MATHEMATICS 300 — FALL 1998

Introduction to Mathematical Reasoning

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1. PROPOSITIONS, STATEMENTS NEGATIONS, CONTRADICTIONS.

Are there big mistakes in the book?

Here I want to discuss some issues on which I think that the book's authors really did a bad job and got things wrong. You should read what I say, read what the book says, **think**, and make up your own mind. If you want to defend the authors and tell me that I am wrong and the book is right, then please do so. **I really like it when people disagree with me**, as long as they do it with **good reasons.** If you tell me "I think the authors are right and you are wrong," then I will ask you why. If you say "I just feel it," then I will know that you have nothing really significant to say, and I will not be very interested in pursuing the conversation. But if you give me a **reason** then I will pay attention, and if your reason is a good one and you get me to change my mind, then I will acknowledge that I had been wrong.

Remember that **in this course nothing and nobody is sacred**. You should not put complete trust into **anything** or **anybody**, not even the authors of the book or the instructor. We all make mistakes. Nothing can be said to be right just "because the book says so," or "because the

teacher said so." **The book can be wrong; the teacher can be wrong, you can be wrong.** When there is a doubt, how do we decide who is right? By **thinking critically!**

What does "thinking critically" mean? It does **not** mean "refusing to accept anything that you cannot prove by yourself." It would be foolish to ask for that much. But it means "being on the lookout for things that could be wrong; questioning what you are told, and if it doesn't make sense to you then trying to settle the question until it is cleared, and doing it without preconceptions."

So let's get started, and do a bit of critical thinking about propositions, negations, and contradictions.

In my view, there is something quite problematic with the way the book defines "proposition" and "negation" in page 1, and then goes on to illustrate these notions with examples. I'll be quite frank and honest with you: **I think the authors really blew it here**.

Here is the problem: if the negation of P is "not P," then the negation of "7 is odd," if we take the authors' words seriously, should be "not 7 is odd." But that's not what the authors say! They say that the negation of "7 is odd" is "It is not the case that 7 is odd." *That does not look at all to me like the same thing!*

Can you think of a way to do this thing **better** than the authors, and to introduce the concepts of "proposition" and "negation" in such a way that this particular problem would be avoided? Don't worry too much if at this point you cannot find a truly satisfactory solution. This is a very delicate problem, and philosophers have devoted quite a lot of time and energy to it. We will come back to this issue later in class. But I want you to see that there is a real problem here, and I want you to understand that *the book can be wrong*, quite wrong in fact. And, maybe, if you think about this, you might be able to come up with a way to take care of this problem.

Here is a hint. Look at the following eight statements:

1. "I will be here tomorrow," asserted by me at home on August 28, 1998,
2. "I will be here tomorrow," asserted by my wife at home on August 28, 1998,
3. "I will be here tomorrow," asserted by me in my office on August 28, 1998,
4. "I will be here tomorrow," asserted by me at home on August 30, 1998,
5. "I was here yesterday," asserted by me at home on August 30, 1998,
6. "My husband will be here tomorrow," asserted by my wife at home on August 28, 1998,
7. "My husband was here yesterday," asserted by my wife at home on August 30, 1998,
8. "I was home yesterday," asserted by me at home on August 30, 1998.

Which of these would you like to regard as "asserting the same thing?" Would you say that 1 and 2, for example, assert the same thing? You wouldn't, would you? (Why do I say this?) Yet, the are the same statement. (Exactly the same words, not one iota of difference.) On the other hand, 1, 5, 6, 7 and 8 say the same thing, albeit in different words. Yet, they are quite different strings of words!

This observation is the key to the solution of our problem. Statements and propositions are two completely different kinds of beasts. A statement is a certain kind of string of words. A proposition is what a statement asserts, what the statement says is the case about the world. Different statements —for example, 1 and 6 in the above list— can assert the same proposition. The same statement can assert different propositions depending on many "contextual" facts: who is saying it, when, where. And there are all kinds of other ways in which a statement can vary but the proposition it asserts will stay the same. For example, "7 is not odd" and "It's not the case that 7 is odd" stand for the same proposition.

Contradictions. The book tells us that "a contradiction is the negation of a tautology." Now,

$$P \wedge (\sim P)$$

ought to be a contradiction. But $P \wedge (\sim P)$ is not the negation of anything! So this is not a good way to define "contradiction."

Here is a good way:

DEFINITION. A *contradiction* is a propositional form that has the truth value "False" for all possible assignments of truth values to its propositional variables.

