)\right)\right)$.
ANSWER: if two men have one parent in common then they have two parents in common.

3. i. Define "tautology." ANSWER: A **tautology** is a propositional form (or "sentence form" or "statement form") whose corrresponding truth function has the value T for all possible assignments of values to the propositional variables. *Note: the book says "a* **tautology** *is a statement whose truth function has all T's as outputs. We discussed this in class in great detail and explained why this is not right.)*

ii. Define "contradiction." ANSWER: A **contradiction** is a propositional form (or "sentence form" or "statement form") whose corrresponding truth function has the value F for all possible assignments of values to the propositional variables.

iii. Explain the distinction between "proposition" and "propositional form." ANSWER: A **proposition** is a declarative sentence that can be true or false. A **propositional form** is an expression in a formal language called "propositional calculus" (more precisely, a well formed formula in a propositional calculus) meant to represent the logical form of a proposition. (Example: "Patricia is 10 years old" is a proposition. "$P \wedge \sim P$" is a propositional form.)

4. In this problem, we use a formal language which is a predicate calculus with the usual logical connectives, that is $\vee$, $\wedge$, $\sim$, $\rightarrow$, $\leftrightarrow$, $\exists$ and $\forall$, as well as the comma, the left and right parentheses —"(" and ")"—the $=$ sign, acting as a binary predicate, and the following additional predicate symbols: a binary (i.e. two-variable) predicate p, two 1-variable predicates q, r, and individual symbols (that is, the letters x, y, z, u, v, n, m, p, q, etc,). A formula is **logically valid** (or a **law of logic**) if it is true for every possible choice of the universe of discourse and every way of assigning a meaning to the predicate symbols. For example: $(\forall x)q(x) \rightarrow \Big(\sim (\exists x)(\sim q(x))\Big)$ is logically valid because, no matter what $q(x)$ might stand for, the statement is true: "everybody eats lettuce" implies "there is no person who does not eat lettuce;" "all people are mortal" implies "nobody is immortal," "every natural number is positive" implies "there is no natural number which is not positive," and so on. On the other hand, $(\exists x)q(x) \rightarrow (\forall x)q(x)$ is **not** logically valid, because we could take $q(x)$ to stand for "x is an elephant," in the universe of animals, and then our sentence would say that "if there are elephants then every animal is an elephant," which is plainly false.

The following five formulas are sentences of this predicate calculus:

$$\forall y \exists x\, p(x,y) \rightarrow \exists x \forall y\, p(x,y)\,; \tag{1}$$

$$\exists x \forall y\, p(x,y) \rightarrow \forall y\, \exists x\, p(x,y)\,; \tag{2}$$

$$\forall x \forall y \Big(x = y \rightarrow (q(x) \rightarrow q(y))\Big)\,. \tag{3}$$

$$\forall x \forall y \Big((q(x) \rightarrow q(y)) \rightarrow x = y\Big)\,. \tag{4}$$

$$\Big(\sim (\exists x) q(x)\Big) \rightarrow \Big((\forall x)(q(x) \rightarrow r(x))\Big)\,. \tag{5}$$

For each of the five formulas, decide whether the formula is logically valid, and give a convincing reason (that is, a proof, but not necessarily a formal proof) for your answer.

ANSWERS: (1) is **not** logically valid. PROOF: let $p(x,y)$ stand for "x is a friend of y." Then (1) says that "if everybody has a friend then there somebody who is everybody's friend." This is clearly not true.

(2) **is** logically valid. PROOF: Assume $\exists x \forall y\, p(x,y)$. We are going to show that $\forall y\, \exists x\, p(x,y)$. Since $\exists x \forall y\, p(x,y)$ is true, we can pick an a such that $\forall y\, p(a,y)$. Let b be arbitrary. Since $\forall y\, p(a,y)$, we can conclude that $p(a,b)$. So $\exists x\, p(x,b)$. Since b was arbitrary,we have shown that $\forall y\, \exists x\, p(x,y)$.

(3) **is** logically valid. PROOF: Let x, y be arbitrary. We prove that $x = y \rightarrow (q(x) \rightarrow q(y))$. Assume $x = y$. We prove that $q(x) \rightarrow q(y)$. Assume $q(x)$. Since $x = y$, we have $q(y)$.

