MATHEMATICS 300 — SPRING 1997

Introduction to Mathematical Reasoning

Instructor: H. J. Sussmann (office: Hill 704; e-mail: sussmann@hamilton.rutgers.edu; phone:445-5407.)

Class schedule: Tue, Thu 7th period (6:10 to 7:30 p.m.), Hardenburgh Hall, A6.

INFORMATION ABOUT THE COURSE, HOMEWORK, GRADES, ETC.

Instructor's office hours: In Hill 704, by appointment. (The best way to schedule an appointment is to talk to the instructor after class.)

Textbook: R. P. Morash, *A Bridge to Abstract Mathematics*, McGraw-Hill.

Handouts: There will be several handouts supplementing the material of the book. **The contents of these handouts are an integral part of the course.**

Grades will be based on the homework, quizzes, midterms, and the final exam:

- The *homework* and the *quizzes*, together, will count for about 1/3 of the grade.
- The *midterms* will count for another third.
- The *final exam* will count for the remaining third.

For the homework:

1. You should read carefully the notes on mathematical writing in the following pages. In this course, writing correctly, clearly and precisely is very important. Whether or not you do it will play a significant role in determining your grade.
2. **Joint work is not permitted.** You can, of course, talk about the problems with your friends, but each assigment should be written individually and should be original. **Copying is not acceptable.**

THE FIRST HOMEWORK ASSIGNMENT.

The first homework assignment is due on Tuesday, January 28. It consists of:

a. Problem 4 in Page 14 of the book,
b. Problem 7 in Page 15,
c. Problem 10 in Page 26,
d. Problem 11 in Page 27,
e. Problems 2 and 8 in Page 37.

WRITING MATHEMATICS

You should write so that you can be easily understood by a properly trained English-speaking individual. In particular, this means that you must

- Use *complete English sentences*, that make clearly identifiable *statements.* (The word *statement* has a precise definition, that you can find in the book and that you ought to be able to repeat, verbally or in writing, if asked. What you write should consist of statements in the very precise sense of this definition.)
- Avoid exaggerated or incorrect use of cryptic mathematical notation.
- Explain what you are doing.
- Make sure that letter "variables" are only used if either: (i) it has been said before what these letters stand for, or (ii) they are "closed variables" (or "dummy variables," or "bound variables") in the sense that will be discussed in detail in class, and also further below in this handout.
- Provide proper connectives between equations as well as between ideas.
- Make sure that all the rules of English grammar (including those of spelling and punctuation) are strictly obeyed.
- Try to say things correctly, following all the rules, but *in your own words.* Please *no rote learning.* If you have to memorize a definition or a statement, this is an indication of lack of understanding.
- **Pay attention to presentation**:
  - If you have to cross out lots of words, then you should rewrite the whole thing anew, cleanly and neatly. If you are not willing to spend some of your time doing this; if what you hand in shows that you were in a hurry and that you did not make the effort to write things neatly and properly, then there is no reason for me or the grader to spend any of our time reading what you wrote, and we will not do it.
  - Make sure that **your name appears in every sheet of paper you hand in**, and that if you are handing in more that one sheet then the sheets are **stapled** and the pages are **numbered**.
  - **Homework assignments that do not meet these requirements will be marked "unacceptable" and returned unread.**

Do not assume that the person reading the paper can read your mind. Do assume that the person reading the paper is intelligent, but also assume that the person reading the paper is busy, and cannot and will not spend an excessive amount of time puzzling out your meaning. Communication is difficult, and written technical communication is close to an art. Effective written exposition will be worth at least 50% of your grade, and, conversely, lousy exposition may be penalized as much as 50% of the grade or even more.

Please proofread carefully what you hand in. Ideally, you should read and reread and revise almost any formal communication. Neatness and clarity count, as you well know if you've tried to read any complicated document.

The best reference known to me on effective writing is *The Elements of Style* by Strunk and White, a very thin paperback published by Macmillan. It isn't expensive, and it is easy to read. I recommend it.

ON THE USE OF LETTERS

Letter variables are only used in mathematics in order to shorten certain statements that would otherwise be too long.

