MATHEMATICS 300 — SPRING 2000

Introduction to Mathematical Reasoning

(640 - 300, Section 03)

H. J. Sussmann

HANDOUT NO. 1 *(January 20, 2000)*

§1. INFORMATION ON THE COURSE

Course schedule. We meet on **Mondays and Thursdays, 3rd period (11:30 to 12:50)**, in Hickman 115 (at Douglass College).

About the instructor. My *name* is **H. J. Sussmann.** My *office* is **Hill 704**. The *phone extension* is 5-5407. The *e-mail address* is **sussmann@math.rutgers.edu**.

Web page. I have set up a *Web page* for our section of Math 300:

http://www.math.rutgers.edu/~sussmann/math300page.html

All the handouts, including this one, will be available there.

Office hours. My office hours will be:

- Tuesday, fifth period, and Thursday, fifth period, in my office, Hill 704;
- any other time, by appointment.

You should let me know that you are coming to see me *even if you are planning to do so during my regular office hours*. REASON: I don't want to have to stay in my office waiting just in case someone might show up. If I know that you are coming then I will be there, but if you don't tell me that I should expect you then I will feel free to go away and do other things.

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§2. HOMEWORK, MIDTERMS, BONUS POINTS, ETC.

Homework will count for about one third of your grade. There will be *two midterms*, which will count—together—for another third of your grade. The *final exam* will count for the remaining third.

Please make sure to read all the handouts carefully, because they contain important information. For example, this handout contains lots of things you need to know in order to do your homework.

I will be awarding extra *bonus points* to any student who finds a mistake in the book or in my handouts. The number of bonus points will depend on how serious the mistake is. Unfortunately, this is a very subjective matter, and you will have to trust my judgement.

By "mistake" I mean **any kind of mistake**. It could just be a typographical error, or a word not spelled correctly, or a missing comma, or a grammatical error, or a mathematical error. They all count. (Naturally, the mathematical mistakes are the most interesting ones, and will get the largest number of bonus points.)

If more than one student spots a mistake, only the first one to bring it to my attention will get the bonus points.

§3. THE TEXTBOOK

We will be using:

- the book *Proof, Logic and Conjecture: the Mathematician's Toolbox*, by **Robert S. Wolf** (W. H. Freeman and Company, New York, 1998),
- the handouts written by the instructor.

The material of the handouts is an integral part of the course, as much as that of the book.

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§4. READING FOR THE FIRST 2&1/2 WEEKS
(that is, for January 20, 24, 27, 31, and February 3)

- the book's "Note to the Student,"
- the book's Chapters 1 and 2,
- the instructor's Handout No. 1.

The instructor's Handouts Nos. 2 and 3 may be added to this list, if they become available on time.

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§5. SOME IMPORTANT REMARKS

REMARK NO. 1: The "natural numbers"—also called "whole numbers," or "strictly positive integers"—are $1, 2, 3, \ldots$. In particular, **the number 0 is not a natural number.** ♢

REMARK NO. 2: Later on, after we have started to work with set notation, we will use $\mathbb{N}$ as a name for the set of all natural numbers, so

$$\mathbb{N} = \{\, 1, 2, 3, 4, \ldots \},$$

while $\mathbb{Z}_+$ wil stand for the set of **nonnegative integers**, that is,

$$\mathbb{Z}_+ = \{\, 0, 1, 2, 3, 4, \ldots \}.$$

Using that language, the statement

$$0 \in \mathbb{N}$$

(meaning "0 belongs to $\mathbb{N}$," that is, "0 belongs to the set of natural numbers," i.e., "0 is a natural number") will be **FALSE**, whereas the statement

$$0 \in \mathbb{Z}_+$$

(meaning "0 belongs to $\mathbb{Z}_+$," that is, "0 belongs to the set of nonnegative integers," i.e., "0 is a nonnegative integer") will be **TRUE**. ♢

REMARK NO. 3: A real number is "positive" if it is > 0. In particular, 0 **is not positive.** ♢

REMARK NO. 4: **In this course, whenever you are given a problem where you are asked to do something, your answer should be either:**

(a) to do what you were asked to do,

or

(b) to show—that is *prove*—that it cannot be done.

(See, for example, Problems 2 and 4 below, where the correct answer is "what you asked me to do cannot be done.")

Notice that, when the answer is that "it cannot be done," it is not enough for you to *say* that it cannot be done. *You have to tell me why.* In other words, you have to *prove* that it cannot be done.

This remark is very important, and will apply throughout the semester, not just during the first week. So please read it and reread it until you are sure you got the point. ♢

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§6. SOME EXAMPLES OF PROBLEMS WITH CORRECT SOLUTIONS

Here are some problems with correct solutions:

PROBLEM 1: Express the number 26 as a sum of two odd natural numbers.

SOLUTION: $26 = 3 + 23$. ♢

PROBLEM 2: Express the number 27 as a sum of two odd natural numbers.

SOLUTION: This is impossible. REASON: the sum of two odd numbers is always even. Since 27 is odd, it cannot be the sum of two odd numbers. ♢

PROBLEM 3: Prove that for every positive real number x there is another real number y such that $0 < y < x$. (Note[1]: Keep Remark No. 2 in mind!.)

