MATHEMATICS 300 — SPRING 1997

Introduction to Mathematical Reasoning

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HANDOUT NO. 14. April 17, 1997

The **second midterm exam** will be on **Thursday, April 24.**

READING ASSIGNMENT FOR APRIL 15, 17 AND 22:

- Chapter 4, especially the part on induction.
- This handout.

(See Handouts 1 to 13 for previous reading assigments.)

HOMEWORK ASSIGNMENTS NOS. 10 AND 11

(See previous handouts for Assigments 1 to 9.)

(No. 10) *Due on Thursday, April 17.* Page 150, Problems 8, 9, 11, 12(a)&(c), and Page 160, Problems 3(a)&(c), 5, 12.

(No. 11) *Due Tuesday, April 22.* Pages 177-180, Problems 3(a),(c),(d), 8, 9, and 17.

Remember that some of the problems in the book could be wrong. If you are asked to prove something, it could happen that what you are asked to prove is not true. In that case, you say it is false, and you prove that it is false.

PROOFS, GUESSES, AND INTELLECTUAL HONESTY

In our discussion of proofs, I have encountered the following situation several times:

- A student has written something that I cannot understand, and he/she claims this is a "proof" of something.
- Since I cannot figure out how the steps in this "proof" can be justified, I ask the student.
- The student then replies telling a long story, that usually begins with the words "well, what I was thinking was ..." or "I was, like, you know, like trying to say ..."
- If pressed further, the student may even say something like: "Since I didn't know what to do, I put down my best guess."

I would like you to read the following discussion very carefully. If you do not agree with my argument, please let us discuss it. Naturally, I believe that I am right, but I do not want you to accept what I am saying just because I am the teacher. This course is about **reasoning**, and reasoning has to consist of **arguments** that are **convincing**, and by that I mean **convincing to the other party**. When you write a proof, you must give an argument that will convince **me**. And when I argue with you, I want to give arguments that will convince **you**. I want to win, of course, but I want to win by **persuasion**, by **convincing you that I am right**. So, please, if you disagree with me, go ahead and make your case. If you have good arguments and manage to persuade me, then I will change my mind. (But please do **not** come to me with statements like "I don't agree with you, that's what I feel." That is not an argument, it's just restating your position. Such statements have no value as arguments. What I would like to hear from you, if you disagree with me, is **reasons**, **arguments**.)

Let us start with some examples that we will all agree on.

EXAMPLE 1. Suppose you are driving somewhere, you are lost, you want to know how to get to the XXX Shopping Mall, you see some people standing on the sidewalk, so you stop and ask them for directions. Suppose they tell you something ("just go straight ahead to the third traffic light and then turn left"), you follow these directions, and then it turns out that they were completely wrong. Suppose you later run into those people again, and you complain to them. Suppose they tell you: "well, we had no idea which way the XXX Mall was, so we just gave you our best guess." Will you say "O.K., that was just the right thing for you to do; thanks for you help"? Or will you be angry and feel like strangling them?

EXAMPLE 2. Suppose you need eyeglasses, you bring your prescription to the optometrist, but then he/she loses your prescription so he gives you some glasses made according to his/her "best guess." Suppose that, when you complain to the optometrist, he/she launches into a long tirade about what he/she was trying to do, and how he/she was trying to help you, and what have you. Once again, you will probably be angry, right?

EXAMPLE 3. Suppose you are in a bus, and the driver does not know how to drive, so when he/she has to stop in order to avoid hitting a wall he/she mistakenly steps on the gas pedal and your bus hits the wall and you and the driver both die. Suppose you run into the driver in the afterlife, and the driver explains to you what happened, by telling you a long story about what he was trying to do, and how he really and truly did want to stop the bus, and he was doing his best, really, please believe me, and he figured that in order to stop the bus there's got to be something somewhere that you step on that will slow it down. (And he got it almost right! He only made one small mistake and stepped on the wrong pedal! But he had the right idea! So he should get a lot of partial credit! Right?) What do you say? Do you say "sure, fine, you did your best"? I bet that what you would say will sound more like: "but if you didn't know how to drive, what on Earth were you doing driving a bus'?'

Do I need more examples? I think not. The point that these examples make is very clear, I hope. **When you do not know something, but pretend that you do, you are being dishonest.** If you yourself believe that you are doing the right thing, then you are being dishonest with yourself, you are deluding yourself into thinking that you are doing the right thing whereas in fact you are not. If you do not believe that you are doing the right thing, then you are being dishonest with others.