For example: $P \wedge (\sim P)$ is a contradiction, because if P is true then $\sim P$ is false so $P \wedge (\sim P)$ is false, and if P is false then $P \wedge (\sim P)$ is false, so $P \wedge (\sim P)$ is false for all possible ways of assigning a truth value to the propositional variable P.

Remark. Notice that this definition of "contradiction" is exactly parallel to the definition of "tautology" given in the book (in page 7).

Remark. Notice that in the above definition of "contradiction," we used exactly the format for a definition that is discussed later on in this handout, in the section on definitions. In particular, the term being defined (the "definiendum"), which is this case is the word "contradiction," is highlighted.

Remark. Notice that the definitions of "tautology" and "contradiction" given in the book (in page 7) also have the right format.

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2. SENTENCES.

Sentences are strings of words that constitute a complete unit of thought. Sentences can be

- **interrogative** (that is, *questions*, for example, "Is 'Titanic' a good movie?"),
- **exclamatory** (for example, "Oh, no!"),
- **imperative** (that is, *commands* or *requests*, for example, "hand in the homework by Monday," or "go away," or "assume that n is a prime number"),

or

- **declarative** (for example, "Math 300 is the most fascinating course that has ever been taught.")

You should take the above explanation with a grain of salt, because the distinction is much more subtle. What makes a sentence belong to one of the above types is not its *meaning* but its *form.* For example: "could you pass me the salt?" is an *interrogative* sentence, even though it expresses a request and certainly does not ask a question. (It wouldn't be right for someone to answer "yes, I could," without lifting a finger to pass you the salt.) And "I would like to know what you think about this" is a **declarative** sentence, even though it is a way of asking a question.

The key point is this: **verbs** have **moods**, each of which has a most common use. For example, the **indicative** ("I know," "you think," "we eat," "they sleep") is the mood we use most of the time to *assert* that something is the case, that is, to make a *statement.* So a sentence whose main verb

is in the indicative mood is a *declarative sentence*. Also, a certain type of expression ("what time is it?", "do you like Indian food?") became the one commonly used to ask questions. But then, as language evolved, people developed all kinds of indirect and roundabout ways to say things. In particular, ways evolved to ask questions or issue commands using the indicative mood (for example: "I want to know what you think about Math 300" is really a way of asking the question, but "I want" is in the indicative mood, so this is a *declarative sentence*) or to issue commands or make requests in the form of a question (for example: "could you pass me the salt?").

What counts to determine the type of a sentence is its *form*. This is why the Concise Oxford Dictionary (8th Ed., Oxford Univ. Press, 1991) says, under "interrogative":

a. of or like a question; used in questions.

b. [Gram.] (of an adjective or pronoun) asking a question (e.g. who?, which?).

c. having the form or force of a question.

Notice the words "or like" and "having the form." The sentence "could you pass me the salt?" **has the form of a question**, so it is interrogative, even though it does not really ask a question.

Remark. The notion of **form** of a sentence or statement is very important and will show up many times later in the course. So *start paying attention to it!*.

Declarative sentences are those that assert that something is the case, and can therefore be true or false. For example, "Is 'Titanic' a good movie?" is a **question** (that is, an interrogative sentence), so it does not make sense to say that this sentence is true or false. On the other hand, " 'Titanic' is a good movie" is a declarative sentence.

Statements are declarative sentences that make clearly identifiable assertions that could be true or false. For example, if I tell you that "she is very smart," but I haven't told you who "she" is, then you will not know what I am talking about, so I haven't made a statement. Similarly, if I tell you "read page x of the book and tell me what it says," then you don't know what I am asking you to do, because I haven't told you what number "x" stands for. If I say "$x > 0$," but I haven't told you who "x" is, then you don't know what I am talking about, so I haven't made a statement.

When you write mathematics, you are trying to **communicate thoughts**, and the appropriate way to do that is to use **declarative sentences that make statements**. Of course, sometimes you may enhance the quality of your writing by including other types of sentences as well. (For example, before you explain how you justify a certain step in a proof, you could say "Why is this step valid? I am going to show you why." It doesn't hurt to say it, maybe it might help the reader follow your train of thought. So it's perfectly fine to say it. But keep in mind that when you ask a question you are not **asserting** anything, you are not making a statement, and the core of a mathematical text is the **statements** it makes.)

