MATHEMATICS 300 — FALL 1998

Introduction to Mathematical Reasoning

H. J. Sussmann

HANDOUT NO. 1

September 2, 1998

Instructor: H. J. Sussmann (office: Hill 704; e-mail: sussmann@hamilton.rutgers.edu; phone: (732)445-5407.)

INFORMATION ABOUT THE COURSE, HOMEWORK, GRADES, ETC.

Class schedule: Mondays and Wednesdays, 7th period (6:10 to 7:30 p.m.), in HH-B2.

Instructor's office hours: In Hill 704, Mondays and Thursdays, 4th period (1:10 to 2:30 p.m.), and by appointment. (The best way to schedule an appointment is to talk to the instructor after class.) Note: **if you are planning to see the instructor during office hours, please let him know ahead of time (for example, after class, or by e-mail) that you are coming.** Reason: the instructor wants to know if anyone is going to see him, because if nobody intends to go then he would not like to be obliged to stay in the office.

Textbook: D. Smith, M. Eggen and R. St. Andre, *A Transition to Advanced Mathematics* (Fourth Edition), Brooks/Cole Publishing Co.

Handouts: There will be several handouts supplementing the material of the book. **The contents of these handouts are an integral part of the course.**

Grades will be based on the *homework*, *quizzes*, *midterms*, and the *final exam*:

- The *homework* and the *quizzes*, together, will count for about 1/3 of the grade.
- The *midterms* will count for another third.
- The *final exam* will count for the remaining third.

Please be assured that there are no grade quotas or curves in our course. **I hope that everyone earns an "A." I want everyone to get a "A." And if you all earn it, you will all get it!**

You will get an "A" if you take the course seriously, learn a lot of the material covered in the course, and in particular can answer questions similar to those of the homework problems and the review questions you will be assigned during the term. You will get a "B" if you take the course seriously and do an adequate amount of work. You will get a "C" if you show serious effort to do the work assigned in the course and learn a reasonable portion of what was covered in it.

In the past, I have been asked several times for a passing grade —and even for a "C"— by students who (a) had done none or almost none of the homework; (b) had a 20% average in the midterms; (c) had a 20% or 25% score in the final exam. Such students asked me for a higher grade on the grounds that they needed it for something (for example, to graduate, or in order not to lose a scholarship). **Please do not do that.** I will be happy to give you a high grade if you have earned it, but if you don't do the work then you should not expect that you will get a high grade just because you need it.

About the homework:

- In this course, **writing correctly, clearly and precisely is very important**. Whether or not you do it will play a significant role in determining your grade.
- You should always use *complete English sentences*, preferably **declarative sentences** (often called **statements**). (See the notes on sentences in Handout No. 2.)
- Avoid exaggerated or incorrect use of cryptic mathematical notation.
- Explain what you are doing.
- Make sure that letter "variables" are only used if either: (i) it has been said before what these letters stand for, or (ii) they are "closed variables" (or "dummy variables," or "bound variables") in the sense that will be discussed in detail in class, and in a later handout.
- Provide proper connectives between equations as well as between ideas.
- Make sure that all the rules of English grammar (including those of spelling and punctuation) are strictly obeyed.
- Try to say things correctly, following all the rules, but *in your own words.* Please *no rote learning.* If you have to memorize a definition or a statement, this is an indication of lack of understanding.
- **Pay attention to presentation**:
  - If you have to cross out lots of words, then you should rewrite the whole thing anew, **cleanly and neatly**. If you are not willing to spend some of your time doing this; if what you hand in shows that you were in a hurry and that you did not make the effort to write things neatly and properly, then there is no reason for me or the grader to spend any of our time reading what you wrote, and we will not do it.
  - Make sure that **your name appears in every sheet of paper you hand in**, and that if you are handing in more that one sheet then the sheets are **stapled** and the pages are **numbered**.
  - **Homework assignments that do not meet these requirements will be marked "unacceptable" and returned unread.** I mean this quite seriously. In the past, I have had students come to me with the complaint that they had handed in a homework assignment that was perfectly fine, except only for the fact that the pages weren't stapled, and then the student got the assignment back unread, marked "unacceptable" (and with a zero grade, of course). Students have complained that "it isn't fair that I should get a zero just because I didn't staple the pages." The answer is: *I am clearly telling you, from the very first day, that if you don't staple the pages you get a zero; so you should know it; moreover, stapling the pages is an easy thing to do; therefore, if you don't do it you should know that this will get you a zero, and this is not "unfair" at all, because you had been told very clearly what to expect.*

