MATHEMATICS 300 — SPRING 2000

Introduction to Mathematical Reasoning

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HOMEWORK ASSIGNMENT NO. 3, DUE ON THURSDAY, FEBRUARY 10

1. Make a precise statement describing which natural numbers can be expressed as the difference of the squares of two natural numbers, and then prove your statement. (In case it is not clear to you what I mean by "make a precise statement," here is an example. Suppose I asked you to "make a precise statement describing which natural numbers can be expressed as the sum of two odd natural numbers." Then a good answer could be:

   – *the natural numbers that can be expressed as the sum of two odd natural numbers are precisely the even natural numbers*,

   or, if you prefer to say it more formally:

   – *let n be a natural number; then n is expressible as a sum of two odd natural numbers if and only if n is even.*

   Just to make sure that we understand each other, here are examples of answer that I would **not** consider satisfactory:

   – *A natural number is expressible as a sum of two odd numbers if and only if n is even.*
   – *If a natural number n is expressible as a sum of two odd numbers then n is even.*

   QUESTION: why are these two answers not right?)

2. Answer the question at the end of the statement of the previous problem.
3. Prove: the product of 5 consecutive natural numbers is always divisible by 120.
4. Problems A and B in page 6 of this handout.

PROPOSITIONAL CALCULI[1]

To specify precisely what a **propositional calculus** is, we have to begin by saying what are the **basic symbols**.

1. THE BASIC SYMBOLS OF A PROPOSITIONAL CALCULUS

A propositional calculus uses **three** kinds of basic symbols:

1. the *propositional variables* (usually letters such as P, Q, R, S),
2. the *logical connectives* $\vee$, $\wedge$, $\longrightarrow$, $\longleftrightarrow$, $\sim$,
3. the *parentheses* (and) .

[1] "Calculi" is the plural of "Calculus."

The logical connectives have **names**:

- " $\wedge$ " is read as "and," and is called "and" or "conjunction;"
- " $\vee$ " is read as "or," and is called "or" or "disjunction;"
- " $\sim$ " is read as "no," or "not," or "it's not the case that," and is called "no" or "not" or "negation;"
- " $\longrightarrow$ " is read as "implies," and is called "implication;"
- " $\longleftrightarrow$ " is read as "if and only if," and is called "biconditional."

The previous list of various ways of reading the logical connectives is not complete. There are many other ways. For example,

- $P \longrightarrow Q$ can be read as "P implies Q," or "if P then Q," or "Q if P."
- $P \longleftrightarrow Q$ can be read as "P if and only if Q," or "P is equivalent to Q."

2. WELL-FORMED FORMULAS

A propositional calculus is a *language*, exactly as English or French are languages. In English or French, there are basic symbols (the letters, the blank space, the punctuation marks), but not every string of such symbols is acceptable or meaningful. In English, for example

- "gfeqpiqew" is meaningless;
- "I am the greatest teacher in the whole world" is meaningful (and obviously true);
- "the square invocation sleeps furiously but no so yesterday why I say" is sort of borderline. (What do you think?)

One thing that distinguishes propositional calculi from "natural languages" such as English is that in a propositional calculus there is a very precise way to tell the "good" strings apart from the "bad" ones. **There are no borderline cases.**

The "good" (or "acceptable," or "meaningful") strings of symbols of a propositional calculus are called **well-formed formulas**[2].

The well-formed formulas are those strings of symbols that can be obtained by applying certain **rules**:

[2] "Well-formed formula" is a generic expression to refer to the "acceptable" strings of any formal language. Sometimes, in a particular formal language, the well-formed formulas of that language have a special name. For example, the well-formed formulas of a propositional calculus are called "propositional forms."

PRECISE DEFINITION OF "PROPOSITIONAL FORM"

Propositional forms are the well-formed formulas of a given propositional calculus. The rules determining which strings are well formed are as follows:

RULE PF1: One-letter strings consisting of a propositional variable are well-formed formulas.