How can you tell if something is a statement? The first thing you should do is check whether or not what you wrote has a **verb** or something that plays the role of a verb. For example, "President Lincoln" is not a statement, because it does not have a verb or, if you prefer, because it does not *assert* anything, that is, because it does not make sense to say that it is true or false. On the other hand, if I write that "President Lincoln was assassinated," or that "President Lincoln was 93 years old when he died," or that "On August 5, 1862, between 6 a.m. and 8 a.m., President Lincoln scratched his head three times," these are all statements, because they assert things and you know perfectly well what it means for them to be true. (The first of these three statements is

true. The second one is false. As for the third one, I have no idea if it it true or false, and I bet you don't either, and we may never be able to find out, but we all know perfectly well what it means for such a sentence to be true.)

A second thing to check is whether all the "variables" have a clear meaning. By "variables" I mean here *words or phrases that refer to something that depends on the context.* For example:

- The pronouns ("I," "you," "he, "she," "we," "they") and possessive adjectives ("my," "your", "his," "her," etc.) are variables, in the sense that the entity that they refer to depends very much on who is talking and what has been said before. For example: the sentences "She is smart" and "Her grades are all A's" do not have a clear meaning unless you tell me who "she" is supposed to be.

- When adverbs or adverbial phrases expressing time ("today," "yesterday," "last week," "four score and seven years ago," etc.) are present, the assertion made by a sentence depends very much on *when* it was uttered. For example, if a magazine article says "Clinton was elected president two years ago," then you cannot decide if that is true or false unless you know when the article was published.

- When adverbs or adverbial phrases expressing place ("here," "there," etc.) are present, the assertion made by a sentence depends very much on *where* it was uttered. For example, if someone says "this is the richest nation on Earth," then you cannot decide if that is true or false unless you know which country the speaker is talking about.

- In mathematics, when letters $(x, y, z, a, b,$ etc.) are present in a sentence, then the assertion being made depends very much on *what the letter stands for.* You cannot decide if the sentence is true or false unless you know this. For example, if I write

  $$x^2 + 3x < 28\,,$$

  just like that, then there is no way to tell if this is true or false unless you know what "x" stands for.

  If I had written, earlier on, something like "Let $x = 5$," then the sentence $x^2 + 3x < 28$ would be making a definite statement, that could be true or false (and, as it happens, would be false in this case).

- Now, what are we to make of "statements" such as

  $$(a + b)^2 = a^2 + 2ab + b^2\,, \tag{1}$$

  and others like it? You probably learned somewhere that this is a "true formula." Now it would seem that, if you take seriously what I have been saying, then (1) is not a statement, because we have not said exactly what it is that "a" and "b" stand for!

  The answer is:

  When we write a formula such as $(a + b)^2 = a^2 + 2ab + b^2$*, usually what we are really trying to say is*

  > For every value of a and b, it is true that $(a + b)^2 = a^2 + 2ab + b^2$.

  or, if you prefer,

If you take any two numbers, add them, and square the sum, then you get the same as when you square each of the numbers, add the two squares, and add to the resulting sum twice the product of the numbers.

Once you say it that way, you see that in this sentence "a" and "b" are not supposed to have specific values, because what we are saying is that the formula is true for **all** *values.*

Really, you should think of "a" and "b" as abbreviated ways to say "the first number" and "the second number." Any other two letters (or other symbols) would do. For example, if you write

$$(a+b)^2 = a^2 + 2ab + b^2 \text{ for all } a, b\,,$$

or

$$(x+y)^2 = x^2 + 2xy + y^2 \text{ for all } x, y\,,$$

or

$$(x_1+x_2)^2 = x_1^2 + 2x_1x_2 + x_2^2 \text{ for all } x_1, x_2\,,$$

or

$$(\varepsilon+\delta)^2 = \varepsilon^2 + 2\varepsilon\delta + \delta^2 \text{ for all } \varepsilon, \delta\,,$$

or

$$(\alpha+\beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2 \text{ for all } \alpha, \beta\,,$$

or

$$(\aleph_1+\aleph_2)^2 = \aleph_1^2 + 2\aleph_1\aleph_2 + \aleph_2^2 \text{ for all } \aleph_1, \aleph_2\,,$$

or

$$(\diamond+\nabla)^2 = \diamond^2 + 2\diamond\nabla + \nabla^2 \text{ for all } \diamond, \nabla\,,$$

or

$$(\diamond+x)^2 = \diamond^2 + 2\diamond x + x^2 \text{ for all } \diamond, x\,,$$

or

$$(Peter+Paul)^2 = Peter^2 + 2\,Peter\,Paul + Paul^2 \text{ for all } Peter, Paul\,,$$

then these are all different ways of saying exactly the same thing.
The "variables" such as a, b, x, y, etc. do not have a *value* that makes sense outside the statement itself. They are **dummy variables**.

