MATHEMATICS 300 — FALL 1998

Introduction to Mathematical Reasoning — H. J. Sussmann

HANDOUT NO. 12 — *November 16, 1998*

THE SECOND MIDTERM EXAM is going to be on **Wednesday, December 2**.

THE FINAL EXAM is going to be on **Wednesday, December 16, 8:00 to 11:00 pm.**

HOMEWORK ASSIGNMENT NO. 9, DUE ON MONDAY, NOV. 23

You will get the list of problems on Wednesday, Nov. 18. (I hope Handout 13 will be ready by then, in which case the list will be included there.)

SOME COMMENTS ON HOMEWORK NO. 7
READ THIS CAREFULLY!!!

In his answer to the problem

Prove that if A and B are sets then $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$,

a student wrote the following:

Let A and B be sets.

Prove that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.
Proof: $S \in \mathcal{P}(A) \vee S \in \mathcal{P}(B)$ iff
$(\forall x)((x \in S \Rightarrow x \in A) \vee (x \in S \Rightarrow x \in B)) \Rightarrow$
$(\forall x)(x \in S \Rightarrow (x \in A \vee x \in B)) \Rightarrow$
$(x \in (A \cup B)) \Rightarrow$
$S \subseteq \mathcal{P}(A \cup B)$

This is completely unintelligble and unacceptable.

- To begin with, the student seems to think that

  $$S \in \mathcal{P}(A) \vee S \in \mathcal{P}(B)$$

  is equivalent to

  $$(\forall x)((x \in S \Rightarrow x \in A) \vee (x \in S \Rightarrow x \in B)).$$

  But that is **FALSE!** Suppose, for example, that A is the set of all male students in our class, B is the set of all female students in our class, and S is the set of all students in our class. Then

  $$(\forall x)((x \in S \Rightarrow x \in A) \vee (x \in S \Rightarrow x \in B))$$

  is true. (Reason: if a is an arbitrary student, then either $a \in A$ or $a \in B$. In the former case, $a \in S \Rightarrow a \in A$ is true. In the latter case, $a \in S \Rightarrow a \in B))$ is true. So $a \in S \Rightarrow a \in A) \vee (a \in S \Rightarrow a \in B))$ in both cases. Since a was an arbitrary student, $(\forall x)((x \in S \Rightarrow x \in A) \vee (x \in S \Rightarrow x \in B))$ is true.) But $S \in \mathcal{P}(A) \vee S \in \mathcal{P}(B)$ is **not true**, because S in neither a subset of A nor a subset of B.

- Second: in the line right before last, the student writes

  $(x \in (A \cup B))$

  but the letter x is not under the scope of any quantifier. So x is a free variable. So this is not a meaningful sentence.

- Third: I cannot figure out exactly what it is that the student is asserting. I can think of at least two possibilities:

  1. The student is trying to say that the formula

     $S \in \mathcal{P}(A) \vee S \in \mathcal{P}(B)$

     is equivalent to the formula

     $(\forall x)((x \in S \Rightarrow x \in A) \vee (x \in S \Rightarrow x \in B)) \Rightarrow (\forall x)(x \in S \Rightarrow (x \in A \vee x \in B))$
     $\Rightarrow (x \in (A \cup B)) \Rightarrow S \subseteq \mathcal{P}(A \cup B)$.

  2. The student is trying to say that

     a. the formula $S \in \mathcal{P}(A) \vee S \in \mathcal{P}(B)$ is equivalent to the formula $(\forall x)((x \in S \Rightarrow x \in A) \vee (x \in S \Rightarrow x \in B))$;

     b. the formula $(\forall x)((x \in S \Rightarrow x \in A) \vee (x \in S \Rightarrow x \in B))$ implies the formula $(\forall x)(x \in S \Rightarrow (x \in A \vee x \in B))$,

     c. the formula $(\forall x)(x \in S \Rightarrow (x \in A \vee x \in B))$, implies the formula $(x \in (A \cup B))$,

     and

     d. the formula $(x \in (A \cup B))$ implies the formula $S \subseteq \mathcal{P}(A \cup B)$.

  My guess is that what the student has in mind is the second possibility, but what is actually **saying** is what's described in the first one.

Another student wrote:
Suppose that a set A has n elements and a set B has m elements. Then $\mathcal{P}(A)$ has 2^n elements and $\mathcal{P}(B)$ has 2^m. Also suppose that $n \leq m$. Then the worst case is that $A \subseteq B$. But then $\mathcal{P}(A) \subseteq \mathcal{P}(B) \Rightarrow \mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(B)$. Hence $\mathcal{P}(B)$ has 2^n elements. On the other side, $A \cup B = B \Rightarrow \mathcal{P}(A \cup B) = \mathcal{P}(B)$. $\mathcal{P}(B)$ also has 2^n elements. So, for this case, $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$ is true. The other boundary or the best case is $A \cap B = \emptyset$. Then $\mathcal{P}(A) \cup \mathcal{P}(B)$ has $2^n + 2^m$ elements while $\mathcal{P}(A \cup B)$ has 2^{n+m} elements. So $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

Once again, this is completely unacceptable. There are lots of mistakes, and lots of things that are unintelligble (for example, I don't understand all this talk of "best case" and "worst case;" I have no idea what the student is talking about), but let us just focus on the most serious problem of all. At the end the student says

> $\mathcal{P}(A \cup B)$ has 2^{n+m} elements. So $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

My question is **WHY?**
The only explanation I can think of is that the student really believes that

> If a set X has fewer elements than another set Y, then $X \subseteq Y$.

But that is **NOT TRUE!** For example, the set

$X = \{0, 1, 2\}$

has three elements, and the set

$Y = \{4, 5, 6, 7\}$

has four elements. Yet, it's not true that $X \subseteq Y$.