- Do not assume that the person reading the paper can read your mind. Do assume that the person reading the paper is intelligent, but also assume that the person reading the paper is busy, and cannot and will not spend an excessive amount of time puzzling out your meaning. Communication is difficult, and written technical communication is close to an art.

- **Effective written exposition will be worth at least 50% of your grade, and, conversely, unclear or incomprehensible exposition may be penalized as much as 50% of the grade or even more.**

- Please **proofread carefully** what you hand in. Ideally, you should read and reread and revise almost any formal communication. **Neatness and clarity count**, as you well know if you have tried to read any complicated document.

(The best reference known to me on effective writing is *The Elements of Style* by Strunk and White, a very thin paperback published by Macmillan. It isn't expensive, and it is easy to read. I recommend it.)

- **Joint work is not permitted.** You can, of course, talk about the problems with your friends, but each assigment should be written individually and should be original. **Copying is not acceptable.**

READING FOR SEPTEMBER 2 AND 9.

Chapter 1, sections 1.1 and 1.2 of the book.

Handouts no. 1 and 2.

THE FIRST HOMEWORK ASSIGNMENT.

The first homework assignment is due on **Monday, September 14**. It consists of:

a. Problem 1, Parts (a), (b), (d), (f), (h), in page 7 of the book,
b. Problem 4, Parts (b), (d), (f), (h), (j), (l), in page 8,
c. Problem 5, Parts (d), (f), (h) and (i), in page 8,
d. Problem 7 in pages 8-9,
e. Problem 9 in page 9,
f. Problem 10 in Page 9,

plus the following:

Each of the following ten mathematical statements is either true or false. In each case, *indicate whether it is true or false, and give a convincing reason why.* The most important thing is that you should give a convincing reason. And I emphasize that the question in each case is whether the statement is true or false, and why. So don't expect all the ten statements to be true.

STATEMENT 1: The sum of two prime numbers is always an even number. (Note: the "positive integers" are the numbers $1, 2, 3, \ldots$. Often, they are also called "natural numbers." You may have heard of them referred to as "whole numbers." A positive integer m is *prime* if m is not equal to 1 and m is not divisible by any positive integers other than 1 and m. For example, 1 is not prime, 2 is prime, 3 is prime, 4 is not prime, 5 is prime, 6 is not prime, 7 is prime.)

STATEMENT 2: The sum of two prime numbers neither one of which is equal to 2 is always an even number.

STATEMENT 3: There exists exactly one positive integer n such that $n > 1$ and the numbers n, $n+2$ and $n+4$ are prime. (Note: If we take $n = 3$ then $n > 1$, and the three numbers n, $n+2$ and $n+4$ —that is, 3, 5 and 7— are prime. So a number n with the desired properties definitely does exist. The question here is whether 3 is the only number with this property. Are there others? Yes? No? Why?)

STATEMENT 4: If x and y are real numbers then $|x - y| \leq |x| - |y|$.

STATEMENT 5: If x and y are positive real numbers then $\sqrt{xy} \leq \frac{x+y}{2}$.

STATEMENT 6: Every real number has a multiplicative inverse.

STATEMENT 7: There exists a real number x that has the property that $x \geq y$ for all real numbers y.

STATEMENT 8: $\lim_{x\to 0} x^2 = 0$.