EXAMPLE 1: Suppose you want to say

> If you take two real numbers, add them, and square the sum, then you get the same result as if you had squared each number separately, added the squares, and added to the result twice the product of the two numbers.

This looks awfully complicated and ugly. You can make it much shorter by saying

> For all real numbers x, y, $(x+y)^2 = x^2 + y^2 + 2xy$.

(Later, we, will learn to say this even more concisely, by writing

$$(\forall x \in \mathbf{R})(\forall y \in \mathbf{R})\Big((x+y)^2 = x^2 + y^2 + 2xy\Big) .$$

Here "$(\forall x \in \mathbf{R})(\forall y \in \mathbf{R})$" is an abbreviation of "for all real numbers x, y.")

EXAMPLE 2: Suppose you want to say

> It is possible to find a real number whose cube, added to its square, is equal to the number itself.

Again, this looks awfully complicated and ugly, and you can make it much shorter by saying

> There exists a real number x such that $x^2 + x^3 = x$,

or

> $x^2 + x^3 = x$ for some real number x.

(Later, we, will learn to say this thing even more concisely, by writing

$$(\exists x \in \mathbf{R})\Big(x^2 + x^3 = x\Big) .$$

Here "$(\exists x \in \mathbf{R})$" is an abbreviation of "for some real number x.")

EXAMPLE 3: suppose you want to say

> For each integer between 2 and 25, take that integer, square it, add 5, and compute the multiplicative inverse of the result. Having done this for all the integers between 2 and 25 (including 2 and 25 themselves), add all the resulting numbers. Then the number you get is smaller than 20.

Once again, this is awful, so you can shorten the whole thing and just say

> Let $N = \sum_{k=2}^{25} \frac{1}{k^2+5}$. Then $N < 20$.

Or, if you prefer, you could just say:

> $\sum_{k=2}^{25} \frac{1}{k^2+5} < 20$.

EXAMPLE 4: Suppose you want to say

> If you take the function that to each real number assigns its cube, and integrate it from 0 to 4, then the result is 64.

Again, this can be made much shorter by saying

> Let $N = \int_0^4 x^3\,dx$. Then $N = 64$.

Or you could just say:

> Let $\int_0^4 x^3\,dx = 64$.

EXAMPLE 5: Suppose I want to say:

> The set of all real numbers whose square is larger than the cube of the number has at least two different members.

We can make this shorter by saying

> Let $S = \{x \in \mathbf{R} : x^2 > x^3\}$. Then S has at least two members.

Or you could just say:

> Let $S = \{x \in \mathbf{R} : x^2 > x^3\}$. Then $(\exists x)(\exists y)\Big(x \in S, y \in S, \text{ and } x \neq y\Big)$.

EXAMPLE 6: Suppose I want to say that "my age is larger than 20 and less than 70, and my brother is younger than I am, but his age is larger than half my age."

Here too we can be more concise, and say

> Let x be my age and let y be my brother's age. Then $20 < x < 70$ and $\frac{x}{2} < y < x$.

In all these cases, we have used **letters** (namely, x, y, k, N, S) as labels, so that we can avoid writing clumsy phrases such as "the number that we have been talking about."

In all the above examples, the statements we wrote using letters only contain the letters as **dummy variables**. The meaning of this is quite simple: the letter is just a label used to keep track of a number (or set, in Example 6) that occurs several times.

The distinctive feature of a dummy variable is that you could have used any other symbol instead, and you would be saying exactly the same thing. For example, if you write

> For all real numbers x, y, $(x + y)^2 = x^2 + y^2 + 2xy$,

then that is exactly the same thing as you had written

> For all real numbers a, b, $(a + b)^2 = a^2 + b^2 + 2ab$,

or

> For all real numbers α, β, $(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$,

or

> For all real numbers m, n, $(m + n)^2 = m^2 + n^2 + 2mn$,

or

> For all real numbers $\diamond$, $\#$, $(\diamond + \#)^2 = \diamond^2 + \#^2 + 2\diamond\#$.

