TWO NEW METHODS FOR MOTION PLANNING FOR CONTROLLABLE SYSTEMS WITHOUT DRIFT

Héctor J. Sussmann*
SYCON- Rutgers Center for Systems and Control
Department of Mathematics, Rutgers University, New Brunswick, NJ 08903

Summary

We describe some recent new approaches for motion planning for nonholonomic systems without drift, developed in joint work of with G. Lafferriere and W. Liu. The main idea is to relate the trajectories of the given system to those of a "Lie bracket extension," and to seek ways to obtain trajectories of the original system that achieve the same effect as a trajectory of the extended system.

Keywords: motion planning, nonholonomic systems.

1. Introduction

The purpose of this note is to describe two new general procedures for constructing (open-loop) control functions for systems without drift, i.e. for solving the *Motion Planning Problem* (MPP) for nonholonomic systems without drift. Following the early work of Brockett and Sastry, cf. [3], [11], many methods have been proposed for solving this problem, e.g. by Sastry, Hauser, Murray and Li. In this note we outline some results, described in more detail in [9], [10], [16], [17], [18], that provide a new approach to the problem.

Consider a control system

$$\Sigma: \quad \dot{x} = \sum_{i=1}^{m} u_i f_i(x), \tag{1}$$

where $f_1, f_2, \ldots, f_m$ are smooth vector fields in $\mathbb{R}^n$. An *admissible control* for System (1) is a Lebesgue integrable function $t \to u(t) = (u_1(t), \ldots, u_m(t)) \in \mathbb{R}^m$ defined on an interval $[0,T]$ for some $T \geq 0$. A *trajectory* for an admissible control $u(\cdot)$ is an absolutely continuous curve $[0,T] \ni t \to x(t) \in \mathbb{R}^n$ that satisfies $\dot{x}(t) = \sum_{i=1}^{m} u_i(t) f_i(x(t))$ for almost all $t \in [0,T]$.

We can state the MPP (Motion Planning Problem) as follows: given two points p, q such that there exists an admissible path from p to q, find one explicitly. Clearly, this is not a purely mathematical question that can be answered by proving a theorem. What is desired is an *algorithm*, and whether a particular algorithm counts as satisfactory is to a large extent a subjective matter.

In this note we will limit ourselves to *real analytic* systems, i.e. systems such that the vector fields f_i are real analytic. Moreover, we will assume that the system is *completely controllable*, i.e. that for every pair p, q there exists an admissible path from p to q. The problem is then to produce an algorithm that computes one such path in a reasonably explicit form.

For real analytic systems, complete controllability is equivalent to the well-known LARC (Lie algebra rank condition): if $L(f)$ denotes the Lie algebra of vector fields generated by $f = \{f_1, \ldots, f_m\}$, and $L(f)(x) \stackrel{\text{def}}{=} \{X(x) : x \in L(f)\}$, then $L(f)(x)$ must be equal to $\mathbb{R}^n$ for all x. Although the LARC guarantees for every p, q the existence of a control function steering p to q, it is not obvious how to construct such a function explicitly. A number of papers (cf. [4], [11], [12], [13]) have proposed various constructions.

The approach proposed here proceeds by first picking a "Lie bracket extension" of (1), i.e. a system

$$\Sigma_{ext}: \quad \dot{x} = \sum_{k=1}^{r} v_k f_k(x) \ , \tag{2}$$

where the first m vector fields $f_1, \ldots, f_m$ are the same as in (1), and the new admissible directions of motion $f_{m+1}, \ldots, f_r$ are Lie brackets of the f_i, $i \in \{1, \ldots, m\}$.

The new f_i are selected in such a way that, for all x in some region $\mathcal{R}$ of interest, the linear span of the $f_i(x)$, $i = 1, \ldots, r$, is the whole space $\mathbb{R}^n$. (Such a choice is always possible, if the LARC holds, when $\mathcal{R}$ is bounded. In many cases of interest, e.g. when the f_i are polynomial vector fields, one can take $\mathcal{R} = \mathbb{R}^n$.)

We then pick an *arbitrary* (not necessarily admissible) smooth curve γ joining p to q, and try to use γ to construct an admissible control $u(\cdot)$ for (1) that gives rise to a trajectory $\tilde{\gamma}$ that in some sense solves our problem.