STATEMENT 9: The equation $x^2 = 2$ has a solution. (Note: in this problem, and also in the one about statement 10, "solution" means "a *real* number x for which the equality is true." In other words, we are only talking about *real number solutions.* If you know about complex numbers then that is very nice, but in this problem the question is about real numbers only.)

STATEMENT 10: The equation $x^4 - 108x + 242 = 0$ has a solution. (See the note for the preceding problem.)

Note: Later on in the course, what I am now calling a "convincing reason" will be called a **proof**.) You have to write something that will persuade a reasonable skeptic that the statement is true. For example, if I ask you to give a "convincing reason" for the statement that "323 is not a prime number," you can assume that I know what a prime number is, and that I know how to multiply numbers, but that I don't know whether 323 is prime. You then tell me that "$323 = 17 \times 19$," and that ought to satisfy me. Suppose I don't accept this. Suppose I say "how do I know that $323 = 17\times 19$?; I am not convinced, please persuade me." Then you could actually do the multiplication and show me how the result is indeed 323. Suppose I say "when you multiplied 17 times 19 you used certain rules; why are those rules you used valid?" (or, maybe, "when you multiplied 17 times 19 you used your calculator; how do I know that what the calculator gives you is right?"). Then maybe you could try to persuade me that the rules are fine. But I could go on asking "why?" and never stop. If I were to behave like that then I would be an **unreasonable skeptic**. (Think of other examples: suppose you are buying something in the supermarket and you claim that the price is $6.95 but the cashier doesn't believe you. Then you may point to the price tag, and show that it says "6.95" and that ought to do it. If the cashier asks "how do I know that the price is what is written in the price tag?" then the cashier is being unreasonable.) When you write a mathematical argument, do not worry about unreasonable skeptics; only about reasonable ones. Reasonable skeptics are people who are just a little bit behind you; they know the things that come "before" what you want to prove, but they do not know that what you are trying to prove is true. So can assume they know almost as much as you know on the question under discussion, but not the answer to the question itself.

REVIEW QUESTIONS FOR SEPTEMBER 2 AND 9.

1. What is a rational number?
2. Explain what —if anything— is wrong with the following purported definition of "rational number":

   A *rational number* is $\frac{m}{n}$.

3. Explain what —if anything— is wrong with the following purported definition of "rational number":

   A *rational number* is $\frac{m}{n}$ where m and n are integers.

4. What is an even number? (Please don't say "it's a number that's not odd." If you say that then I might ask you "what is an odd number?" and, if you tell me that "it's a number that's not even," then your definitions will be *circular*, and that is very bad. Remember that *a definition of something should explain what that "something" is to a person who does not already know what it means.* Now, if you have to explain to someone what an even number is, you have to assume that the person you are talking to does not know what an even number is. Usually, people who don't know what an even number is would not know what an odd number is either, because the concepts of "even" and "odd" are so closely related. So, for you to assume that the reader knows what an "odd number" is is not reasonable. **Read the notes on "definitions" in Handout No. 2.**)
5. Explain what —if anything— is wrong with the following purported definition of "even number":

   An *even number* is a number that can be divided by 2.

6. According to the definition proposed in the previous question, is 5 an even number?
7. What is a prime number?
8. Is the number 6,095,812,773 prime?
9. Explain what —if anything— is wrong with the following purported definition of "prime number":

   A *prime number* is a number that is only divisible by 1 and by itself.