Similarly, instead of

$$(\exists x \in \mathbf{R})\Big(x^2 + x^3 = x\Big)\,,$$

you could have said

$$(\exists z \in \mathbf{R})\Big(z^2 + z^3 = z\Big)\,,$$

or

$$(\exists a \in \mathbf{R})\Big(a^2 + a^3 = a\Big)\,,$$

or

$$(\exists \omega \in \mathbf{R})\Big(\omega^2 + \omega^3 = \omega\Big)\,,$$

or

$$(\exists \aleph \in \mathbf{R})\Big(\aleph^2 + \aleph^3 = \aleph\Big)\,,$$

or

$$(\exists \diamond \in \mathbf{R})\Big(\diamond^2 + \diamond^3 = \diamond\Big)\,.$$

Just one more example:

Instead of

$$\sum_{k=2}^{25} \frac{1}{k^2+5} < 20\,,$$

you could have said

$$\sum_{n=2}^{25} \frac{1}{n^2+5} < 20\,,$$

or

$$\sum_{\rho=2}^{25} \frac{1}{\rho^2+5} < 20\,.$$

So the "dummy variables" in the above examples are really just **internal tags**, using to keep track of numbers **within a particular text**. Outside this text, they variables do not have any meaning. For example, after you have written

$$\sum_{k=2}^{25} \frac{1}{k^2+5} < 20\,,$$

it would not make sense for me to ask you "what is the value of k?" The letter k is just a tag to specify a *range of values*, as part of a set of instructions on how to compute a sum. Any other letter could be used instead, so in fact "k isn't really there." You can persuade yourself of this in another way, by remembering that

$$\sum_{k=2}^{25} \frac{1}{k^2+5} < 20$$

is just another way of saying

> For each integer between 2 and 25, take that integer, square it, add 5, and compute the multiplicative inverse of the result. Having done this for all the integers between 2 and 25 (including 2 and 25 themselves), add all the resulting numbers. Then the number you get is smaller than 20.

As you can see, there is no k there!

This way of speaking and writing, using dummy variables, is very useful, especially when what we are trying to say is quite long. But **you have to be careful** when writing this way, and have to make sure that you are using the letters correctly.

THE FIRST FUNDAMENTAL RULE

A letter symbol can only be used if either its value has been declared before or it is a dummy variable.

For example: if you write

$$x^2 + x = 90$$

just like that, without any words before or after explaining what x is, or making it a dummy variable, then what you wrote is **meaningless**. We have no way to tell if what you wrote is true or false, because we have no idea what you are talking about.

If you write this, then I will write in red **what is x?**

For comparison, here are some perfectly meaningful things you could have said:

1. Let $x = 9$. Then $x^2 + x = 90$.
2. Let $x = 6$. Then $x^2 + x = 90$.
3. Let x be the number of planets in the solar system. Then $x^2 + x = 90$.
4. $x^2 + x = 90$ for every real number x.
5. $x^2 + x = 90$ for some real number x.
6. $(\forall x \in \mathbf{R})(x^2 + x = 90)$.
7. $(\exists x \in \mathbf{R})(x^2 + x = 90)$.
8. The equation $x^2 + x = 90$ has a solution $x \in \mathbf{R}$. (Or "the equation $x^2 + x = 90$ has a real solution.")

Some of these are true, while the others are false. (Precisely: 1,3,5,7,8 are true; 2,4,6 are false. Moreover: 6 is just another way of saying the same thing as 4, and 7 and 8 are just alternative ways of stating 5.)

Notice that in 1, 2 and 3 the letter x has a value that was declared before. In 4, 5, 6, 7 and 8 the letter x is a dummy variable.

Again for comparison, let us now write some things that do **not** have a precise meaning:

A. The equation $x^2 = 2$ has a solution.

B. The equation $x^2 + 1 = 0$ has a solution.

C. The equation $3x = 5$ has a solution.

Why are these things bad? Because they do not specify the **range** of values within which the solution x is to be sought. For example, A is true if we look for solutions $x \in \mathbf{R}$, but false if we look for solutions $x \in \mathbf{Q}$. B is true if we look for solutions $x \in \mathbf{C}$ (the set of complex numbers) but false if we look for solutions $x \in \mathbf{R}$. C is true if we look for solutions $x \in \mathbf{Q}$ (or $x \in \mathbf{R}$) but false if we look for solutions $x \in \mathbf{Z}$.