(4) is **not** logically valid. PROOF: let $q(x)$ stand for "x is very smart." We will prove that, in this particular interpretation, (4) is false. For this purpose, we must exhibit x, y such that $(q(x) \rightarrow q(y)) \rightarrow x = y$ is false. Let $x =$Bill Bradley, $y =$Héctor Sussmann. Then $q(y)$ is (obviously) true, so $q(x) \rightarrow q(y)$ is true. (It does not matter whether $q(x)$ is true or not.) But $x = y$ is not true. So $(q(x) \rightarrow q(y)) \rightarrow x = y$ is false.

(5) **is** logically valid. PROOF: Assume $\sim (\exists x)q(x)$, We are going to prove that $(\forall x)(q(x) \rightarrow r(x))$. Let x be arbitrary. We are going to prove $q(x) \rightarrow r(x)$. Now, $q(x)$ is false, because we are assuming that there aren't any u's for which $q(u)$. So $q(x) \rightarrow r(x)$ is true.

5. In this problem, the universe of discourse is **R**, the set of real numbers. **"Number" means "real number,"** that is, member of **R**. We use LERNA, the **language of elementary real-number arithmetic**. This is a **formal language** which is a predicate calculus containing the usual logical connectives —$\wedge$, $\vee$, $\sim$, $\to$, $\leftrightarrow$, $\exists$ and $\forall$— as well as the usual binary predicate symbols $=$, $>$, $<$, $\geq$, $\leq$, with their usual meanings, the constant symbols 0, 1, the symbols $+$, $\cdot$ for addition and multiplication, the left and right parentheses —"(" and ")", individual symbols (that is, the letters x, y, z, u, v, n, m, p, q, etc,), ¡**and nothing else**. (For example, we do not have the symbol 2. If you want to say 2, you have to say $1+1$. Also, we do not have the symbol $-$ for subtraction, so if you want to say $x - y = z$ you have to manage without using "$-$", e.g. by saying $x = y + z$ instead.)
Translate each of the following English sentences into our formal language, and decide if it is true or false. (Note: "positive" here means "> 0.")

1. Every number has a multiplicative inverse. TRANSLATION: $\forall x \exists y(x.y = 1)$. This is FALSE, because if $x = 0$ then y does not exist.
2. Every positive number has a fourth root. TRANSLATION: $\forall x(x > 0 \to \exists y((y.y).(y.y) = x)$. This is TRUE.
3. There exists a number whose square added to the number is equal to 1.
TRANSLATION: $\exists x(x.x+x = 1)$. This is TRUE. (REASON: the function f given by $f(x) = x^2+x$ is continuous. Moreover, $f(0) = 0$ and $f(1) = 2$. So by the intermediate value theorem there exists an x such that $f(x) = 1$. ANOTHER PROOF: you can compute x directly, by solving the quadratic equation $x^2 + x - 1 = 0$. The solutions are $x = \frac{-1\pm\sqrt{5}}{2}$.)
4. For any two numbers, there exists a unique number that added to the second number gives the first number. TRANSLATION: $\forall x \forall y \exists z(y + z = x \wedge \forall u(y + u = x \to u = z))$. This is TRUE.
5. There exists a smallest number. TRANSLATION: $\exists x \forall y(x \leq y)$. This is FALSE.
6. For any two distinct numbers, there is a number that is strictly between them (that is, between them and not equal to either of them).
TRANSLATION: $\forall x \forall y(\sim x = y \to \exists z((x < z \wedge z < y) \vee (y < z \wedge z < x)))$.

6. In this problem, the universe of discourse is **N**, the set of natural numbers, so **"number" means "natural number."** (The natural numbers are the integers that are strictly greater than zero, that is, $1, 2, 3, \ldots$.) We use LENNA, the **formal language of elementary natural number aithmetic.** This is a **formal language** which is a predicate calculus containing the usual logical connectives —$\wedge$, $\vee$, $\sim$, $\to$, $\leftrightarrow$, $\exists$ and $\forall$— as well as the usual binary predicate symbols $=$, $>$, $<$, $\geq$, $\leq$, with their usual meanings, the constant symbol 1, the symbols $+$ and $\cdot$ for addition and multiplication, the left and right parentheses—"(" and ")", individual symbols (that is, the letters x, y, z, u, v, n, m, p, q, etc,) **and nothing else**.