For this purpose, we begin by writing

$$\dot{\gamma}(t) = \sum_{k=1}^{r} v_k(t) f_k(\gamma(t)) \ . \tag{3}$$

(This is possible because the span of the vectors $f_k(\gamma(t))$, $k \in \{1, \ldots, r\}$, is the whole space $\mathbb{R}^n$.)

We refer to the collection of coefficient functions v_k as an "extended input." We can now restate our problem as that of producing, for a given extended input v, an ordinary input u that achieves the same result as v. (We require that the interval $[0, T]$ on which u is defined be the same as the domain of definition of v.)

The precise meaning of this depends on what we mean by "achieving the same result." Moreover, we will be content with "achieving the same result within a prescribed error margin ε." The two possibilities we will consider here are:

I. we want u to give rise to a trajectory $\tilde{\gamma}$ that goes from p to q, or at least to a point q' such that $||q' - q|| \leq \varepsilon$.

II. we want u to give rise to a trajectory $\tilde{\gamma}$ such that $||\tilde{\gamma}(t) - \gamma(t)|| \leq \varepsilon$ for all $t \in [0, T]$.

The work of [9], [10], [14] deals with Problem **I**. In [14] it was shown how to compute a control $u(\cdot) \in L^1([0, T], \mathbb{R}^m)$ for (1) in such a way that:

A. if the Lie algebra $L(f)$ is nilpotent, and the vector fields f_i, $i \in \{1, \ldots, m\}$ are complete, then $u(\cdot)$ steers p to q *exactly*,

B. in the general case, $u(\cdot)$ steers p to a point q' that satisfies the error bound $||q' - q|| \leq C||p - q||^\rho$, where $\rho > 1$ and C is a constant depending only on the region $\mathcal{R}$.

This is done as follows: pick an integer N such that all the Lie brackets f_k that occur in (2) are of degree $\leq N$. Then solve the "formal equation"

$$\dot{S}(t) = S(t)\Big(\sum_{k=1}^{r} v_k(t) X_k\Big) \ , \ S(0) = 1 \ , \qquad (4)$$

in $\mathbf{A}_N(X_1, \ldots, X_m)$, the free nilpotent associative algebra of step $N+1$ in the m noncommuting indeterminates $X_1, \ldots, X_m$. (Here the expressions X_k for $k > m$ are chosen so that $E(X_k) = f_k$, where E is the map that, for $i \in \{1, \ldots, m\}$, plugs in each f_i for the corresponding X_i. For example, if (2) is the system $\dot{x} = \Big(v_1 f_1 + v_2 f_2 + v_3[f_1, f_2] + v_4[f_1, [f_1, f_2]]\Big)(x)$, so that $f_3 = [f_1, f_2]$ and $f_4 = [f_1, [f_1, f_2]]$, then we can take $N = 3$, $X_3 = [X_1, X_2]$ and $X_4 = [X_1, [X_1, X_2]]$, so (4) becomes $\dot{S} = S\Big(v_1 X_1 + v_2 X_2 + v_3[X_1, X_2] + v_4[X_1, [X_1, X_2]]\Big)$.) It is then easy to prove that $S(T)$ is an exponential Lie series, and from this it follows that, if B is a suitably chosen basis of $\mathbf{L}_N(X_1, \ldots, X_m)$ —the free nilpotent Lie algebra of step $N+1$ in the indeterminates $X_1, \ldots, X_m$— then $S(T)$ can be written as a product of exponentials $\exp(\alpha_B B)$. One can then compute u by simply doing it separately for each factor and then concatenating the results. A detailed explanation is given in [14].

Let us refer to the above as Algorithm 1.