On the other hand, the following are perfectly meaningful statements:

A1. The equation $x^2 = 2$ has a solution $x \in \mathbf{R}$. (That is: $(\exists x \in \mathbf{R})(x^2 = 2)$.)

A2. The equation $x^2 = 2$ has a solution $x \in \mathbf{Q}$. (That is: $(\exists x \in \mathbf{Q})(x^2 = 2)$.)

B1. The equation $x^2 + 1 = 0$ has a solution $x \in \mathbf{Z}$. (That is: $(\exists x \in \mathbf{Z})(x^2 + 1 = 0)$.)

B2. The equation $x^2 + 1 = 0$ has a solution $x \in \mathbf{R}$. (That is: $(\exists x \in \mathbf{R})(x^2 + 1 = 0)$.)

B3. The equation $x^2 + 1 = 0$ has a solution $x \in \mathbf{C}$. (That is: $(\exists x \in \mathbf{C})(x^2 + 1 = 0)$.)

C1. The equation $3x = 5$ has a solution $x \in \mathbf{Z}$. (That is: $(\exists x \in \mathbf{Z})(3x = 5)$.)

C2. The equation $3x = 5$ has a solution $x \in \mathbf{Q}$. (That is: $(\exists x \in \mathbf{Q})(3x = 5)$.)

C3. The equation $3x = 5$ has a solution $x \in \mathbf{R}$. (That is: $(\exists x \in \mathbf{R})(3x = 5)$.)

C4. The equation $3x = 5$ has a solution $x \in \mathbf{C}$. (That is: $(\exists x \in \mathbf{C})(3x = 5)$.)

(Actually, A1, B3, C2, C3 and C4 are true, whereas A2, B1, B2 and C1 are false. Of course, in each case you could say the same thing in lots of different ways. For example, instead of C2 you could say "the equation $3x = 5$ has a rational solution," or "the equation $3x = 5$ can be solved in the set of rational numbers," or "there is a rational number that satisfies the equation $3x = 5$," or "$3x = 5$ is solvable in the set of rational numbers.")

A sentence whose meaning is imprecise, because it can be interpreted in more than one way, is said to be *ambiguous*. For example, the sentences

> Flying planes can be dangerous
>
> College students are turning to vegetables
>
> The University president promised to stop drinking on campus

are ambiguous. Why? Think of other examples of ambiguous sentences.

Similarly, A, B, C above are ambiguous. Another example of ambiguous sentence is a sentence where a letter does not have a declared value and is not a dummy variable either. For example, if I tell you to "read chapter x," then that's ambiguous —unless I have told you before what "x" stands for, or given you clues so you can find out, or perhaps I have said "read Chapter x for all x" (that is "read all the chapters") or "read Chapter x for some x" (that is "read one chapter")— because you don't know what x is supposed to be.

THE SECOND FUNDAMENTAL RULE

Avoid ambigous sentences!

THE THIRD FUNDAMENTAL RULE

Formulas without words are usually meaningless and should be avoided like the plague.

It's the **words** that glue the formulas together and give a structure and a meaning to mathematical writing. If you just write a bunch of formulas then you probably do not know what you are saying, and if you don't know then you should not expect that the instructor or the grader will know.

Look, for example, at Statements 1, 2, 4 and 5 above. In all cases, the "formula" $x^2 + x = 90$ is the same, but the statements are quite different and have totally different meanings. That's because the words are different!

The three "fundamental rules" can in fact be subsumed into a single principle:

BE PRECISE!

AN EXAMPLE

Here is an example of the kind of thing that you should *not* do when you write the solution of a homework problem.

QUESTION: describe the set $\{x \in \mathbf{R} : 1 < x^2 < 2\}$ as a union of intervals.