6.a Define "prime number" in terms of a sentence in the formal language. (That is, write a sentence in the formal language that says "n is a prime number.")
ANSWER: "n is prime" means "$n > 1 \wedge \forall p \forall q(n = p.q \to (p = 1 \vee q = 1))$".

6.b Translate each of the following English sentences into the formal language, and indicate whether it is true or false.

1. Every number has an additive inverse.
ANSWER: $\forall x \exists y(x + y + 1 = 1)$. (Comment: I do not say $\forall x \exists y(x + y = 0)$ because 0 is not in the list of permitted symbols.) This statement is FALSE, because if $x = 1$ then y does not exist.
2. There exist at least two prime numbers.
ANSWER: $$\exists n \exists m\Big((n > 1 \wedge \forall p \forall q(n = p.q \to (p = 1 \vee q = 1))) \wedge(m > 1 \wedge \forall p \forall q(m = p.q \to (p = 1 \vee q = 1)))\wedge \sim n = m\Big).$$ This is TRUE.
3. Every number which is greater than 1 is divisible by some prime number.
ANSWER: $\forall n(n > 1 \to \exists m(m > 1 \wedge \forall p \forall q(m = p.q \to (p = 1 \vee q = 1))))$. This is TRUE.
4. If two numbers are not equal then one of them is greater than or equal to 1 plus the other one.
ANSWER: $\forall n \forall m(\sim n = m \to (n \geq m + 1 \vee m \geq n + 1))$. This is TRUE.

5. There is no smallest number.
   ANSWER: $\sim \exists n \forall m (n \leq m)$. This is FALSE. REASON: there does exist a smallest number, namely, 1.
6. There is no largest number.
   ANSWER: $\sim \exists n \forall m (n \geq m)$. This is TRUE, because if n was such that $\forall m (n \geq m)$, then we could take $m = n + 1$ and get a contradiction.
7. The sum of four consecutive numbers is always divisible by 4.
   ANSWER: $\forall n \exists m (n + (n+1) + (n+1+1) + (n+1+1+1) = 4m)$. This is FALSE, because if $n = 1$ then $n + (n+1) + (n+2) + (n+3) = 10$ which is not divisible by 6.

7. In each of the following cases, check the appropriate box, and *justify your answer*. (In most cases there is no need to write a complete truth table. Recall that a **contingency** is a formula of the propositional calculus that in neither a tautology nor a contradiction.)

1. $P \to (Q \to (R \to (S \to (P \wedge Q \wedge R \wedge S))))$ is a ☐Tautology ☐Contradiction ☐Contingency .
   ANSWER: It's a **tautology**. REASON: for $P \to (Q \to (R \to (S \to (P \wedge Q \wedge R \wedge S))))$, P has to be T and $Q \to (R \to (S \to (P \wedge Q \wedge R \wedge S)))$ has to be F. But then Q has tgo be T and $R \to (S \to (P \wedge Q \wedge R \wedge S))$ has to be F. Then R is T and $S \to (P \wedge Q \wedge R \wedge S)$ has to be F. So S is T and $P \wedge Q \wedge R \wedge S$ is F. So we have seen that P, Q, R and S must be T but $P \wedge Q \wedge R \wedge S$ is F. This is impossible.

2. $(P \wedge (\sim P)) \to Q$ is a ☐Tautology ☐Contradiction ☐Contingency .
   ANSWER: It's a **tautology**. REASON: $P \wedge (\sim P)$ is always F.

3. $P \to (\sim P)$ is a ☐Tautology ☐Contradiction ☐Contingency .
   ANSWER: It's a **contigency**. REASON: $P \to (\sim P)$ is T if P is F, and F is P is T.

4. $(P \wedge (\sim Q)) \to (P \to Q)$ is a ☐Tautology ☐Contradiction ☐Contingency .
   ANSWER: It's a **contigency**. REASON: $(P \wedge (\sim Q)) \to (P \to Q)$ is T if P is F, and is F is P is T and Q is F.