It follows from **B** that there is a $\Delta > 0$ (that was called the *critical distance* in [9], [10], [14]) such that, if $||p - q|| \leq \Delta$, then the control $u(\cdot)$ obtained using Algorithm 1 steers p to a q' such that $||q' - q|| \leq \frac{1}{2}||p - q||$. Using this fact, we gave in [9], [10], [14] a rapidly convergent iterative algorithm for the MPP, that will be referred to here as Algorithm 2:

I. set $\tilde{p} := p$,

II. let $\tilde{q}$ be the point in the straight-line segment from $\tilde{p}$ to q, but at a distance from $\tilde{p}$ equal to $\min(\Delta, ||\tilde{p} - q||)$,

III. apply Algorithm 1 to the pair $(\tilde{p}, \tilde{q})$, and let $\bar{q}$ be the point to which the resulting control steers $\tilde{p}$,

IV. if $||\bar{q} - q|| \leq \varepsilon$ then set $q' := \bar{q}$ and stop; else set $\tilde{p} := \bar{q}$ and go to **II**.

Here $\varepsilon > 0$ is our prescribed error margin. It was shown in [14] that this algorithm will produce in at most $2\left[\frac{||p-q||}{\Delta}\right] + \log_2\left[\frac{\Delta}{\varepsilon}\right]$ steps a trajectory leading from p to a q' such that $||q' - q|| \leq \varepsilon$.

The work of [16], [17], [18] deals with Problem **II**, i.e. with a method for constructing a sequence of admissible trajectories γ_j that converge uniformly to γ. The existence of such an approximating sequence was already proved by G.W. Haynes and H. Hermes in [5], but we provide a simple constructive algorithm, that will be referred to here as Algorithm 3. A more detailed description of this algorithm is given in §2 below.

One can then combine Algorithms 2 and 3 as follows: we use Algorithm 3 to approximate the given trajectory within an error not greater than the critical distance, and then use Algorithm 2 to steer to q the terminal point $\tilde{q}$ of the approximating trajectory $\tilde{\gamma}$.

Notice that this method is applicable to solving the *MPP with obstacles*. Indeed, suppose we want to find an admissible path that avoids some closed set C. If we can find any path γ at all, i.e. if we can solve the MPP with obstacle avoidance for the totally unconstrained dynamics $\dot{x} = v$, v arbitrary, then we obtain a solution of the same problem for our system with constrained dynamics because, for large j, the curve γ_j will necessarily avoid C as well.

2. The approximation algorithm

We now begin the description of an algorithm to approximate a given path by admissible trajectories of (1). We will concentrate on outlining the main ideas. The details and proofs, which are quite technical and long, are given in [17] and [16].

Our main technical tool is a convergence theorem proved in [17], that we now review. This result establishes the convergence, as $j \to \infty$, of the solutions x^j of an initial value problem

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{m} u_i^j(t) f_i(x(t)) \ , \qquad (5) \\ x(0) &= x_0 \ , \end{aligned}$$

where $\{u^j\}_{j=1}^{\infty}$ is a sequence of control functions $u^j = (u_1^j, u_2^j, \ldots, u_m^j)$ belonging to $L^1([0,T], \mathbb{R}^m)$, to the solution of

$$\begin{aligned} \dot{x}(t) &= \sum_{|I| \le r} u_I^{\infty}(t)[f_I](x(t)) \ , \qquad (6) \\ x(0) &= x_0, \end{aligned}$$

where, for any index $I = (i_1, \ldots, i_k)$, $1 \le i_j \le m$, we define $|I| = k$ and write $[f_I]$ to denote the Lie bracket $[\cdots[[f_{i_1}, f_{i_2}], \cdots, f_{i_k}]$. Here the $u_I^{\infty}(t)$ are certain functions in $L^1[0,T]$ to be defined below. Convergence of the solutions follows if $\{u^j\}_{j=1}^{\infty}$ satisfies a convergence condition involving the u_I^{∞}. The main part of this condition is the requirement that all the iterated integrals

$$U_I^j(t) \stackrel{\text{def}}{=} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} u_{i_1}^j(t_1) \ldots u_{i_k}^j(t_k) dt_k \ldots dt_1$$

should converge uniformly, as $j \to \infty$, to the indefinite integrals H_I of certain L^1 functions h_I, for all multiindices I such that $k \le r$.