10. According to the definition proposed in the previous question, is 5 a prime number?
11. What is a statement?
12. Name four different types of sentences and give two examples for each type.
13. What is a propositional form?
14. Explain the distinction between "proposition" and "propositional form."
15. What is a tautology?
16. What is a contradiction?
17. Explain what —if anything— is wrong with the following purported explanation of what a tautology is: "a *tautology* is a proposition that is true for all possible assignments of truth values to its arguments."
18. Explain what —if anything— is wrong with the following purported explanation of what a tautology is: "a *tautology* is a statement that is true for all possible assignments of truth values to its arguments."
19. Explain what —if anything— is wrong with the following purported explanation of what a tautology is: "a tautology is a propositional form that is true for all possible assignments of truth values to its arguments."
20. Explain what —if anything— is wrong with the following purported explanation of what a tautology is: "a tautology is a true statement."
21. Explain what —if anything— is wrong with the following purported explanation of what a tautology is: "**Tautology** is when a propositional form is true for all possible assignments of truth values to its arguments."
22. Define "conjunction." (Note: see the remarks on definitions in Handout No. 2. In particular, do not forget that the first thing to do is to figure out the number of arguments. Take a look at the definition of "conjunction" in the book, in page 2, and you will see that the authors scrupulously followed my suggestion that one has to spell out carefully how many arguments there are and what they are.)
23. Define "brother." (Assume you know what "male," "female," "father" and "mother" mean. Do not allow a half-brother to count as a brother.)

24. Define "disjunction."
25. Define "negation."
26. Define "denial."
27. If you follow the book's definition of "negation," is it true that the negation of the negation of a proposition P is again P?
28. Is "He was here yesterday" a statement? What extra information —if any— would have to be supplied for it to deserve to be called a statement? What words in "He was here yesterday" should be thought of as "variables"?
29. Is "$x^2 = 36$" a statement? What extra information —if any— would have to be supplied for "$x^2 = 36$" to deserve to be called a statement? What words or symbols in "$x^2 = 36$" should be thought of as "variables"?
30. Is "there exists a real number x such that $x^2 = 36$" a statement? What extra information —if any— would have to be supplied for it to deserve to be called a statement? What words or symbols in this expression should be thought of as "variables"? Are there any dummy variables? Which ones?
31. Is "there exists a real number y such that $x^2 = 36$" a statement? What extra information —if any— would have to be supplied for it to deserve to be called a statement? What words or symbols in this expression should be thought of as "variables"? Are there any dummy variables? Which ones?
32. Is "$x + y = y + x$ for all real numbers x and y" a statement? What extra information—if any— would have to be supplied for it to deserve to be called a statement? What words or symbols in this expression should be thought of as "variables"? Are there any dummy variables? Which ones?
33. Is "$x + y = y + x$ for all real numbers a and b" a statement? What extra information—if any— would have to be supplied for it to deserve to be called a statement? What words or symbols in this expression should be thought of as "variables"? Are there any dummy variables? Which ones?
34. Is "$\int_0^1 x^2 dx = 6$" a statement? What extra information—if any—would have to be supplied for it to deserve to be called a statement? What words or symbols in this expression should be thought of as "variables"? Are there any dummy variables? Which ones?
35. Is "$\int_0^1 x^2 dx = y$" a statement? What extra information—if any—would have to be supplied for it to deserve to be called a statement? What words or symbols in this expression should be thought of as "variables"? Are there any dummy variables? Which ones?
36. Is "There exists a real number such that $x^2 + x^3 = x$" a statement? What extra information— if any—would have to be supplied for it to deserve to be called a statement? What words or symbols in this expression should be thought of as "variables"? Are there any dummy variables? Which ones?
37. What—if anything—is wrong with the formula "$\int_0^1 x^2 dx = x$"?
38. What—if anything—is wrong with the formula $\sum_{k=2}^{375} \frac{1}{k^2+5} < k$?
39. Is it true that $\sum_{k=2}^{375} \frac{1}{k^2+5} < 60$?
40. What is the value of $\sum_{k=1}^{5} k^2$?
41. Is "$\sum_{k=1}^{5} k^2 = 23$" a statement? What extra information—if any—would have to be supplied for it to deserve to be called a statement? What words or symbols in this expression should be thought of as "variables"? Are there any dummy variables? Which ones?
42. Is "$\sum_{n=1}^{5} k^2 = 23$" a statement? What extra information—if any—would have to be supplied for it to deserve to be called a statement? What words or symbols in this expression should be thought of as "variables"? Are there any dummy variables? Which ones?