In intellectual life, dishonesty often takes takes the form of **intellectual dishonesty**. Intellectual dishonesty occurs when somebody pretends to have a proof or a justification for what he/she is saying, but in fact does not. For example:

- If a scientist makes an experiment in the lab and then tampers with the results in order to make them fit the outcome he/she wants to get and publishes a paper with the doctored results, then this scientist is being dishonest with his readers. If a scientist is convinced that his/her experiment will give the right results, but because he/she is in a hurry he/she actually fails to make the experiment, and just says that his theory has been verified, is deluding himself, and cheating the others.

- If a historian claims to have seen documents proving, say, that Thomas Jefferson liked strawberries, but in fact the documents do not exist, or they exist but do not prove that Jefferson liked strawberries, then this historian is being dishonest with his readers. (Usually, the readers will not go and check the documents themselves, so they have to rely on the honesty and good faith of the historian.)

- If a philosopher or literary critic or art critic writes impressive-sounding words that he knows are just plain gibberish, expecting that the readers will think that these must be profound thoughts, then he is being dishonest with the readers.

In mathematics, when you write a proof, you should have a sound justification for every step. ("I was, like, you know, like trying to figure out what to do, and I thought of doing this" is **not** a justification. "It was my best guess" is **not** a justification.) On the other hand, it would be impossible in a real mathematical proof for you to write down absolutely every justification of every step. So the normal thing to do is this: you write your proof, but you don't give all the details, because you assume that the proof will be read by your teacher or your grader, who are moderately intelligent people, and will be able to figure out the details by themselves.

Now, this works as long as

1. the missing justification really exists,

2. what you wrote is clear enough that when others read it they can figure out what the missing justification may be.

When you write something whose justification is not clear to me, I will start by giving you the benefit of the doubt, and try to figure out by myself what the justification might be. But, if you had no justification to begin with, then you are being **dishonest with me** (and maybe with yourself as well, if you truly believe that what you are doing is right).

You are behaving exactly like the people who gave you directions without really knowing which way you should go, or like the optometrist who did not exactly know what glasses you needed so he gave you some glasses made according to his "best guess," or like the bus driver that did not know how to drive but pretended that he did. The person who gave you wrong directions, the optometrist, the phony driver, they are making you waste time you shouldn't have to waste, and maybe even endangering your life.

Similarly, when you pretend that you have a proof but you don't, you are making me or the grader spend time trying to figure out what your justification may have been, until we come to the conclusion that there wasn't any, and when we reach this conclusion we feel cheated.

If you do not know how to justify what you wrote, then do not write it. Or, if you are convinced that it must be O.K. but you just do not know how to give a good justification, then please be honest and write something: "I am not sure of how to justify this, but it seems to me that..." **Do not say that you have a proof if you do not have a proof.**

For example, I had a student in my Math 300 course last year who asked me point blank: "suppose I have to do a proof by induction, and in the inductive step I need an algebraic identity but I cannot figure out how to get it. Can I just say it?" My answer was quite simple: "You are asking me whether you can cheat. The answer is **NO, YOU CANNOT CHEAT**. And to say that you are proving something when you yourself know that you don't have a justification is outright cheating."

Or, consider the case of the student who goes from the inequalities

$$\sqrt{x} \leq \frac{x}{2} \tag{1}$$

and

$$\sqrt{y} \leq \frac{y}{2} \tag{2}$$

to

$$\sqrt{xy} \leq \frac{x}{2} + \frac{y}{2}\,. \tag{3}$$

This is certainly a very weird thing to do. The student may have seen before situations in which one takes two inequalities and *adds them*, or situations in which one *multiplies them*. But here what seems to be involved is taking two inequalities, and *multiplying one side while adding the other side!*

Is it possible that this student seriously believes that one can do this? That the student thinks one can take two inequalities, and then combine them by multiplying one side and adding the other one? (For example, that you can go from $5 < 7$ and $8 < 9$ to $5.8 < 7 + 9$, i.e. to $40 < 16$.) I find this hard to believe.

Actually, I have had a lot of trouble figuring out what this student might have been thinking, but here are two of my own "best guesses":

1. *The student thought it was O.K. to go from (1) and (2) to (3).* In that case, the student probably did not ask himself/herself *why* this was O.K. In other words, the student became convinced of something without having a valid justification. If this is the case, then the student was being **dishonest with himself or herself**.

2. *The student knew it was not O.K. to go from (1) and (2) to (3), but wrote it anyhow.* In that case, the student was being **dishonest with the teacher and the grader,** by asserting that he/she had a proof when in fact he/she knew he/she did not.

CONCLUSION. *Honesty is the best policy, and to be honest with others you have to begin by being honest with yourself. The first and most important way of being honest with yourself is to admit that when you do not know something then you just do not know, and you do not lie to yourself and tell yourself that you do know. And the first and most important way of being honest with others is to make sure that when you do not know something then you do not pretend that you do, and when you do not have a justification for what you say then you do not pretend that you have it.*