MATHEMATICS 300 — FALL 1999

Introduction to Mathematical Reasoning

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HANDOUT NO. 2 *(September 9, 1999)*

READING FOR THE FIRST THREE DAYS OF CLASS

(that is, for Sept. 2-9-13)

- the book's "Preface for the Student,"
- the book's Part I, Chapter 1, first four sections (that is, up to and including the section on "functions");
- the instructor's Handouts Nos. 1 and 2.

HOMEWORK ASSIGNMENT NO. 2

This assignment is due on Thursday Sept. 16.

Before you start writing your homework, read carefully the parts of Handouts 1 and 2, on "writing mathematics."

- Problems 1.5, 1.8, 1.9, 1.18, 1.19, 1.27, 1.30 in pages 21-22-23-24 of the book.
- Rewrite the statement of Proposition 1.4 in page 5 of the book in English, *without using any letter variables.*

MORE ON WRITING MATHEMATICS

ON THE USE OF LETTERS

Letter variables are only used in mathematics in order to shorten certain statements that would otherwise be too long.

In principle, any time you say something with letters it should also be possible to say the same thing without letters. *If you cannot translate what you are saying into a statement without letters it probably means that what you are saying is meaningless.*

EXAMPLE 1: Suppose you want to state Pythagoras' theorem. You can say it with words:

- The sum of the squares of the two legs of a right triangle is equal to the square of the hypothenuse.

or, using letters:

- If a and b are the lengths of the two legs of a right triangle, and c is the length of its hypothenuse, then $a^2 + b^2 = c^2$.

> AN EXAMPLE OF WHAT YOU SHOULD **NOT** WRITE
>
> QUESTION: State Pythagoras' theorem.
>
> ANSWER: $a^2 + b^2 = c^2$.
>
> WHY IS THIS BAD? Because nobody knows what the letters a, b, c are supposed to stand for!

EXAMPLE 2: Suppose you want to say

- If you take two real numbers, add them, and square the sum, then you get the same result as if you had squared each of the two numbers separately, added the squares, and added to the result twice the product of the two numbers.

This looks awfully complicated and ugly. You can make it much shorter by saying

- For all real numbers x, y, $(x+y)^2 = x^2 + y^2 + 2xy$.

(Later, we, will learn to say this even more concisely, by writing

$$(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})\Big((x+y)^2 = x^2 + y^2 + 2xy\Big) .$$

Here "$(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})$" is an abbreviation of "for all real numbers x, y.")

EXAMPLE 3: Suppose you want to say

- It is possible to find a real number whose cube, added to its square, is equal to the number itself plus two.

Again, this looks awfully complicated and ugly, and you can make it much shorter by saying

- There exists a real number x such that $x^2 + x^3 = x + 2$,

or

$x^2 + x^3 = x$ for some real number x.

(Later, we, will learn to say this thing even more concisely, by writing

$$(\exists x \in \mathbb{R})\Big(x^2 + x^3 = x + 2\Big) .$$

Here "$(\exists x \in \mathbb{R})$" is an abbreviation of "for some real number x.")

RECOMMENDED QUESTION FOR YOU TO THINK ABOUT: if I were to say "There exists a real number such that $x^2 + x^3 = x + 2$." This would **not** be acceptable. Why?

QUESTION: Is it true or false that $(\exists x \in \mathbb{R})\Big(x^2 + x^3 = x + 2\Big)$?

EXAMPLE 4: suppose you want to say

- For each integer between 2 and 25, take that integer, square it, add 5, and compute the multiplicative inverse of the result. Having done this for all the integers between 2 and 25 (including 2 and 25 themselves), add all the resulting numbers. Then the number you get is smaller than 20.

Once again, this is awful, so you can shorten the whole thing and just say

- Let $N = \sum_{k=2}^{25} \frac{1}{k^2+5}$. Then $N < 20$.

Or, if you prefer, you could just say:

- $\sum_{k=2}^{25} \frac{1}{k^2+5} < 20$.

RECOMMENDED QUESTIONS FOR YOU TO THINK ABOUT: is it true or false that $\sum_{k=2}^{25} \frac{1}{k^2+5} < 20$? How about $\sum_{k=2}^{25} \frac{1}{k^2} < 20$? And $\sum_{k=2}^{25} 1 < 20$? And what would you say if I ask you whether it's true that $\sum_{n=2}^{25} \frac{1}{k^2+5} < 20$?