SOLUTION: Let x be a real number such that $x > 0$. Take $y = \frac{x}{2}$. Then $y > 0$. On the other hand, $y < x$, because $y + y = x$ and $y > 0$, so $y = x - y < x$. Therefore $0 < y < x$. ♢

[1] A QUESTION FOR THE STUDENT: Why did I feel it was necessary to have this note here?

PROBLEM 4: Prove that for every natural number x there is another natural number y such that $0 < y < x$. (Note: here too, you must keep Remark No. 2 in mind!")

SOLUTION: This cannot be done. REASON: the statement that we are being asked to prove is false, and a false statement cannot be proved.

The reason that the statement is false is as follows: 1 is a natural number, and there is no natural number y such that $0 < y < 1$, because 1 is the smallest natural number. So if we take x to be 1 then the conclusion is not true. So it is not true that "for *every* natural number x there is another natural number y such that $0 < y < x$." ♢

PROBLEM 5: Express the number 12 as the difference of the squares of two natural numbers.

SOLUTION: $12 = 16 - 4$, so $12 = 4^2 - 2^2$. ♢

PROBLEM 6: Express the number 20 as the sum of the squares of two natural numbers.

SOLUTION: $20 = 4 + 16 = 2^2 + 4^2$. ♢

PROBLEM 7: Express the number 4 as the sum of the squares of two natural numbers.

SOLUTION: This cannot be done. REASON: here are all the ways to express 4 as a sum of two natural numbers:

$$4 = 1 + 3\,, \qquad 4 = 2 + 2\,.$$

In neither of these two equalities are the two terms on the right-hand side squares.

NOTE: One can also write $4 = 0 + 4$, and then $4 = 0^2 + 2^2$, so it would appear that 4 **is** a sum of two squares. Notice, however, that we have made it very clear in Remark No. 1 above that 0 **is not a natural number.** ♢

PROBLEM 8: Express the number 12 as the sum of the squares of two natural numbers.

SOLUTION: This cannot be done. REASON: here are all the ways to express 12 as a sum of two natural numbers:

$$12 = 1 + 11\,,\ \ 12 = 2 + 10\,,\ \ 12 = 3 + 9\,,\ \ 12 = 4 + 8\,,\ \ 12 = 5 + 7\,,\ \ 12 = 6 + 6\,.$$

In none of these expressions are the two terms on the right-hand side squares. ♢

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§7. HOMEWORK ASSIGNMENT NO. 1

This assignment is due on Thursday, January 27.

Before you start writing your homework, read carefully the rest of this handout, in particular §5 and §6, and §8, on "writing mathematics."

Don't worry if some of these problems sem too hard. Just do as much as you can.

- Problems 1, 2, and 5 in the book's pages 12 and 13.
- Prove that every year must have a Friday the 13th.
- Write a correct statement of the Mean Value Theorem[2].
- Express the number 28 as the difference of the squares of two natural numbers.
- Express the number 29 as the difference of the squares of two natural numbers.
- Express the number 30 as the difference of the squares of two natural numbers.

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§8. WRITING MATHEMATICS

You should write so that you can be easily understood by a properly trained English-speaking individual. In particular, this means that you must

- Use *complete English sentences*, that make clearly identifiable *statements* with a *clear meaning* that can can be understood by anyone reading what you wrote. For example:
  - If you tell me that "she is very smart," but you haven't told me who "she" is, then I don't know what you are talking about, so you haven't made a statement with a clear meaning.
  - If you write "$x > 0$," but you haven't told me who "x" is, then I don't know what you are talking about, so you haven't made a statement.
  - If I ask you to state Pythagoras' theorem and you answer "$a^2+b^2 = c^2$," then nobody will know what you are talking about, because you have not said what "a," "b," and "c" are supposed to be.
  - If I ask you to state the Mean Value Theorem and you answer

    $$f'(c) = \frac{f(b) - f(a)}{b - a}\,,$$

    then I have no idea what you are talking about, because you have not told me who "f," "a," "b" and "c" are supposed to be.)

    As an example, let us take Rolle's theorem. Suppose the question is "WRITE A STATEMENT OF ROLLE'S THEOREM." Here are some wrong answers and some right ones.

    WRONG ANSWER NO. 1. $f'(c) = 0$.

    WRONG ANSWER NO. 2. Rolle's theorem says that $f'(c) = 0$.

    WRONG ANSWER NO. 3. For c in the interval $[a, b]$, $f'(c) = 0$.

    WRONG ANSWER NO. 4. For some c in the interval $[a, b]$, $f'(c) = 0$.

    WRONG ANSWER NO. 5. Let f be a function defined on a closed interval $[a, b]$. Assume that f is continuous on the interval $[a, b]$, and differentiable at every point x such that $a < x < b$. Assume, moreover, that $f(a) = f(b)$. Then $a < c < b$ and $f'(c) = 0$.

[2]This is a theorem you must have seen in your Calculus course. If you don't remember what it says, look it up in your Calculus textbook.