- There will be a lot more discussion of sentences and statements and variables and dummy variables later on.

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3. ON DEFINITIONS.

If you are asked to "define" something, then you are expected to *explain what it means in such a way that a person who does not know will be able to find out from your definition.* For example,

Elephants have four legs.

is not an acceptable definition of "elephant," because if I do not know what an elephant is then this definition will not tell me.

Similarly,

An even number is a number like 2, 4, 6, etc.

is not an acceptable definition of "even number," because if I do not know what an even number is then this definition will not tell me. (The crux of the matter here is the word "like." What does it mean for a number to be "like" 2, 4, 6, etc.? Could it mean "being a one-digit number"? If this was it, then 5 and 7 would also be "even," but 12 would not be. Surely that can't be what the author had in mind. But: *how do you know that this is not what the author had in mind?* You know it because *you already know what an even number is.*. If you didn't know, then you would not be able to figure it out from what the author wrote. And that's precisely why this purported definition of "even number" is wrong.)

The number of arguments. One of the first things you have to do to give a definition of something is to figure out on *how many arguments* the term you are defining depends. For example: if I ask you to define "elephant" then you have to explain to me what it means for something —that is, for *one* object—- to be an elephant: "an *elephant* is a huge grey four-legged animal that has a trunk, enormous ears and tusks." But if I ask you to define "cousin," then that kind of thing will NOT do. You don't define what it means for someone to be a *cousin*. You define what it means for someone to be *somebody else's cousin*. So in your definition you have to talk about *two* objects. For example, you could say: "a person is a *cousin* of another person if one of the parents of the first person is a brother or sister of one of the parents of the other person." Later we will be more formal, and we will say that "elephant" is a *one-variable attribute* and "cousin" is a *two-variable attribute*. The "variables" are letters that one uses in order to avoid the clumsy repetition of complicated ways of designating the arguments. For example, here is how you could define "cousin" using letter variables: "If x and y are persons, we say that x and y are *cousins* if one of x's parents is a brother or sister of one of y's parents." See how the use of letters makes things easier?

The form of a definition. A definition should be stated as a *complete sentence*. If I ask you "what is a pig?" then I want you to write something like

> A *pig* is an omnivorous hoofed mammal, usually domesticated, that has a long broad snout and a thick, fat body covered with coarse bristles.

Similarly, if I ask you "what is a rational number?" then I expect you to write something like

> A *rational number* is a real number which is the quotient of two integers.

In both cases, the **form** of the definition is very much the same. Notice that

a. the definitions are formulated as complete sentences;

b. in each case, the term being defined ("pig," "rational number") is **highlighted**; I did it here by using *italic*. I could have used **boldface** instead, or underlining. For you, in your homework assignments and quizzes and midterm and final exams, since you are not using fancy word-processing, underlining is obviously the best way.

Remark. The term being defined ("pig," "rational number") is called the *definiendum*.

Remark. Notice that the previous remark is a definition of the word "definiendum." Do you now understand why the word is highlighted?

4. SOME REMARKS ON MATHEMATICAL WRITING

ON THE USE OF LETTERS:

Letter variables are only used in mathematics in order to shorten certain statements that would otherwise be too long.

EXAMPLE 1: Suppose you want to say

> If you take two real numbers, add them, and square the sum, then you get the same result as if you had squared each number separately, added the squares, and added to the result twice the product of the two numbers.

This looks awfully complicated and ugly. You can make it much shorter by saying

> For all real numbers x, y, $(x+y)^2 = x^2 + y^2 + 2xy$.

(Later, we, will learn to say this even more concisely, by writing

$$(\forall x \in \mathbf{R})(\forall y \in \mathbf{R})\Big((x+y)^2 = x^2 + y^2 + 2xy\Big) .$$

Here "$(\forall x \in \mathbf{R})(\forall y \in \mathbf{R})$" is an abbreviation of "for all real numbers x, y.")

EXAMPLE 2: Suppose you want to say

> It is possible to find a real number whose cube, added to its square, is equal to the number itself.

Again, this looks awfully complicated and ugly, and you can make it much shorter by saying

> There exists a real number x such that $x^2 + x^3 = x$,

or

> $x^2 + x^3 = x$ for some real number x.