5. $((\sim P) \vee (\sim Q)) \to (P \to (Q \to R))$ is a ☐Tautology ☐Contradiction ☐Contingency .
   ANSWER: It's a **tautology**. REASON: for $((\sim P) \vee (\sim Q)) \to (P \to (Q \to R))$ to be F, $(\sim P) \vee (\sim Q)$ has to be T and $P \to (Q \to R)$ has to be F. But if $(\sim P) \vee (\sim Q)$ is T, then P or Q must be F. If P is F then $P \to (Q \to R)$ is T. If Q is F then $Q \to R$ is T, so $P \to (Q \to R)$ is T.

6. $((\sim P) \wedge (\sim Q)) \wedge (P \vee Q)$ is a ☐Tautology ☐Contradiction ☐Contingency .
   ANSWER: It's a **contradiction**. REASON: for $((\sim P) \wedge (\sim Q)) \wedge (P \vee Q)$ to be T, $\sim P$, $\sim Q$ and $P \vee Q$ must be T. But then P and Q must be F, so $P \vee Q$ is F.

8. *In this problem, "number" means "natural number," as in problem 6 above.*

i. Prove or disprove: The sum of 6 consecutive numbers is always divisible by 6.
   ANSWER. This is FALSE. PROOF: $1 + 2 + 3 + 4 + 5 + 6 = 21$, and 21 is not divisible by 6.

ii. Prove or disprove: The sum of 7 consecutive numbers is always divisible by 7.
   ANSWER. This is TRUE. PROOF: let n be an arbitrary natural number. Then

$$n+(n+1)+(n+2)+(n+3)+(n+4)+(n+5)+(n+6) = 7n+1+2+3+4+5+6 = 7n+21 = 7(n+3)\,,$$

so $n + (n+1) + (n+2) + (n+3) + (n+4) + (n+5) + (n+6)$ is divisible by 7.

9. In this problem, the range of all the variables is the set $\mathbf{R}$ of real numbers. Translate each of the following English sentences into the language LERNA described in Problem 5.

i. x is the absolute value of y. ANSWER: $x \geq 0 \wedge (x = y \vee x + y = 0)$.

ii. x is the largest of y and z. ANSWER: $x \geq y \wedge x \geq z \wedge (x = y \vee x = z)$.

iii. The function $x^2 - x$ attains a maximum value on the closed interval $[0, 1]$. ANSWER:
   $\exists x \Big(0 \leq x \wedge x \leq 1 \wedge \forall y((0 \leq y \wedge y \leq 1) \to x.x + y \geq y.y + x)\Big)$.

10. Decide whether each of the following sentences is true, over the universe of real numbers, and justify your statement by giving a reason. *Note: For Parts (v), (vi) and (vii) you may need to use some of the things you learned in your Calculus courses.*

(i) $\forall x \exists y (y > x)$. ANSWER: It is TRUE. PROOF: let x be arbitrary; pick $y = x + 1$; then $y > x$.

(ii) $\forall x \exists y (y+1 = x)$. ANSWER: It is TRUE. PROOF: let x be arbitrary; pick $y = x-1$; then $y+1 = x$.

(iii) $\exists y \forall x (y + 1 = x)$. ANSWER: It is FALSE. PROOF: Suppose y exists. Pick one such y. Then $\forall x (y + 1 = x)$. Now apply this to $x = y$. We get $y + 1 = y$, which is false.

(iv) $\forall x \Big(x \geq 0 \to \exists y (y^2 = x) \Big)$. ANSWER: It is TRUE. REASON: this statement says that every nonnegative number has a square root.

(v) $\exists x (x^4 + 32x + 50 = 0)$. ANSWER: It is FALSE. REASON: the function f given by $f(x) = x^4 + 32x$ has a derivative $f'(x) = 4x^3 + 32$. Then $f'(x) < 0$ of $x < -2$ and $f'(x) > 0$ if $x > -2$. So f is decreasing for $x < -2$ and increasing for $x > -2$. So the minimum value of f is attained at $x = -2$. Now, $f(-2) = -48$. Therefore $x^4 + 32x \geq -48$ for all x. So $x^4 + 32x + 50 \geq 2$ for all x. So there is no x for which $x^4 + 32x + 50 = 0$.