EXAMPLE 5: Suppose you want to say

- If you take the function that to each real number assigns its cube, and integrate it from 0 to 4, then the result is 64.

Again, this can be made much shorter by saying

- Let $N = \int_0^4 x^3\,dx$. Then $N = 64$.

Or you could just say:

- $\int_0^4 x^3\,dx = 64$.

RECOMMENDED QUESTIONS FOR YOU TO THINK ABOUT: is it true or false that $\int_0^4 x^3\,dx = 64$? How about $\int_0^4 y^3\,dx = 64$?

EXAMPLE 6: Suppose I want to say:

- The set of all real numbers whose square is larger than the cube of the number has at least two different members.

We can make this shorter by saying

- Let $S = \{x \in \mathbb{R} : x^2 > x^3\}$. Then S has at least two members.

Or you could just say:

- Let $S = \{x \in \mathbb{R} : x^2 > x^3\}$. Then $(\exists x)(\exists y)\Big(x \in S, y \in S, \text{ and } x \neq y\Big)$.

EXAMPLE 7: Suppose I want to say that "my age is larger than 20 and less than 70, and my brother is younger than I am, but his age is larger than half my age."

Here too we can be more concise, and say

- Let x be my age and let y be my brother's age. Then $20 < x < 70$ and $\frac{x}{2} < y < x$.

In all these cases, we have used **letters** (namely, x, y, k, N, S) as labels, so that we can avoid writing clumsy phrases such as "the number that we have been talking about."

In all the above examples, the statements we wrote using letters only contain the letters as **dummy variables** or as **constants**. The meaning of the phrase "dummy variable" is quite simple: it means that the letter is just a label used to keep track of an object that occurs several times.

For example, if I say

- If x and y are real numbers then $xy \leq \frac{x^2+y^2}{2}$,

then I am just saying

- The product of any two real numbers is less than or equal to one half the sum of their squares.

I am not talking about "x" or about "y". The letters "x" and "y" are just **internal tags**, used within my statement to keep track of these two numbers. I could have used any other pair of symbols, for example

- If a and b are real numbers then $ab \leq \frac{a^2+b^2}{2}$,

and I would have been saying exactly the same thing.

That's why in these statements we call x, y, a and b "dummy variables." It's as if they were not there. They have no value outside the statement where they are used.

Constants are symbols that have a fixed value. For example, the symbol "1" has a value that we all agree upon once and for all: it stands for the number "one." Similarly, the Greek letter "π" stands for the number "pi," the ratio between the length of a circle and its diameter. So symbols like 1, 2, 35, pi are **fixed constants**, because they have values that are fixed once and for all.

In mathematics, we also use the device of **temporary constants**. That is, we take a symbol (usually a letter) and **declare** it to have a particular value for a while. Then this symbol has the declared value until the end of the particular text where this declaration is in force.

For example, suppose I say:

(A) Let $x = \frac{1+\sqrt{5}}{2}$. Then $x^2 = x + 1$.

In the first sentence I am declaring x to have the value $\frac{1+\sqrt{5}}{2}$. That is, I am announcing that I am going to use the letter x as an abbreviation for the number $\frac{1+\sqrt{5}}{2}$. Then in the second sentence I am making the assertion that $x^2 = x + 1$, so I am saying

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+\sqrt{5}}{2} + 1$$

in abbreviated form.

If you look at the whole paragraph (A), then **within** (A) **the letter x is a dummy variable,** because I coudl have said equally well

(B) Let $y = \frac{1+\sqrt{5}}{2}$. Then $y^2 = y + 1$.

or

(C) Let $\aleph = \frac{1+\sqrt{5}}{2}$. Then $\aleph^2 = \aleph + 1$.

and I would have been saying exactly the same thing.

On the other hand, once I have said "Let $x = \frac{1+\sqrt{5}}{2}$," then from that moment on x is a constant, because it has a fixed value. So, **in (A), within the sentence "Then $x^2 = x + 1$," the letter x is a constant.** If I changed x to another letter—say y— **only within that sentence**, then I would get something like

(*) Let $x = \frac{1+\sqrt{5}}{2}$. Then $y^2 = y + 1$.

which is completely meaningless, because we do not know what y is.