RULE PF2: If A is a well-formed formula consisting of a single letter then $\sim A$ is a well-formed formula.

RULE PF3: If A is a well-formed formula consisting of more than a single letter then $\sim (A)$ is a well-formed formula.

RULE PF4: If A and B are well-formed formulas consisting of a single letter then $A \wedge B$, $A \vee B$, $A \longrightarrow B$ and $A \longleftrightarrow B$ are well-formed formulas.

RULE PF5: If A and B are well-formed formulas such that A consists of a single letter but B does not, then $A \wedge (B)$, $A \vee (B)$, $A \longrightarrow (B)$ and $A \longrightarrow (B)$ are well-formed formulas.

RULE PF6: If A and B are well-formed formulas such that B consists of a single letter but A does not, then $(A) \wedge B$, $(A) \vee B$, $(A) \longrightarrow B$ and $(A) \longrightarrow B$ are well-formed formulas.

RULE PF7: If A and B are well-formed formulas neither one of which consists of a single letter, then $(A) \wedge (B)$, $(A) \vee (B)$, $(A) \longrightarrow (B)$ and $(A) \longrightarrow (B)$ are well-formed formulas.

RULE PF8: Only those strings obtained by applying Rules PF1, PF2, PF3, PF4, PF5, PF6, and PF7 are well-formed formulas.

Here are some examples of propositional forms (assuming that the letters P, Q, R are propositional variables):

- P
- Q
- $\sim P$
- $(\sim P) \wedge Q$
- $\sim ((\sim P) \wedge Q)$
- $(P \vee R) \longrightarrow (\sim ((\sim P) \wedge Q))$.

And now let us look at a more complicated example of a propositional form:

AN EXAMPLE OF THE USE OF RULES PF1 TO PF9

Let us look at the string

$$(\sim Q) \longrightarrow (\sim (\sim (P \wedge (((\sim (\sim Q)) \vee R) \longleftrightarrow (R \longrightarrow (P \wedge Q)))))) \ .$$

Is this a well-formed formula? The answer is YES. To prove this, we exhibit a "derivation" of our string from the rules (using "WFF" to stand for "well-formed formula");

STEP 1: P is a WFF by Rule PF1.
STEP 2: Q is a WFF by Rule PF1.
STEP 3: $\sim Q$ is a WFF by Rule PF2.
STEP 4: $\sim (\sim Q)$ is a WFF by Rule PF3.
STEP 5: $P \wedge Q$ is a WFF by Rule PF4.
STEP 6: R is a WFF by Rule PF1.
STEP 7: $R \longrightarrow (P \wedge Q)$ is a WFF by Rule PF5.
STEP 8: $(\sim (\sim Q)) \vee R$ is a WFF by Rule PF6.
STEP 9: $((\sim (\sim Q)) \vee R) \longleftrightarrow (R \longrightarrow (P \wedge Q))$ is a WFF by Rule PF6.
STEP 10: $P \wedge (((\sim (\sim Q)) \vee R) \longleftrightarrow (R \longrightarrow (P \wedge Q)))$ is a WFF by Rule PF5.
STEP 11: $\sim (P \wedge (((\sim (\sim Q)) \vee R) \longleftrightarrow (R \longrightarrow (P \wedge Q))))$ is a WFF by Rule PF3.
STEP 12: $\sim (\sim (P \wedge (((\sim (\sim Q)) \vee R) \longleftrightarrow (R \longrightarrow (P \wedge Q)))))$ is a WFF by Rule PF3.
STEP 13: $(\sim Q) \longrightarrow (\sim (\sim (P \wedge (((\sim (\sim Q)) \vee R) \longleftrightarrow (R \longrightarrow (P \wedge Q))))))$ is a WFF

by Rule PF7.

QUESTION 1: Is $((P \wedge Q) \longrightarrow Q$ is well-formed?