(Later, we, will learn to say this thing even more concisely, by writing

$$(\exists x \in \mathbf{R})\Big(x^2 + x^3 = x\Big) .$$

Here "$(\exists x \in \mathbf{R})$" is an abbreviation of "for some real number x.")

EXAMPLE 3: suppose you want to say

> For each integer between 2 and 375, take that integer, square it, add 5, and compute the multiplicative inverse of the result. Having done this for all the integers between 2 and 375 (including 2 and 375 themselves), add all the resulting numbers. Then the number you get is smaller than 60.

Once again, this is awful, so you can shorten the whole thing and just say

> Let $N = \sum_{k=2}^{375} \frac{1}{k^2+5}$. Then $N < 60$.

Or, if you prefer, you could just say:

> $\sum_{k=2}^{375} \frac{1}{k^2+5} < 60$.

EXAMPLE 4: Suppose you want to say

> If you take the function that to each real number assigns its cube, and integrate it from 0 to 4, then the result is 64.

Again, this can be made much shorter by saying

> Let $N = \int_0^4 x^3\,dx$. Then $N = 64$.

Or you could just say:

> Let $\int_0^4 x^3\,dx = 64$.

EXAMPLE 5: Suppose I want to say:

> The set of all real numbers whose square is larger than the cube of the number has at least two different members.

We can make this shorter by saying

> Let $S = \{x \in \mathbf{R} : x^2 > x^3\}$. Then S has at least two members.

Or you could just say:

> Let $S = \{x \in \mathbf{R} : x^2 > x^3\}$. Then $(\exists x)(\exists y)\Big(x \in S, y \in S, \text{ and } x \neq y\Big)$.

EXAMPLE 6: Suppose I want to say that "I am older than 20 and younger than 70, and my brother is younger than I am, but his age is larger than half my age."

Here too we can be more concise, and say

> Let x be my age and let y be my brother's age. Then $20 < x < 70$ and $\frac{x}{2} < y < x$.

In all these cases, we have used **letters** (namely, x, y, k, N, S) as labels, so that we can avoid writing clumsy phrases such as "the number that we have been talking about."

In all the above examples, the statements we wrote using letters only contain the letters as **dummy variables**. The meaning of this is quite simple: the letter is just a label used to keep track of a number (or set, in Example 6) that occurs several times.

The distinctive feature of a dummy variable is that you could have used any other symbol instead, and you would be saying exactly the same thing. For example, if you write

> $x^2 \geq 0$ for all real numbers x,

then that is exactly the same thing as you had written

> $y^2 \geq 0$ for all real numbers y,

or

> $a^2 \geq 0$ for all real numbers a,

or

> $b^2 \geq 0$ for all real numbers b,

or

$\alpha^2 \geq 0$ for all real numbers α,

or

$m^2 \geq 0$ for all real numbers m,

or

$n^2 \geq 0$ for all real numbers n,

or

$\varepsilon^2 \geq 0$ for all real numbers ε,

or

$\aleph^2 \geq 0$ for all real numbers $\aleph$,

or

$\diamond^2 \geq 0$ for all real numbers $\diamond$,

or

$\nabla^2 \geq 0$ for all real numbers ∇,

or

$g^2 \geq 0$ for all real numbers g,

or

$A^2 \geq 0$ for all real numbers A.

These are just alternative ways of saying

The square of any real number is greater than or equal to zero.

Notice that when you say it that way there is no letter variable there.

Similarly, instead of

there exists a real number x such that $x = x^2 + x^3$,

you could have said

there exists a real number z such that $z = z^2 + z^3$,

or

there exists a real number a such that $a = a^2 + a^3$,

or

there exists a real number ω such that $\omega = \omega^2 + \omega^3$,

or

there exists a real number $\aleph$ such that $\aleph = \aleph^2 + \aleph^3$,

or

there exists a real number $\diamond$ such that $\diamond = \diamond^2 + \diamond^3$,

These are just alternative ways of saying

There exists a real number which is equal to the sume of its square and its cube.

Notice that when you say it that way there is no letter variable there.

Just one more example:

Instead of

$$\sum_{k=2}^{375} \frac{1}{k^2+5} < 60\,,$$

you could have said

$$\sum_{n=2}^{375} \frac{1}{n^2+5} < 60\,,$$

or

$$\sum_{\rho=2}^{375} \frac{1}{\rho^2+5} < 60\,.$$

These are just alternative ways of saying

For each integer between 2 and 375, take that integer, square it, add 5, and compute the multiplicative inverse of the result. Having done this for all the integers between 2 and 375 (including 2 and 375 themselves), add all the resulting numbers. Then the number you get is smaller than 60.