(vi) $\exists x (x^4 + 32x + 45 = 0)$. ANSWER: It is TRUE. REASON: let f be the same function as in the previous part. Then $f(-2) = -48$ and $f(0) = 0$. It then follows from the intermediate value theorem that $f(x) = -45$ for some x, that is, $x^4 + 32x + 45 = 0$ for some x.

(vii) $\exists x (x^7 + 93x^6 + 3x^4 + 32x + 50 = 0)$ ANSWER: It is TRUE. REASON: let $f(x) = x^7 + 93x^6 + 3x^4 + 32x + 50$. Then $f(0) = 50$ and $f(-100) < 0$ (because, if $x = 100$, then $x^7 + 93x^6 + 3x^4 + 32x + 50 = (x + 93)x^6 + 3x^4 + 32x + 50 = -7000000000000 + 300000000 - 3200 + 50 < 0$). It then follows from the intermediate value theorem that $f(x) = 0$ for some x, that is, $x^7 + 93x^6 + 3x^4 + 32x + 50 = 0$ for some x.

HOMEWORK DUE ON THURSDAY MARCH 23

1. Do all the problems in the midterm, in your own handwriting.
2. Copy all the rules for proofs in Chapter 4 of the book.
3. Write down the following three times:
   a. To prove an existential statement I can do it by exhibiting an example (that is, a "witness"). But to prove a universal statement I **cannot** do it by exhibiting an example; I need an **argument**. For example: suppose $h(x)$ stands for "x is wearing a hat." To prove $\exists x\, h(x)$ is suffices for me to point to someone who is wearing a hat. But to prove $\forall x\, h(x)$ is cannot do it by gving an example of a person wearing a hat. I need a **general argument.** In exactly the same way: the statement that some sentence is logically valid is a universal statement (for **all possible interpretations** ...); therefore, if I want to prove that a statement is law of logic I cannot do it by giving an example of an interpretation where the statement is true; I have to show that it is true for **all possible interpretations**, and that requires a general argument. But to prove that the statement is *not* a law of logic I have to show that there is *some* interpretation for which it is false, and that I can do by exhibiting a particular example of an interpretation.
   b. If I want to translate a sentence involving certain free variables, then the translation should have the same free variables. For example, if I want to translate "x has a son" into the formal language of Problem 1 of the midterm, then the translation should have x as a free variable and nothing else; so something like $s(y, x)$ cannot possibly be a correct translation because it involves the fgree variable y which was not in the original statement. (A correct translation is "$\exists y\, s(y, x)$.")
   c. If I am asked to translate a statement into a sentence involving certain symbols **and nothing else**, then what I should do is translate the statement into a sentence involving those certain symbols **and nothing else**.

NOTE: you are exempted from doing Problem 3 if you can show that in your mditerm test you never violated the principles laid out in Problem 3. If you wish to avail yourself of that exemption, then hand in your test together with the homework.

4. For what natural numbers k is it true that "the sum of k consecutive natural numbers is always divisible by k"? (In case you don't see what I mean by this question, let me give you an example: if I ask you "for what natural numbers k is it true that k^2 is divisble by 3?", you answer should be "the numbers whose square is divisble by 3 are exactly the numbers that are themselves divisble by 3.") JUSTIFY YOUR ANSWER (that is, give a proof.)

5. For all the "translation" problems in the mditerm, every time you translated something wrong, take the translation you gave and translate it back into the original language. (For example, if you were asked to translate "everybody has a son" from English to the formal language of problem 1, and your translation was "$s(x, y)$", then when you translate back you get "x is y's son," which is not at all the same thing. If you were asked to translate $\forall x \exists y (y.y = x)$ from formal language to English, and you translated this as "x has a square root," then when you translate back you get $\exists y (y.y = x)$,which is not at all the same thing.)

6. Problems 11 and 12 in page 70 of the book.

7. Problems 6 and 10 in page 90.

8. Problems 9, 16, 17, 18 in pages 106, 107.