ANSWER: NO. It is a good exercise to try to understand *why*, that is,

How on Earth can anyone ever possibly know for sure that the string $((P \wedge Q) \longrightarrow Q$ is not well-formed?

First of all, you could try to produce $((P \wedge Q) \longrightarrow Q$ using the three rules, and no matter how hard you try you will not be able to. But,

Just because you have not been able to do something it does not follow that it cannot be done. (Maybe God can say: "if I cannot do something then it sure means it cannot be done." But certainly no limited, fallible, mortal human being would be entitled to make such a claim.)

NOTE: In Homework assignment No. 1 some students, in order to prove that the number 30 cannot be expressed as the difference of the squares of two natural numbers, wrote things like "It cannot be done; reason: I tried several ways and it did not work." Could those students really have meant what they said? If you are one of them, please think this over and do some soul-searching. THIS IS IMPORTANT!

So, how do we know that $((P \wedge Q) \longrightarrow Q$ **cannot possibly** be obtained using the rules? We need an *argument*. We need *reasoning*. We need a **proof.**

Here is a real proof. First, we prove a small theorem.

THEOREM 1. *In every well-formed formula the number of left parentheses is equal to the number of right parentheses.*

PROOF. Call a string "balanced" if it has the same number of left and right parentheses. Any string obtained using Rule 1 is balanced, because it has zero left parentheses and zero right parentheses. An application of Rules PF2 or PF4 will not produce any new parentheses, so if the strings to which you applied these rules were balanced then the new string will also be balanced. An application of Rules PF3, PF5, or PF6 will produce two new parentheses, one left and one right. So, once again, if the strings you applied these rules to were balanced then the new string will also be balanced. Finally, if you apply Rules PF7 then you will produce four new parentheses, two left and two right, and then in this case too if the strings to which you applied the rule were balanced then the new string will also be balanced. So the application of Rules PF1 to PF7 can never yield unbalanced strings. END OF PROOF.

PROOF THAT THE ANSWER TO QUESTION 1 IS "NO": The string $((P \wedge Q) \longrightarrow Q$ has two left parentheses and one right parenthesis. So Theorem 1 tells us that it is not well-formed. END OF PROOF.

QUESTION 2: Is the following "proof that $((P \wedge Q) \longrightarrow Q$ is not well-formed" correct?

PURPORTED PROOF THAT THE ANSWER TO QUESTION 1 IS "NO": The string $((P \wedge Q) \longrightarrow Q$ has two left parentheses and one right parenthesis. So $((P \wedge Q) \longrightarrow Q$ is not balanced. But Theorem 1 tells us that a well-formed string has to be balanced. So $((P \wedge Q) \longrightarrow Q$ is not well-formed. END OF PROOF.

ANSWER: NOT REALLY[3]. If you think that this proof is correct, then please think some more before you read the next paragraph.

Why is this "proof" not really correct? What is the problem? The problem is this: when you look at the statement of Theorem 1, you will notice that it does not say anything about strings being "balanced." Therefore, in the "purported proof" the author is talking about "balanced strings" without having said what "balanced" means.

Now, you may object that the meaning of "balanced" is explained quite clearly in the proof of Theorem 1. That is indeed true, but the word "balanced" was introduced *within the proof*, for use within the proof, and *outside the proof we do not know what "balanced" means.*

Remark. Theorem 1 gives an example of a **necessary condition** for a string to be well formed. (We say that A is a *necessary condition* for B if A happens whenever B does, that is, if B implies A. For example: "he took the final exam" is a necessary condition for "he passed the course," because if he passed the course then he must have taken the final exam. But is of course not a "sufficient condition." Just taking the final exam is not enough to pass the course.) The theorem says that if a string is well formed then it has to have the same number of left and right parentheses. This condition is not sufficient: there are many strings that have the same number of left and right parentheses but are not well formed, as we will soon see. $\diamondsuit$

QUESTION 3: Is the string $P)($ well-formed?