A TRULY AWFUL SOLUTION (*the final answer is correct, so this would get some partial credit, say 40%*):

$$1 < x^2 < 2$$
$$1 < x < \sqrt{2} \quad \text{or} \quad -\sqrt{2} < x < -1$$
$$A = (-\sqrt{2}, -1) \cup (1, \sqrt{2})$$

WHAT MAKES THIS BAD? Well, lots of things. For example:

1. To begin with, I do not see any *statements* here. Remember that a *statement* is supposed to be something that is either true or false. When the student writes the inequality "$1 < x^2 < 2$," what is the student asserting? What statement is the student making? (I know, you are probably thinking that what the student is saying is that "x^2 is between 1 and 2." But that does not do me any good. I still do not know what this means, because I have no idea what x is supposed to be. Do you? And please don't tell me "x is supposed to be a real number." What good does that do to me? I still don't know whether $1 < x^2 < 2$ is true or false, unless you tell me *which* number

it is.) Is this true or is it false? If you cannot tell, it means that there is no statement there. (There is a standard convention used very often in mathematics: when you say something involving variables, and you don't specify the values of the variables or bound the variables with a quantifier, it means that you are saying that your formula is true for *all* values of the variables. In this case: could the student be trying to say that $1 < x^2 < 2$ is true for *all* values of x? I sure hope not! Because if that was it, then it would be **false**! For example, $1 < x^2 < 2$ is not true for $x = 3$.)

2. The letters x and A are used, but we are never told what they are supposed to stand for. (I know: you will tell me that A is the set $\{x \in \mathbf{R} : 1 < x^2 < 2\}$. But the student did not say that anywhere!)
3. Throughout this proposed "solution," there are no words explaining what is being done. And there is not a single punctuation mark!
4. There is nothing connecting each line to the next one.

Now, here is an example of a good solution of the same problem. (The words in italics are my own comments. They are not part of the student's solution.)

A GOOD SOLUTION (*which gets full credit*):
Let $A = \{x \in \mathbf{R} : 1 < x^2 < 2\}$. (*Comment: the student is announcing what the letter A is going to mean in this solution.*)
Let $x \in \mathbf{R}$. (*Comment: the student is announcing that x is going to stand for an arbitrary real number. This means that, from now on, whatever assertion the student makes about x should be true for* **all** *real numbers x. For example, if in the next line the student said "$1 < x^2 < 2$," then that would be bad, because "$1 < x^2 < 2$" is* **not** *true for all x.*)
Then "$x \in A$" is equivalent to "$1 < x^2 < 2$." (*Comment: the student is* **not** *saying that $x \in A$, or that $1 < x^2 < 2$. He/she is just saying that those two statements are equivalent, that is, that they are either both true or both false. So the statement being made here is perfectly true.*)
And "$1 < x^2 < 2$" is equivalent to "either $1 < x < \sqrt{2}$ or $-\sqrt{2} < x < -1$." (*Comment: again, a true statement.*)
Therefore "$x \in A$" is equivalent to "either $x \in (1, \sqrt{2})$ or $x \in (-\sqrt{2}, -1)$." So, finally,

$$A = (1, \sqrt{2}) \cup (-\sqrt{2}, -1), \text{ and this is the desired answer.}$$

GENERAL COMMENT: notice the use of the words "so", "therefore," "then." These words give continuity to the whole thing. They tell you how each paragraph is related to the previous one. English is a very rich language. Take advantage of this to write well! Notice also the nice touch of putting the word "finally," to announce that we are getting to the end. Why not use "then" all the time? **Because it's nicer when you don't repeat the same words over and over again.** *It is not nice to give the impression that you have a very poor vocabulary.*

And here is another example of a correct solution to the same problem.

ANOTHER PERFECTLY FINE SOLUTION (*which also gets full credit*):

A real number belongs to the set under discussion if and only if the square of this number is larger than one and less than two. This will happen if either the number itself is strictly between one and the square root of two (*Comment: why "strictly"? Because I am saying that it cannot be equal to one or square root of two.*), or its negative is strictly between minus the square root of two and minus one. So our set is $(-\sqrt{2}, -1) \cup (1, \sqrt{2})$. *General comment: notice that the student did everything with words, without using any letter variables. Is that bad? Not at all! There is nothing wrong with using words rather than math symbols. The reason that we usually don't do it is that math symbols make it* **easier** *to say things, as we saw before.*

A FINAL COMMENT. The main point of mathematical writing is not the use of weird symbols. It is the use of **precise language.** You can be precise with words or with symbols. But symbols are just a way to say more concisely certain things that would take too long to say with words. As a rule, **if you say something with mathematical symbols, then you ought to know how to say exactly the same thing using only words.**

ANOTHER EXAMPLE

Let us look at the definition of "interval," and let us compare some correct ways of stating it and some incorrect ones.