The u_I^{∞} are then computed in terms of the h_I by the algebraic formulas

$$v_I(t) = \sum_k (-1)^k \sum_{J_0 J_1 \cdots J_k = I} \dot{H}_{J_0} H_{J_1} \ldots H_{J_k} \ , \qquad (7)$$

$$u_I^{\infty} = \frac{(-1)^{|I|-1}}{|I|} v_I \ , \qquad (8)$$

where the inner summation in (7) is over all ways of expressing the multiindex I as a concatenation $J_0 J_1 \cdots J_k$ of $k+1$ indices.

The convergence condition contains, in addition, the requirement that the L^1 norms of the integrands of some iterated integrals $\tilde{u}_I^j$ of order r be uniformly bounded.

The general definition of the $\tilde{u}_I^j$ is most easily formulated in terms of the *chronological product* operation $(u, v) \to u * v$ (cf. [1]) from $L^1([0,T]) \times L^1([0,T])$ to $L^1([0,T])$, defined by $(u * v)(t) = Iu(t)v(t)$ where $Iu(t) = \int_0^t u(s)\, ds$. We have

$$\tilde{u}_{i_1 i_2 \ldots i_k}^j \stackrel{\text{def}}{=} \qquad (9)$$

$$\sum_{r=1}^{k} (-1)^{r+1} v_{i_1 i_2 \ldots i_{k-r}} * u_{i_{k-r+1}}^j * u_{i_{k-r+2}}^j * \ldots * u_{i_k}^j,$$

where we are using the convention that $w_1 * w_2 * \ldots * w_k$ means $(\ldots(w_1 * w_2) * \ldots) * w_k$.

Formula (9) then implies the recursive relation

$$\tilde{u}_{i_1 i_2 \ldots i_{k+1}}^j = (v_{i_1 i_2 \ldots i_k} - \tilde{u}_{i_1 i_2 \ldots i_k}) * u_{i_{k+1}}^j \qquad (10)$$

which, together with the formulas $\tilde{u}_i^j \stackrel{\text{def}}{=} u_i^j$, constitute an alternative way of defining the $\tilde{u}_I^j$.

The precise statement of the convergence theorem is:

Theorem 1 *Let $\{u^j\}_{j=1}^{\infty}$ be a sequence of input functions belonging to $L^1([0,T], \mathbb{R}^m)$. Let r be a positive integer. Assume that*

(1) *the iterated integrals U_I^j, for $|I| \le r$, converge uniformly to absolutely continuous functions H_I;*

(2) *the functions v_I, $u_I^{\infty} : [0,T] \to \mathbb{R}$ are defined, for $|I| \le r$, by (7) and (8);*

(3) *there exists a finite constant C such that the bound $\int_0^T |\tilde{u}_I^j(t)| dt \le C$ holds for all multi-indices I such that $|I| = r$, where the $\tilde{u}_I^j$ are defined by (9).*

Assume that $f_1, f_2, \ldots, f_m$ are vector fields of class C^{r-1} on a manifold M of class C^r. Assume also that the initial value problem (6) has a unique solution $t \to x^{\infty}(t)$ which is defined on the interval $[0,T]$. Then

I. *for sufficiently large j, there exists a solution $t \to \{x^j(t)\}$ of (5), defined on the whole interval $[0,T]$,*

II. *the $x^j(t)$ converge to $x^{\infty}(t)$ as $j \to \infty$, uniformly with respect to t for $t \in [0,T]$.*

The important point is that, in the limit as $j \to \infty$, we may obtain trajectories of an "extended equation" (2). For a simple illustration of this phenomenon, fix a $T > 0$ and consider the system

$$\begin{aligned} \dot{x}_1 &= u_1 \ , \\ \dot{x}_2 &= u_2 \ , \\ \dot{x}_3 &= u_2 x_1 \ , \end{aligned}$$

on $[0,T]$, with initial condition $x_0 = (0,0,0)$. If $u^j(t) = \sqrt{j}(\cos jt, \sin jt)$, then the u^j "converge to 0" in the sense that their indefinite integrals converge to 0 uniformly. However, a simple calculation shows that the trajectories $x^j(t)$ do not converge to zero, but converge uniformly to the curve $\gamma : t \to (0,0,\frac{t}{2})$. On the other hand, the convergence condition of Theorem 1 is satisfied with $r=2$. The vector fields f_1, f_2 have components $(1,0,0)$ and $(0,1,x_1)$, respectively, and the Lie bracket $f_3=[f_1,f_2]$ has components $(0,0,1)$. The theorem then implies that $x^j(t)$ converges uniformly to the solution x^∞ of $\dot{x}=\frac{1}{2}f_3(x)$, $x(0)=x_0$, i.e. precisely to the curve γ.