The distinctive feature of a dummy variable is that you could have used any other symbol instead, and you would be saying exactly the same thing. For example, if you write

- For all real numbers x, y, $(x + y)^2 = x^2 + y^2 + 2xy$,

then that is exactly the same thing as you had written

- For all real numbers a, b, $(a + b)^2 = a^2 + b^2 + 2ab$,

or

- For all real numbers α, β, $(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$,

or

- For all real numbers m, n, $(m + n)^2 = m^2 + n^2 + 2mn$,

or

- For all real numbers $\diamond$, $\#$, $(\diamond + \#)^2 = \diamond^2 + \#^2 + 2 \diamond \#$.

Similarly, instead of

$(\exists x \in \mathbb{R})\Big(x^2 + x^3 = x\Big)$,

you could have said

$(\exists z \in \mathbb{R})\Big(z^2 + z^3 = z\Big)$,

or

$(\exists a \in \mathbb{R})\Big(a^2 + a^3 = a\Big)$,

or

$(\exists \omega \in \mathbb{R})\Big(\omega^2 + \omega^3 = \omega\Big)$,

or

$(\exists \aleph \in \mathbb{R})\Big(\aleph^2 + \aleph^3 = \aleph\Big)$,

or

$(\exists \diamond \in \mathbb{R})\Big(\diamond^2 + \diamond^3 = \diamond\Big)$.

Just one more example:

Instead of

$$\sum_{k=2}^{25} \frac{1}{k^2+5} < 20\,,$$

you could have said

$$\sum_{n=2}^{25} \frac{1}{n^2+5} < 20\,,$$

or

$$\sum_{\rho=2}^{25} \frac{1}{\rho^2+5} < 20\,.$$

So the "dummy variables" in the above examples are really just **internal tags**, using to keep track of numbers **within a particular text**. Outside this text, they variables do not have any meaning. For example, after you have written

$$\sum_{k=2}^{25} \frac{1}{k^2+5} < 20\,,$$

it would not make sense for me to ask you "what is the value of k?" The letter k is just a tag to specify a *range of values*, as part of a set of instructions on how to compute a sum. Any other letter could be used instead, so in fact "k isn't really there." You can persuade yourself of this in another way, by remembering that

$$\sum_{k=2}^{25} \frac{1}{k^2+5} < 20$$

is just another way of saying

- For each integer between 2 and 25, take that integer, square it, add 5, and compute the multiplicative inverse of the result. Having done this for all the integers between 2 and 25 (including 2 and 25 themselves), add all the resulting numbers. Then the number you get is smaller than 20.

As you can see, there is no k there!

This way of speaking and writing, using dummy variables, is very useful, especially when what we are trying to say is quite long. But **you have to be careful** when writing this way, and have to make sure that you are using the letters correctly.

IMPORTANT RULE NO. 1

A letter symbol can only be used if either its value has been declared before or it is a dummy variable.

For example: if you write

$$x^2 + x = 90$$

just like that, without any words before or after explaining what x is, or making it a dummy variable, then what you wrote is **meaningless**. We have no way to tell if what you wrote is true or false, because we have no idea what you are talking about.

If you write this, then I will write in red **what is x?**

For comparison, here are some perfectly meaningful things you could have said:

1. Let $x = 9$. Then $x^2 + x = 90$.
2. Let $x = 6$. Then $x^2 + x = 90$.
3. Let x be the number of planets in the solar system. Then $x^2 + x = 90$.
4. $x^2 + x = 90$ for every real number x.
5. $x^2 + x = 90$ for some real number x.
6. $(\forall x \in \mathbb{R})(x^2 + x = 90)$.
7. $(\exists x \in \mathbb{R})(x^2 + x = 90)$.
8. The equation $x^2 + x = 90$ has a solution $x \in \mathbb{R}$. (Or "the equation $x^2 + x = 90$ has a real solution.")

Some of these are true, while the others are false. (Precisely: 1,3,5,7,8 are true; 2,4,6 are false. Moreover: 6 is just another way of saying the same thing as 4, and 7 and 8 are just alternative ways of stating 5.)

Notice that in 1, 2 and 3 the letter x has a value that was declared before. In 4, 5, 6, 7 and 8 the letter x is a dummy variable.