WRONG ANSWER NO. 6. Let a, b be real numbers such that $a < b$. Let f be a function defined on the closed interval $[a, b]$. Assume that f is continuous on $[a, b]$, and differentiable at every point x such that $a < x < b$. Assume, moreover, that $f(a) = f(b)$. Then, for c such that $a < c < b$, $f'(c) = 0$.

RIGHT ANSWER[3] NO. 1. Let a, b be real numbers such that $a < b$. Let f be a function defined on the closed interval $[a, b]$. Assume that f is continuous on $[a, b]$, and differentiable at every point x such that $a < x < b$. Assume, moreover, that $f(a) = f(b)$. Then, for some c such that $a < c < b$, $f'(c) = 0$.

RIGHT ANSWER NO. 2. Let a, b be real numbers such that $a < b$. Let f be a function defined on the closed interval $[a, b]$. Assume that f is continuous on $[a, b]$, and differentiable at every point x such that $a < x < b$. Assume, moreover, that $f(a) = f(b)$. Then there exists a real number c such that $a < c < b$ and $f'(c) = 0$.

RIGHT ANSWER NO. 3[4]. Suppose a function is continuous on a closed bounded interval of positive length, and has a derivative at every point of this interval, except possibly at the endpoints. Suppose the values of the functions at both endpoints are equal. Then the value of the derivative of the function is zero somewhere inside the interval.

- Avoid exaggerated or incorrect use of cryptic mathematical notation.
- Explain what you are doing.
- Make sure that letter "variables" are only used if either: (i) it has been said before what these letters stand for, or (ii) they are "closed variables" (or "dummy variables," or "bound variables") in the sense that will be discussed in detail in class, and also in subsequent handouts.
- Provide proper connectives between equations as well as between ideas.
- Make sure that all the rules of English grammar (including those of spelling and punctuation) are strictly obeyed.
- Try to say things correctly, following all the rules, but *in your own words.* Please *no rote learning.* If you have to memorize a definition or a statement, then this is an indication of lack of understanding.
- Please proofread carefully what you hand in. Ideally, you should read and reread and revise almost any formal communication. **Neatness and clarity count**, as you well know if you've tried to read any complicated document.

[3] A QUESTION FOR THE STUDENT: Why am I saying that "Right answer No. 1" is right, while "Wrong answer No. 6" is wrong? Aren't they basically the same? What's the difference?

[4] Pay attention to this example. The lesson you should learn from it is that, in order to be right, it is not necessary to use mathematical symbols. Actually, *anything you say with symbols you should also be able to say it with just plain English words.* We use mathematical symbols because they make things shorter, and for no other reason. For example: the statement "if a, b are real numbers then $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$" is just a shorter way of saying "if you take two numbers, add them, and compute the cube of the sum, then you get the same result as if you had added the cube of the first number, plus three times the square of the first number times the second number, plus three times the first number times the square of the second number, plus the cube of the second number." It is much shorter, of course, when you say it with letters, but you ought to be able to say with words if asked.

- *Do not assume that the people reading your paper can read your mind. Do assume that they are intelligent, but also assume that they are busy, and cannot and will not spend an excessive amount of time puzzling out your meaning. Communication is difficult, and written technical communication is close to an art.*

Effective written exposition will be worth at least 50% of your grade. Conversely, bad or unclear exposition may be penalized as much as 50% of the grade or even more.

- The best reference known to me on effective writing is *The Elements of Style* by Strunk and White, a very thin paperback published by Macmillan. It isn't expensive, and it is easy to read. I recommend it.

Pay attention to presentation, especially for the homework:

- If you have to cross out lots of words, then you should rewrite the whole thing anew, **cleanly and neatly**. If you are not willing to spend some of your time doing this; if what you hand in shows that you were in a hurry and that you did not make the effort to write things neatly and properly, then there is no reason for me to spend any of my time reading what you wrote, and I will not do it.
- Use a pen. Never use a pencil.
- Use any color other than red (for example, black, blue, or green), but DO NOT USE RED. (Reason: The use of red is reserved for the instructor's comments.)
- If your handwriting is too hard to read, please consider typing or using a word processor.
- Make sure that **your name appears in every sheet of paper you hand in**, and that if you are handing in more than one sheet then the sheets are **stapled** and the pages are **numbered**.
- If you tear off the sheets from a spiral notebook, please make sure before you hand them in that there are none of those ugly shreds of paper at the margins. Use scissors, or a cutter, if necessary.

Homework assignments that fail to meet the above requirements will be marked "unacceptable" and returned unread.

Keep in mind that

If you hand in a homework assignment that

- **is written carelessly or in a hurry,**

or

- **has lots of words crossed out,**

or

- **has unreadable handwriting,**

or

- **has pages that are unstapled or unnumbered or fail to show your name,**

or

- **has shreds of paper at the margins**

(in other words, anything that looks ugly, or anything that you would be ashamed to hand in, say, if you were writing a letter to apply for a job) then this is a form of disrespect towards your instructor, and that's exactly how you should expect that your instructor will interpret it.