Remark. The convergence theorem is a generalization of the results of Kurzweil and Jarnik, cf. [6], [7], [8]. These in turn are far reaching generalizations of a classical result on continuous dependence of trajectories with respect to the input, according to which, if a sequence of inputs $u^j \in L^1([0,T],\mathbb{R}^m)$ converges to a limit u^∞ in the weak topology of $L^1([0,T],\mathbb{R}^m)$, then the corresponding trajectories (for some fixed initial condition) also converge.

3. An Approximate Tracking Algorithm.

Let us now assume that our control system (1) is completely controllable. Then the LARC implies that the right-hand side of (2) contains (if r is large enough) sufficiently many brackets so as to span all possible directions at each point in some prescribed compact region $\mathcal{R}$. Thus, given any smooth path in $\mathcal{R}$, we can express it as a solution of (2) for *some* choice of the functions v_i. The collection $\{v_i\}_{i=1}^r$ will be called the *family of coefficient functions* associated to (2). A control sequence u^j will be said to *converge* to a family of coefficient functions $\{v_i\}_{i=1}^r$ associated to (2) if the admissible trajectories corresponding to u^j converge to those of (2). The problem considered here is then: given a family $\{v_i\}_{i=1}^r$ of coefficient functions, find explicitly a sequence of inputs u^j that converges to $\{v_i\}_{i=1}^r$.

In order to solve this problem, the basic idea is to use "highly oscillatory inputs," i.e. inputs whose components are superpositions of functions of the form $j^\alpha \eta_\omega(t)\cos j\omega t$ and $j^\alpha \eta_\omega(t)\sin j\omega t$ for some frequencies ω. Roughly speaking, we will treat each term $v_i f_i$ in the right-hand side of (2) separately, and produce for each part a sequence of control functions that will converge to it. Although the input to trajectory map is nonlinear, it turns out that, if the frequencies associated to each part are independent (in a sense that will be made clear below), then an "asymptotic high frequency superposition principle" holds, namely, the control sequences associated to each part do not interfere with one another in the limit.

In order to show the main ideas, let us consider the case of a two-input system in $\mathbb{R}^5$, i.e. a system

$$\dot{x}(t) = u_1(t)f_1(x(t)) + u_2(t)f_2(x(t)),$$

where f_1, f_2 are smooth vector fields in $\mathbb{R}^5$. Assume that $\{f_1,f_2,[f_1,f_2],[f_1,[f_1,f_2]],[f_2,[f_1,f_2]]\}$ spans $\mathbb{R}^5$ everywhere. For any given p,q, choose a smooth path $t\to\gamma(t)\in\mathbb{R}^5$ on $[0,1]$ connecting p and q. Then, by the span condition, there exist smooth functions $v_1(t),\ldots,v_5(t)$ on $[0,1]$ such that $t\to\gamma(t)$ is a solution of the equation

$$\begin{aligned}\dot{x}(t) &= v_1(t)f_1(x(t)) + v_2(t)f_2(x(t))\\ &+ +v_3(t)[f_1,f_2](x(t)) + v_4(t)[f_1,[f_1,f_2]](x(t)\\ &+ v_5(t)[f_2,[f_1,f_2]](x(t)) \qquad (11)\end{aligned}$$

with initial condition $x(0)=p$.

There are 5 terms in the right hand side of (11). We will handle $v_3[f_1,f_2]$, $v_4[f_1,[f_1,f_2]]$, $v_5[f_2,[f_1,f_2]]$ separately and write down a control sequence for each of them.