Again for comparison, let us now write some things that do **not** have a precise meaning:

A. The equation $x^2 = 2$ has a solution.

B. The equation $x^2 + 1 = 0$ has a solution.

C. The equation $3x = 5$ has a solution.

Why are these things bad? Because they do not specify the **range** of values within which the solution x is to be sought. For example, A is true if we look for solutions $x \in \mathbb{R}$, but false if we look for solutions $x \in \mathbb{Q}$. B is true if we look for solutions $x \in \mathbf{C}$ (the set of complex numbers) but false if we look for solutions $x \in \mathbb{R}$. C is true if we look for solutions $x \in \mathbb{Q}$ (or $x \in \mathbb{R}$) but false if we look for solutions $x \in \mathbb{Z}$.

On the other hand, the following are perfectly meaningful statements:

A1. The equation $x^2 = 2$ has a solution $x \in \mathbb{R}$. (That is: $(\exists x \in \mathbb{R})(x^2 = 2)$.)

A2. The equation $x^2 = 2$ has a solution $x \in \mathbb{Q}$. (That is: $(\exists x \in \mathbb{Q})(x^2 = 2)$.)

B1. The equation $x^2 + 1 = 0$ has a solution $x \in \mathbb{Z}$. (That is: $(\exists x \in \mathbb{Z})(x^2 + 1 = 0)$.)

B2. The equation $x^2+1=0$ has a solution $x \in \mathbb{R}$. (That is: $(\exists x \in \mathbb{R})(x^2+1=0)$.)

B3. The equation $x^2+1=0$ has a solution $x \in \mathbf{C}$. (That is: $(\exists x \in \mathbf{C})(x^2+1=0)$.)

C1. The equation $3x=5$ has a solution $x \in \mathbb{Z}$. (That is: $(\exists x \in \mathbb{Z})(3x=5)$.)

C2. The equation $3x=5$ has a solution $x \in \mathbb{Q}$. (That is: $(\exists x \in \mathbb{Q})(3x=5)$.)

C3. The equation $3x=5$ has a solution $x \in \mathbb{R}$. (That is: $(\exists x \in \mathbb{R})(3x=5)$.)

C4. The equation $3x=5$ has a solution $x \in \mathbf{C}$. (That is: $(\exists x \in \mathbf{C})(3x=5)$.)

(Actually, A1, B3, C2, C3 and C4 are true, whereas A2, B1, B2 and C1 are false. Of course, in each case you could say the same thing in lots of different ways. For example, instead of C2 you could say "the equation $3x=5$ has a rational solution," or "the equation $3x=5$ can be solved in the set of rational numbers," or "there is a rational number that satisfies the equation $3x=5$," or "$3x=5$ is solvable in the set of rational numbers.")

A sentence whose meaning is imprecise, because it can be interpreted in more than one way, is said to be *ambiguous*. For example, the sentences

Flying planes can be dangerous

College students are turning to vegetables

The University president promised to stop drinking on campus

are ambiguous. Why? Think of other examples of ambiguous sentences.

Similarly, A, B, C above are ambiguous. Another example of ambiguous sentence is a sentence where a letter does not have a declared value and is not a dummy variable either. For example, if I tell you to "read chapter x," then that's ambiguous —unless I have told you before what "x" stands for, or given you clues so you can find out, or perhaps I have said "read Chapter x for all x" (that is "read all the chapters") or "read Chapter x for some x" (that is "read one chapter")— because you don't know what x is supposed to be.

IMPORTANT RULE NO. 2

Avoid ambigous sentences!

IMPORTANT RULE NO. 3

Formulas without words are usually meaningless and should be avoided like the plague.

It's the **words** that glue the formulas together and give a structure and a meaning to mathematical writing. If you just write a bunch of formulas then you probably do not know what you are saying, and if you don't know then you should not expect that the instructor or the grader will know.

Look, for example, at Statements 1, 2, 4 and 5 above. In all cases, the "formula" $x^2 + x = 90$ is the same, but the statements are quite different and have totally different meanings. That's because the words are different!

The three "fundamental rules" can in fact be subsumed into a single principle:

BE PRECISE!

AN EXAMPLE

Here is an example of the kind of thing that you should *not* do when you write the solution of a homework problem.

QUESTION: describe the set $\{x \in \mathbb{R} : 1 < x^2 < 2\}$ as a union of intervals.