ANSWER: NO. This may look almost obvious to you but, once again, it is a good exercise to try to understand *how we can know for sure that this string is not well-formed.*

Theorem 1 will not help us, because this string has one left parenthesis and one right parenthesis, so the necessary condition of Theorem 1 is satisfied. So we need another theorem, giving us another necessary condition for well-formedness.

[3] If you wrote this in a paper then the people grading it might give you a very high grade anyhow, because this argument shows that you got the point. But they might not, because after all the proof does contain a serious expository flaw, and if the greaders decide to be strict (for example, because the instructor has told them to do so) then they might take advantage of that flaw to give you a low grade, perhaps even a zero. The best way to be sure that a grader will not use the logical flaws in your reasoning to give you a lower grade is to **write perfect proofs**, with no flaws. **Never rely on the kindness of graders!** It's much better to rely on your own ability to be precise and clear and to give complete arguments.

THEOREM 2. *In every well-formed formula, if the formula contains parentheses, then the first parenthesis—if you read the formula from left to right—must be a left parenthesis.*

PROOF. Call a string "bad" if it contains parentheses and the first parenthesis from left to right is a right parenthesis. Call a string "good" if it is not bad. A string obtained using Rule 1 is good, because it has no parentheses at all. An application of Rules PF2 or PF4 will not produce any new parentheses, so if the strings you applied these rules to were good then the new string will also be good. An application of Rule PF3 will yield a string whose first symbol is $\sim$ and whose second symbol is (, so this string is clearly good. An application of Rule PF5 will yield a string whose first symbol is a letter, whose second symbol is one of the logical connectives, and whose third symbol is (, so this string is clearly good. An application of Rule PF6 or PF7 will yield a string whose first symbol is (, so this string is clearly good. So the application of Rules PF1 to PF7 can never yield bad strings. END OF PROOF.

Remark. Theorem 2 gives another example of a **necessary condition** for a string to be well formed: it says that if a string is well formed and contains parentheses then it first parenthesis—goin from left to right—has to be a left parenthesis. This condition is not sufficient, even if you use it together with that of Theorem 1: there are many strings that satisfy the conditions of Theorems 1 and 2 but are not well formed. $\diamondsuit$

QUESTION 4: Is the string $P()$ well-formed?

ANSWER: NO. Again, you may think this is obvious, but we want to have a real proof.

Theorem 1 will not help us, because this string has one left parenthesis and one right parenthesis, so the necessary condition of Theorem 1 is satisfied. Theorem 2 will not help us either, because the first parenthesis of our string is a left parenthesis.

So we need another theorem, giving us another necessary condition for well-formedness.

THEOREM 3. *A well-formed formula cannot contain a left parenthesis immediately followed by a right parenthesis.*

PROOF. Call a string "bad" if it contains a left parentheses immediately followed by right parenthesis. Call a string "good" if it is not bad. A string obtained using Rule 1 is good, because it has no parentheses at all. An application of Rules PF2 or PF4 will not produce any new parentheses, so if the strings you applied these rules to were good then the new string will also be good. An application of Rule PF3 will yield a string whose first symbol is $\sim$ and whose second symbol is (, so this string is clearly good. An application of Rule PF5 will yield a string whose first symbol is a letter, whose second symbol is one of the logical connectives, and whose third symbol is (, so this string is clearly good. An application of Rule PF6 or PF7 will yield a string whose first symbol is (, so this string is clearly good. So the application of Rules PF1 to PF7 can never yield bad strings. END OF PROOF.

Remark. Theorem 3 gives us a third example of a necessary condition for a string to be well formed. $\diamondsuit$

PROBLEM A. Is $((P \vee Q) \wedge \sim Q) \longrightarrow (P \vee Q)$ well formed? Justify your answer.

PROBLEM B. Is $((P \vee Q) \wedge \sim Q)(\longrightarrow (P \vee Q))$ well formed? Justify your answer.