Notice that when you say it that way there is no letter variable there.

The "dummy variables" in the above examples are really just **internal tags**, using to keep track of numbers (or other things, such as statements or sets) **within a particular text**. Outside this text, the variables do not have any meaning. For example:

- After you have written

  $a^2 \geq 0$ for all real numbers a,

  it would not make sense for me to ask you "what is the value of a?" The letter a is just a tag to specify a *range of values* ("all real numbers"). Any other letter could have been used instead, so in fact "a isn't really there."

- After you have written

  there exists a real number $\aleph$ such that $\aleph = \aleph^2 + \aleph^3$,

  it would not make sense for me to ask you "what is the value of $\aleph$?" The letter $\aleph$ is just a tag to specify a *range of values* ("all real numbers") within which the solution of our equation is to be sought. Any other letter could have been used instead, so in fact "$\aleph$ isn't really there."

- After you have written

$$\sum_{k=2}^{375} \frac{1}{k^2+5} < 60\,,$$

it would not make sense for me to ask you "what is the value of k?" The letter k is just a tag to specify a *range of values*, as part of a set of instructions on how to compute a sum. Any other letter could have been used instead, so in fact "k isn't really there."

This way of speaking and writing, using dummy variables, is very useful, especially when what we are trying to say is quite long. But **you have to be careful** when writing this way, and have to make sure that you are using the letters correctly.

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5. THE FIRST FUNDAMENTAL RULE

A letter symbol can only be used if either its value has been declared before or it is a dummy variable.

For example: if you write

$$x^2 + x = 90$$

just like that, without any words before or after explaining what x is, or making it a dummy variable, then what you wrote is **meaningless**. We have no way to tell if what you wrote is true or false, because we have no idea what you are talking about.

If you write this, then I will write in red **what is x?**

For comparison, here are some perfectly meaningful things you could have said:

1. Let $x = 9$. Then $x^2 + x = 90$.
2. Let $x = 6$. Then $x^2 + x = 90$.
3. Let x be the number of planets in the solar system. Then $x^2 + x = 90$.
4. $x^2 + x = 90$ for every real number x.
5. $x^2 + x = 90$ for some real number x.
6. $(\forall x \in \mathbf{R})(x^2 + x = 90)$.
7. $(\exists x \in \mathbf{R})(x^2 + x = 90)$.
8. The equation $x^2 + x = 90$ has a solution $x \in \mathbf{R}$. (Or "the equation $x^2 + x = 90$ has a real solution.")

Some of these are true, while the others are false. (Precisely: 1,3,5,7,8 are true; 2,4,6 are false. Moreover: 6 is just another way of saying the same thing as 4, and 7 and 8 are just alternative ways of stating 5.)

Notice that in 1, 2 and 3 the letter x has a value that was declared before. In 4, 5, 6, 7 and 8 the letter x is a dummy variable.

Again for comparison, let us now write some things that do **not** have a precise meaning:

A. The equation $x^2 = 2$ has a solution.

B. The equation $x^2 + 1 = 0$ has a solution.

C. The equation $3x = 5$ has a solution.

Why are these things bad? Because they do not specify the **range** of values within which the solution x is to be sought. For example, A is true if we look for solutions $x \in \mathbf{R}$, but false if we look for solutions $x \in \mathbf{Q}$. B is true if we look for solutions $x \in \mathbf{C}$ (the set of complex numbers) but false if we look for solutions $x \in \mathbf{R}$. C is true if we look for solutions $x \in \mathbf{Q}$ (or $x \in \mathbf{R}$) but false if we look for solutions $x \in \mathbf{Z}$.

On the other hand, the following are perfectly meaningful statements:

A1. The equation $x^2 = 2$ has a solution $x \in \mathbf{R}$. (That is: $(\exists x \in \mathbf{R})(x^2 = 2)$.)

A2. The equation $x^2 = 2$ has a solution $x \in \mathbf{Q}$. (That is: $(\exists x \in \mathbf{Q})(x^2 = 2)$.)

B1. The equation $x^2 + 1 = 0$ has a solution $x \in \mathbf{Z}$. (That is: $(\exists x \in \mathbf{Z})(x^2 + 1 = 0)$.)

B2. The equation $x^2 + 1 = 0$ has a solution $x \in \mathbf{R}$. (That is: $(\exists x \in \mathbf{R})(x^2 + 1 = 0)$.)