Take three groups of frequencies $\Omega_k = \{\omega_{k,1},\omega_{k,2}\}$, $k=1,2,3$. Assume that $\omega_{k,l}$ satisfy

1. $\omega_{1,1}+\omega_{1,2}=0$, $2\omega_{2,1}+\omega_{2,2}=0$, $\omega_{3,1}+2\omega_{3,2}=0$;
2. $\Omega_1,\Omega_2,\Omega_3$ are independent in the sense that $\omega_{k,l}$ do not satisfy any relation

$$\alpha_1\omega_1+\alpha_2\omega_2+\alpha_3\omega_3=0,\ \omega_k\in\Omega_k, k=1,2,3,$$

where the α_k are integers between -2 and 2, other than the trivial one that corresponds to $\alpha_1=\alpha_2=\alpha_3=0$.

For any functions $\eta_1,\eta_2,\eta_{\omega_{k,l}}$, $k=1,2,3, l=1,2$, of class C^1 on $[0,1]$, let

$$\begin{aligned}u_1^j(t) &= \eta_1(t) + j^{\frac{1}{2}}\eta_{\omega_{1,1}}(t)\sin\omega_{1,1}jt\\ &+ j^{\frac{2}{3}}\eta_{\omega_{2,1}}(t)\cos\omega_{2,1}jt + j^{\frac{2}{3}}\eta_{\omega_{3,1}}(t)\cos\omega_{3,1}jt\\ u_2^j(t) &= \eta_2(t) + j^{\frac{1}{2}}\eta_{\omega_{1,2}}(t)\cos\omega_{1,2}jt\\ &+ j^{\frac{2}{3}}\eta_{\omega_{2,2}}(t)\cos\omega_{2,2}jt + j^{\frac{2}{3}}\eta_{\omega_{3,2}}(t)\cos\omega_{3,2}jt\,.\end{aligned}$$

Then we have the following

Theorem 2 *For any frequencies $\omega_{k,l}, k=1,2,3, l=1,2$, satisfying 1 and 2 and any functions $\eta_1,\eta_2,\eta_{\omega_{k,l}}$, $k=1,2,3, l=1,2$, of class C^1 on $[0,1]$, $u^j=(u_1^j,u_2^j)$ converges to the family of coefficient functions*

$$(\eta_1,\eta_2,-\frac{\eta_{\omega_{1,1}}\eta_{\omega_{1,2}}}{2\omega_{1,1}},-\frac{\eta^2_{\omega_{2,1}}\eta_{\omega_{2,2}}}{8\omega^2_{2,1}},-\frac{\eta_{\omega_{3,1}}\eta^2_{\omega_{3,2}}}{4\omega_{3,1}\omega_{3,2}}) \qquad (12)$$

associated to (11).

Now it is easy to see how to choose $\eta_k, \eta_{\omega_{k,l}}$ to produce any given $(v_1, v_2, v_3, v_4, v_5)$. We can simply let

$$\begin{aligned} \eta_1 &= v_1, & \eta_2 &= v_2, \\ \eta_{\omega_{1,1}} &= 1, & \eta_{\omega_{1,2}} &= -2\omega_{1,1}v_3, \\ \eta_{\omega_{2,1}} &= 1, & \eta_{\omega_{2,2}} &= -8\omega_{2,1}^2 v_4, \\ \eta_{\omega_{3,1}} &= -4\omega_{3,1}\omega_{3,2}v_5, & \eta_{\omega_{3,2}} &= 1. \end{aligned}$$

From (12) we see that $\omega_{1,l}, \eta_{\omega_{1,l}}$ affect only the coefficient of $[f_1, f_2]$ and $\omega_{k,l}, \eta_{\omega_{k,l}}$ affect only the coefficients of $[f_1,[f_1,f_2]], [f_2,[f_1,f_2]]$ respectively for $k = 2$ and 3. This is because the frequencies corresponding to each part are independent. The proof of this Theorem follows easily from the more general results of [17].

For the general extended system (2), a similar idea works, although the algebra becomes much more complicated. (Some of the reasons for the algebraic difficulties are: (a) that the expression of a vector as a linear combination of brackets is not unique, even if the special brackets $[f_I]$ are used, so it becomes important to make a suitable choice of a basis; (b) that the "asymptotic superposition principle" only makes it possible to separate contributions that correspond to brackets that do not have the same degree in all the components. In order to separate brackets that are linearly independent but have the same degree in all the components, further work is required.)