The definition says:

(D1) an *interval* is a set I of real numbers such that, whenever $a, b \in I$ and $c \in \mathbf{R}$ are such that $a < c < b$, it follows that $c \in I$.

Again, as in our previous examples, all this can be said without letters. For example, here is one possible way:

(D2) an *interval* is a set of real numbers with the property that any time you take two elements of the set then every real number lying between those two elements is also in the set.

It should be clear to you that (D2) and (D1) say exactly the same thing. (If it isn't clear, then you should go on thinking and rethinking this whole thing until it becomes clear.)

Now let us look at some bad definitions of "interval":

EXAMPLE B1: trying to define "interval," a student wrote:

(W1) for a set to be an interval I, $(\forall a \in \mathbf{R}), (\forall b \in \mathbf{R}), (\forall c \in \mathbf{R})$ If $(a \in I), (b \in I)$, there exists a c such that $a \leq c \leq b$, If $c \in I$, then the set I is an interval.

(NOTE: *This a real example. I actually took it from a midterm exam written by a Math 300 student.*) Can you figure out what this means? Can you say what this means without letters? I cannot figure it out. When I grade, this definition of interval is worth 0. (Can you make sense of things like "for all c there exists a c"? I cannot.)

EXAMPLE B2: also trying to define "interval," another student writes:

(W2) an *interval*, I, exists when real numbers a, b are in I, and there is a real number c in the *interval* I if and only if $a < c < b$.

Again, see if you can figure out what this means. As before, ask yourself: how would you say this without letters? I cannot figure it out. When I grade, this definition of interval is also worth 0.

EXAMPLE B3: here is another answer:

(W3) an *interval* I is defined by the following rule where for all real numbers a and b such that $a \in I$ and $b \in I$, there exists a real number c such that $a < c < b$.

Again, let's see if we can figure out what this means. How can we say this without letters? Well, in this case I am afraid we *can* say it. (There are some very wrong things that the student is doing here, like for instance using the expression "the following rule where ..." which is completely ungrammatical English. Perhaps "namely" would have been better than "where"? Also, the student never said that an interval is supposed to be a set of real numbers, so for all we know the set of all apples could be an interval. But let's ignore all this.)

What this student is saying is this: *for a set to be an interval, what has to happen is that whenever you pick two members of this set then there has to be a third number strictly between these two members.* (Why do I say "strictly"? Because the student wrote $a < c < b$, so c cannot be equal to one of the two numbers you picked.)

We could also say what the student seems to be saying using the symbols that we introduced earlier. An interval, according to the student, is a set I such that

$$(\forall a \in I)(\forall b \in I)(\exists c \in \mathbf{R})(a < c < b)\,.$$

(Look at this formula, think it over, and see whether or not it is an adequate translation of what the student wrote. If you *don't* agree, tell us why and we can discuss this in class.)

Now, let us see what sets turn out to be "intervals" according to this definition. Suppose I is a set of real numbers. Suppose I is not the empty set. So I has at least one member. Now, nothing prevents me from picking a to be one of the members of I, and picking b to be exactly that very same element. (The student said "for all $a \in I$ and $b \in I$" so this certainly includes the possibility of taking a and b to be equal.) But if I pick a and b this way, then of course there cannot exist a c such that $a < c < b$. So the definition of interval is not satisfied. So: **according to this student's definition of "interval," the only set of real numbers that is a true interval is the empty set!**

So this definition of "interval" also gets a 0, but this time my reason is different. This time the definition does say something. But what it says is wrong! (Probably, the student did not ask himself/herself the question: "what am I saying here? What does this say in plain English?" You should always ask yourself that question!)

I am sure the student does not believe that the only set of real numbers that is an interval is the empty set. So what we have got here is not failure to know, but **failure to communicate**. This needs serious work, and it's you, the students, who have to do the work. I cannot teach you how to express your thoughts. I can help you, but most of the work has to be done by you. And it will require **hard work** on your part.