A TRULY AWFUL SOLUTION (*the final answer is correct, so this would get some partial credit, say 40%*):

$$1 < x^2 < 2$$
$$1 < x < \sqrt{2} \quad \text{or} \quad -\sqrt{2} < x < -1$$
$$A = (-\sqrt{2}, -1) \cup (1, \sqrt{2})$$

WHAT MAKES THIS BAD? Well, lots of things. For example:

- 1. To begin with, I do not see any *statements* here. Remember that a *statement* is supposed to be something that is either true or false. When the student writes the inequality "$1 < x^2 < 2$," what is the student asserting? What statement is the student making? (I know, you are probably thinking that what the student is saying is that "x^2 is between 1 and 2." But that does not do me any good. I still do not know what this means, because I have no idea what x is supposed to be. Do you? And please don't tell me "x is supposed to be a real number." What good does that do to me? I still don't know whether $1 < x^2 < 2$ is true or false, unless you tell me *which* number it is.) Is this true or is it false? If you cannot tell, it means that there is no statement there. (There is a standard convention used very often in mathematics: when you say something involving variables, and you don't specify the values of the variables or bound the variables with a quantifier, it means that you are saying that your formula is true for *all* values of the variables. In this case: could the student be trying to say that $1 < x^2 < 2$ is true for *all* values of x? I sure hope not! Because if that was it, then it would be **false**! For example, $1 < x^2 < 2$ is not true for $x = 3$.)
- 2. The letters x and A are used, but we are never told what they are supposed to stand for. (I know: you will tell me that A is the set $\{x \in \mathbb{R} : 1 < x^2 < 2\}$. But the student did not say that anywhere!)
- 3. Throughout this proposed "solution," there are no words explaining what is being done. And there is not a single punctuation mark!
- 4. There is nothing connecting each line to the next one.

Now, here is an example of a good solution of the same problem. (The words in italics are my own comments. They are not part of the student's solution.)

A GOOD SOLUTION (*which gets full credit*):
Let $A = \{x \in \mathbb{R} : 1 < x^2 < 2\}$. (*Comment: the student is announcing what the letter A is going to mean in this solution.*)
Let $x \in \mathbb{R}$. (*Comment: the student is announcing that x is going to stand for an arbitrary real number. This means that, from now on, whatever assertion the student makes about x should be*

true for **all** *real numbers x. For example, if in the next line the student said "$1 < x^2 < 2$," then that would be bad, because "$1 < x^2 < 2$" is* **not** *true for all x.)*
Then "$x \in A$" is equivalent to "$1 < x^2 < 2$." (*Comment: the student is* **not** *saying that $x \in A$, or that $1 < x^2 < 2$. He/she is just saying that those two statements are equivalent, that is, that they are either both true or both false. So the statement being made here is perfectly true.*)
And "$1 < x^2 < 2$" is equivalent to "either $1 < x < \sqrt{2}$ or $-\sqrt{2} < x < -1$." (*Comment: again, a true statement.*)
Therefore "$x \in A$" is equivalent to "either $x \in (1, \sqrt{2})$ or $x \in (-\sqrt{2}, -1)$." So, finally,

$$A = (1, \sqrt{2}) \cup (-\sqrt{2}, -1), \text{ and this is the desired answer.}$$

GENERAL COMMENT: notice the use of the words "so", "therefore," "then." These words give continuity to the whole thing. They tell you how each paragraph is related to the previous one. English is a very rich language. Take advantage of this to write well! Notice also the nice touch of putting the word "finally," to announce that we are getting to the end. Why not use "then" all the time? **Because it's nicer when you don't repeat the same words over and over again.** *It is not nice to give the impression that you have a very poor vocabulary.*

And here is another example of a correct solution to the same problem.

ANOTHER PERFECTLY FINE SOLUTION (*which also gets full credit*):

A real number belongs to the set under discussion if and only if the square of this number is larger than one and less than two. This will happen if either the number itself is strictly between one and the square root of two (*Comment: why "strictly"? Because I am saying that it cannot be equal to one or square root of two.*), or its negative is strictly between minus the square root of two and minus one. So our set is $(-\sqrt{2}, -1) \cup (1, \sqrt{2})$. *General comment: notice that the student did everything with words, without using any letter variables. Is that bad? Not at all! There is nothing wrong with using words rather than math symbols. The reason that we usually don't do it is that math symbols make it* **easier** *to say things, as we saw before.*

A FINAL COMMENT. The main point of mathematical writing is not the use of weird symbols. It is the use of **precise language.** You can be precise with words or with symbols. But symbols are just a way to say more concisely certain things that would take too long to say with words. As a rule, **if you say something with mathematical symbols, then you ought to know how to say exactly the same thing using only words.**