Let r be the largest degree of the brackets occurring in (2). We take our approximating sequence to be of the form

$$u_k^j(t) = \zeta_{k,0}(t) + \sum_{n=2}^{r} j^{\frac{n-1}{n}} \sum_{\omega\in\Omega(n,k)} \zeta_{k,\omega}(t)e^{ij\omega t},$$

where $\zeta_{k,0}, \zeta_{k,\omega}$ are complex-valued functions of class C^1 on some interval $[0,T]$, and the $\Omega(n,k)$ are suitably chosen finite sets of frequencies. (It is convenient in general to use $e^{ij\omega t}$ instead of $\cos\omega jt$ and $\sin\omega jt$. Of course we need to impose conditions on the $\zeta_{k,\omega}(t)$ to make sure that $u_k^j(t)$ is real.) It turns out that under similar conditions on the frequencies as 1 and 2 in the example above, for any functions $\zeta_{k,0}, \zeta_{k,\omega}$ of class C^1, u^j will converge to some coefficient functions $\{\tilde{v}_i\}_{i=1}^r$ associated to (2). And it is always possible to choose $\zeta_{k,0}, \zeta_{k,\omega}$ to make $\{\tilde{v}_i\}_{i=1}^r$ equal to any prescribed $\{v_i\}_{i=1}^r$, provided that v_i are differentiable. For details about how to choose the frequencies in each $\Omega(n,k)$ and the functions $\zeta_{k,0}, \zeta_{k,\omega}$, the reader is referred to [16].

Remark. As explained above, delicate algebraic problems arise from the need to make a suitable choice of basis. It turns out to be important to use a *P. Hall basis*. (Cf. [2]. Some of the advantages of using P. Hall basis for the Motion Planning Problem were pointed out in [14]. The specific reason why they are important for our approximation algorithm is the "product expansion of the Chen series" given in [15].)

4. A Path Planning Algorithm.

For controllable systems without drift we have described in Section 3 an algorithm to approximate a given smooth curve by admissible trajectories. Let us call it Algorithm 3. Given any pair p, q in $\mathcal{R}$, we then have the following algorithm to produce an admissible control u that steers p to some point $\tilde{q}$ in a ball $B_\delta(p)$:

Algorithm 4:

Step 1: Find a control $v = \{v_i\}_{i=1}^r$ for the extended system (2) steering p to q. For this we can choose any smooth curve $\gamma : [0,T] \to \mathbb{R}^n$ in $\mathcal{R}$ connecting p and q. Having chosen γ, then write the tangent vector $\dot{\gamma}(t)$ as a linear combination of $f_1(\gamma(t)), \ldots, f_r(\gamma(t))$. The resulting coefficients are the v_i.

Step 2: Find a control sequence u^j that converges to $\{v_i\}_{i=1}^r$. Then u^j will steer p to some point q_j near q. Take j large enough such that $\|q - q_j\| \le \delta$. Let $u = u^j$.

Now, for any bounded region $\mathcal{R}$ in $\mathbb{R}^n$, let us use $\Delta(\mathcal{R})$ to denote the critical distance corresponding to $\mathcal{R}$. We can then combine Algorithms 2 and 4 to get the following algorithm that gives an alternative solution of solving the MPP for nonholonomic systems without drift.

Algorithm 5 (a path finding algorithm):

Input: Two points p and q in some bounded region $\mathcal{R}$ in $\mathbb{R}^n$ and a prescribed positive error margin ε.

Step 1: Use Algorithm 4 with $\delta = \Delta(\mathcal{R})$ to find an admissible control $\tilde{u}$ that steers p to $\tilde{q}$

Step 2: Apply Algorithm 2 to find a control $\hat{u}$ that steers $\tilde{q}$ to some q' with $\|q - q'\| < \varepsilon$.

Output: the control u that results from concatenating $\tilde{u}$ and $\hat{u}$.

Remark 1. From the theoretical point of view one can achieve the same result by applying Algorithm 4 with $\delta = \varepsilon$. But this will in general require a very large j. So, instead, if p and q are far apart, we first apply Algorithm 4 to steer p to a point near q. Once we are near q, we use Algorithm 2, which converges very quickly if the two points are near (precisely if $\|p-q\| \le \Delta(\mathcal{R})$), to move even closer to q.