B3. The equation $x^2 + 1 = 0$ has a solution $x \in \mathbf{C}$. (That is: $(\exists x \in \mathbf{C})(x^2 + 1 = 0)$.)

C1. The equation $3x = 5$ has a solution $x \in \mathbf{Z}$. (That is: $(\exists x \in \mathbf{Z})(3x = 5)$.)

C2. The equation $3x = 5$ has a solution $x \in \mathbf{Q}$. (That is: $(\exists x \in \mathbf{Q})(3x = 5)$.)

C3. The equation $3x = 5$ has a solution $x \in \mathbf{R}$. (That is: $(\exists x \in \mathbf{R})(3x = 5)$.)

C4. The equation $3x = 5$ has a solution $x \in \mathbf{C}$. (That is: $(\exists x \in \mathbf{C})(3x = 5)$.)

(Actually, A1, B3, C2, C3 and C4 are true, whereas A2, B1, B2 and C1 are false. Of course, in each case you could say the same thing in lots of different ways. For example, instead of C2 you could say "the equation $3x = 5$ has a rational solution," or "the equation $3x = 5$ can be solved in the set of rational numbers," or "there is a rational number that satisfies the equation $3x = 5$," or "$3x = 5$ is solvable in the set of rational numbers.")

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6. AMBIGUOUS SENTENCES AND THE SECOND FUNDAMENTAL RULE

A sentence whose meaning is imprecise, because it can be interpreted in more than one way, is said to be *ambiguous*. For example, the sentences

Flying planes can be dangerous

College students are turning to vegetables

The University president promised to stop drinking on campus

are ambiguous. Why? Think of other examples of ambiguous sentences.

Similarly, A, B, C above are ambiguous. Another example of ambiguous sentence is a sentence where a letter does not have a declared value and is not a dummy variable either. For example, if I tell you to "read chapter x," then that's ambiguous —unless I have told you before what "x" stands for, or given you clues so you can find out, or perhaps I have said "read Chapter x for all x" (that is "read all the chapters") or "read Chapter x for some x" (that is "read one chapter")— because you don't know what x is supposed to be.

THE SECOND FUNDAMENTAL RULE

Avoid ambiguous sentences!

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7. THE THIRD FUNDAMENTAL RULE

Formulas without words are usually meaningless and should be avoided like the plague.

Words are what glues the formulas together and gives structure and meaning to mathematical writing. If you just write a bunch of formulas, then you are not communicating complete thoughts, you are not making statements, and you should not expect that the instructor or the grader will understand you, or that they will give you the benefit of the doubt if they do not.

Look, for example, at Statements 1, 2, 4 and 5 above. In all cases, the "formula" $x^2 + x = 90$ is the same, but the statements are quite different and have totally different meanings. That's because the words are different!

The three "fundamental rules" can in fact be subsumed into a single principle:

BE PRECISE!

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8. HOW NOT TO WRITE: AN EXAMPLE

Here is an example of the kind of thing that you should *not* do when you write the solution of a homework problem.

QUESTION: describe the set $\{x \in \mathbf{R} : 1 < x^2 < 2\}$ as a union of intervals.

A TRULY AWFUL SOLUTION (*the final answer is correct, so this would get some partial credit, say 40%*):

$$1 < x^2 < 2$$
$$1 < x < \sqrt{2} \quad \text{or} \quad -\sqrt{2} < x < -1$$
$$A = (-\sqrt{2}, -1) \cup (1, \sqrt{2})$$

WHAT MAKES THIS BAD? Well, lots of things. For example:

1. To begin with, I do not see any *statements* here. Remember that a *statement* is supposed to be something that is either true or false. When the student writes the inequality "$1 < x^2 < 2$," what is the student asserting? What statement is the student making? (I know, you are probably thinking that what the student is saying is that "x^2 is between 1 and 2." But that does not do me any good. I still do not know what this means, because I have no idea what x is supposed to be. Do you? And please don't tell me "x is supposed to be a real number." What good does that do to me? I still don't know whether $1 < x^2 < 2$ is true or false, unless you tell me *which* number it is.) Is this true or is it false? If you cannot tell, it means that there is no statement there. (There is a standard convention used very often in mathematics: when you say something involving variables, and you don't specify the values of the variables or bound the variables with a quantifier, it means that you are saying that your formula is true for *all* values of the variables. In this case: could the student be trying to say that $1 < x^2 < 2$ is true for *all* values of x? I sure hope not! Because if that was it, then it would be **false**! For example, $1 < x^2 < 2$ is not true for $x = 3$.)
2. The letters x and A are used, but we are never told what they are supposed to stand for. (I know: you will tell me that A is the set $\{x \in \mathbf{R} : 1 < x^2 < 2\}$. But the student did not say that anywhere!)
3. Throughout this proposed "solution," there are no words explaining what is being done. And there is not a single punctuation mark!
4. There is nothing connecting each line to the next one.