Remark 2. It was pointed out in [14] that the method proposed there has the drawback that one needs knowledge of the critical distance in order to apply it. Unfortunately, the same applies to the path-finding algorithm presented above.

Acknowledgment. We thank G. Lafferriere, W. Liu and L. Gurvits for their useful comments.

References

[1] Agrachev, A.A., and R.V. Gamkrelidze, "The Exponential Representation of Flows and Chronological Calculus," *Math. Sbornik* **107** (149) (1978), pp. 467-532; Engl. Transl. in *Math. USSR Sbornik* **35** (1979), pp. 727-785.

[2] N. Bourbaki, *Elements of Mathematics: Lie Groups and Lie Algebras*, Hermann (1975) Paris.

[3] Brockett, R.W., "Control Theory and Singular Riemannian Geometry," *New Directions in Applied Mathematics*, P.H. Hilton and G.S. Young Eds., Springer-Verlag, New York, 1981, pp 11-27.

[4] Hauser, J., S.S. Sastry and P.V. Kokotovic, "Nonlinear Control Via Approximate Input-output Linearization: The Ball and Beam Example," *Proc. 28th IEEE CDC, Tampa, Florida, Dec. 1989,* IEEE Publications, 1989, pp. 1987-1993.

[5] Haynes, G.W., and H. Hermes, "Nonlinear Controllability Via Lie Theory," *SIAM J. Control and Optimization,* **8**, 1970, pp 450-460.

[6] Kurzweil, J., and J. Jarnik, "Limit Process in Ordinary Differential Equations," *Journal of Applied Mathematics and Physics* **38**, March 1987, pp 241-256.

[7] Kurzweil, J. and J. Jarnik, "Iterated Lie Brackets in Limit Processes in Ordinary Differential Equations," *Results in Mathematics* **14**, 1988, pp 125-137.

[8] Kurzweil, J. and J. Jarnik, "A Convergence Effect in Ordinary Differential Equations," in *Asymptotic Methods of the Mathematical Physics*, (Russian), 301, 'Naukova Pumka', Kiev, 1989.

[9] Lafferriere, G., "A General Strategy for Computing Steering Controls of Systems without Drift," Submitted to the 1991 CDC.

[10] Lafferriere, G., and Sussmann, H.J., "Motion Planning for Controllable Systems without Drift," *Proc. IEEE Int. Conf. Robotics and Automation,* Sacramento, CA, April 1991.

[11] Murray, R.M., and S.S. Sastry, "Steering Controllable Systems," *Proc. 29th IEEE CDC, Honolulu, Dec. 1990,* IEEE Publications, 1990, pp. 2097-2101.

[12] Murray, R.M., and S.S. Sastry, "Grasping and Manipulations Using Multifingered Robot Hands," Memorandum No. UCB/ERL M90/24, Electronics Research Laboratory, Univ. of California, Berkeley, March 1990.

[13] Sastry, S.S., and Zexiang Li, "Robot Motion Planning with Nonholonomic Constraints," *Proc. 28th IEEE CDC, Tampa, Florida, Dec. 1989,*, IEEE Publications, 1989, pp 211-216.

[14] Lafferriere, G. and H.J. Sussmann, "Motion Planning for Controllable Systems without Drift: A Preliminary Report," Rutgers University Systems and Control Center Report Sycon 90-04, June 1990.

[15] Sussmann, H.J., "A Product Expansion for the Chen Series," in *Theory and Applications of Nonlinear Control Systems*, C. Byrnes and A. Lindquist Eds., North-Holland, 1986, pp. 323-335.

[16] Sussmann, H.J., and Wensheng Liu, " Limits of Highly Oscillatory Controls and the Approximation of General Paths by Admissible Trajectories," Rutgers University Systems and Control Center Report Sycon 91-02, February 1991. Submitted to the 1991 IEEE Conference on Decision and Control.

[17] Sussmann, H. J., and W. Liu, "Limiting Behavior of Trajectories for Highly Oscillating Controls," in preparation.

[18] Sussmann, H. J., and W. Liu, "Construction of Approximating Sequences for Trajectories of Extended Control Systems," in preparation.