Now, here is an example of a good solution of the same problem. (The words in italics are my own comments. They are not part of the student's solution.)

A GOOD SOLUTION (*which gets full credit*):
Let $A = \{x \in \mathbf{R} : 1 < x^2 < 2\}$. (*Comment: the student is announcing what the letter A is going to mean in this solution.*)
Let $x \in \mathbf{R}$. (*Comment: the student is announcing that x is going to stand for an arbitrary real number. This means that, from now on, whatever assertion the student makes about x should be true for* **all** *real numbers x. For example, if in the next line the student said "$1 < x^2 < 2$," then that would be bad, because "$1 < x^2 < 2$" is* **not** *true for all x.*)
Then "$x \in A$" is equivalent to "$1 < x^2 < 2$." (*Comment: the student is* **not** *saying that $x \in A$, or that $1 < x^2 < 2$. He/she is just saying that those two statements are equivalent, that is, that they are either both true or both false. So the statement being made here is perfectly true.* **Another point***: it is not good writing to start a sentence with a formula. This is one of the reasons for having the word "then" there. The other reason for having the word "then" is that it serves to link this sentence with what was said before. We are trying to say that, if we pick any real number x, then "$x \in A$" is equivalent to "$1 < x^2 < 2$."*)
And "$1 < x^2 < 2$" is equivalent to "either $1 < x < \sqrt{2}$ or $-\sqrt{2} < x < -1$." (*Comment: again, a true statement.*)
Therefore "$x \in A$" is equivalent to "either $x \in (1, \sqrt{2})$ or $x \in (-\sqrt{2}, -1)$." So, finally,

$$A = (1, \sqrt{2}) \cup (-\sqrt{2}, -1), \text{ and this is the desired answer.}$$

(*Comment: Notice that at the end the author is announcing to the reader that "of all the things we have done, here, finally, is the answer."*)

GENERAL COMMENT: notice the use of the words "so", "therefore," "then." These words give continuity to the whole thing. They tell you how each paragraph is related to the previous one. English is a very rich language. Take advantage of this to write well! Notice also the nice touch of putting the word "finally," to announce that we are getting to the end. Why not use "then" all the time? **Because it's nicer when you don't repeat the same words over and over again.** *It is not nice to give the impression that you have a very poor vocabulary.*

Now, here is the same perfectly correct solution, without the superfluous comments:

A GOOD SOLUTION:
Let $A = \{x \in \mathbf{R} : 1 < x^2 < 2\}$. Let $x \in \mathbf{R}$. Then "$x \in A$" is equivalent to "$1 < x^2 < 2$." And "$1 < x^2 < 2$" is equivalent to "either $1 < x < \sqrt{2}$ or $-\sqrt{2} < x < -1$." Therefore "$x \in A$" is equivalent to "either $x \in (1, \sqrt{2})$ or $x \in (-\sqrt{2}, -1)$."

So, finally,

$$A = (1, \sqrt{2}) \cup (-\sqrt{2}, -1)\,,$$

and this is the desired answer.

And here is another example of a correct solution to the same problem.

ANOTHER PERFECTLY FINE SOLUTION (*which also gets full credit*):

A real number belongs to the set under discussion if and only if the square of this number is larger than one and less than two. This will happen if either the number itself is strictly between one and the square root of two (*Comment: why "strictly"? Because I am saying that it cannot be equal to one or square root of two.*), or its negative is strictly between minus the square root of two and minus one. So our set is $(-\sqrt{2}, -1) \cup (1, \sqrt{2})$. *General comment: notice that the student did everything with words, without using any letter variables. Is that bad? Not at all! There is nothing wrong with using words rather than math symbols. The reason that we usually don't do it is that math symbols make it* **easier** *to say things, as we saw before.*

A FINAL COMMENT. The main point of mathematical writing is not the use of weird symbols. It is the use of **precise language.** You can be precise with words or with symbols. But symbols are just a way to say more concisely certain things that would take too long to say with words. As a rule, **if you say something with mathematical symbols, then you ought to know how to say exactly the same thing using only words.**

END OF HANDOUT NO. 2