Some worked out examples of logical proofs

Example 1. Suppose $P(x)$ and $Q(x)$ are one-variable predicates. Prove the following:

$$(\forall x)(P(x) \wedge Q(x)) \Leftrightarrow ((\forall x)P(x) \wedge (\forall x)Q(x)) \ .$$

SOLUTION. Here is a proof,

Step 1.	Assume $(\forall x)(P(x) \wedge Q(x))$.	[Assumption]
Step 2.	Let a be arbitrary.	[Declaration]
Step 3.	$P(a) \wedge Q(a)$	[Rule $\forall_{use}$, from 1 & 2]
Step 4.	$P(a)$	[Rule $\wedge_{use}$, from 3]
Step 5.	$(\forall x)P(x)$	[Rule $\forall_{get}$, from 2 & 4]
Step 6.	Let a be arbitrary.	[Declaration]
Step 7.	$P(a) \wedge Q(a)$	[Rule $\forall_{use}$, from 1 & 6]
Step 8.	$Q(a)$	[Rule $\wedge_{use}$, from 7]
Step 9.	$(\forall x)Q(x)$	[Rule $\forall_{get}$, from 6 & 8]
Step 10.	$(\forall x)P(x) \wedge (\forall x)Q(x)$	[Rule $\wedge_{get}$, from 5 & 9]
Step 11.	$(\forall x)(P(x) \wedge Q(x)) \Rightarrow ((\forall x)P(x) \wedge (\forall x)Q(x))$	[R. $\Rightarrow_{get}$, fr. 1 & 10]
Step 12.	Assume $(\forall x)P(x) \wedge (\forall x)Q(x)$.	[Assumption]
Step 13.	$(\forall x)P(x)$	[Rule $\wedge_{use}$, from 12]
Step 14.	$(\forall x)Q(x)$	[Rule $\wedge_{use}$, from 12]
Step 15.	Let a be arbitrary.	[Declaration]
Step 16.	$P(a)$.	[Rule $\forall_{use}$, from 13 & 15]
Step 17.	$Q(a)$	[Rule $\forall_{use}$, from 14 & 15]
Step 18.	$P(a) \wedge Q(a)$	[Rule $\wedge_{get}$, from 16 & 17]
Step 19.	$(\forall x)(P(x) \wedge Q(x))$	[Rule $\forall_{get}$, from 15 & 18]
Step 20.	$((\forall x)P(x) \wedge (\forall x)Q(x)) \Rightarrow (\forall x)(P(x) \wedge Q(x))$	[R. $\Rightarrow_{get}$, fr. 12 & 19]
Step 21.	$(\forall x)(P(x) \wedge Q(x)) \Leftrightarrow ((\forall x)P(x) \wedge (\forall x)Q(x))$	[R. $\Leftrightarrow_{get}$, fr. 11 & 20]

THE END

Example 2. Suppose $P(x, y)$ is a two-variable predicate. Prove the following:

$$(\forall x)(\forall y)P(x, y) \Rightarrow (\forall y)(\forall x)P(x, y) \, .$$

SOLUTION. Here is a proof.

Step 1.	Assume $(\forall x)(\forall y)P(x, y)$	[Assumption]
Step 2.	Let a be arbitrary	[Declaration]
Step 3.	Let b be arbitrary	[Declaration]
Step 4.	$(\forall y)P(b, y)$	[Rule $\forall_{use}$, from 1 & 3.]
Step 5.	$P(b, a)$	[Rule $\forall_{use}$, from 2 & 4.]

Step 6.	$(\forall x)P(x,a)$	[Rule $\forall_{get}$, from 3 & 5.]
Step 7.	$(\forall y)(\forall x)P(x,y)$	[Rule $\forall_{get}$, from 2 & 6.]
Step 8.	$(\forall x)(\forall y)P(x,y) \Rightarrow (\forall y)(\forall x)P(x,y)$.	[Rule $\Rightarrow_{get}$, from 1 & 7.]

THE END

Example 3. Suppose $P(x,y)$ is a two-variable predicate. Prove the following:

$$(\exists x)(\exists y)P(x,y) \Rightarrow (\exists y)(\exists x)P(x,y)\,.$$

SOLUTION. Here is a proof.

Step 1.	Assume $(\exists x)(\exists y)P(x,y)$.	[Assumption]
Step 2.	Pick a such that $(\exists y)P(a,y)$.	[Rule $\exists_{out}$, from 1]
Step 3.	Pick b such that $P(a,b)$.	[Rule $\exists_{out}$, from 2]
Step 4.	$(\exists x)P(x,b)$.	[Rule $\exists_{get}$, from 3]
Step 5.	$(\exists y)(\exists x)P(x,y)$.	[Rule $\exists_{get}$, from 4]
Step 6.	$(\exists y)(\exists x)P(x,y)$.	[Rule $\exists_{use}$, from 2, 3 & 5]
Step 7.	$(\exists y)(\exists x)P(x,y)$.	[Rule $\exists_{use}$, from 1, 2, & 6]
Step 8.	$(\exists x)(\exists y)P(x,y) \Rightarrow (\exists y)(\exists x)P(x,y)$.	[Rule $\Rightarrow_{get}$, from 1 & 7.]

THE END

Example 4. Suppose $P(x)$ and $Q(x)$ are one-variable predicates. Prove the following:

$$(\forall x)(P(x) \vee Q(x)) \Rightarrow ((\forall x)P(x) \vee (\forall x)Q(x))\ . \qquad (0.0.1)$$

SOLUTION. This cannot be proved because it need not be true. For example, suppose we take the universe of discourse to be the set of all U.S. senators. excluding the independents, if there are any. Suppose $P(x)$ stands for "x is a Democrat", and $Q(x)$ stands for "x is a Republican". Then the sentence "$(\forall x)(P(x) \vee Q(x))$" says that "every senator is a Democrat or a Republican", which is true, whereas "$(\forall x)P(x)$" says that "every senator is a Democrat", which is false, and "$(\forall x)Q(x)$" says that "every senator is a Republican", which is also false. Therefore the disjunction "$(\forall x)P(x) \vee (\forall x)Q(x)$" is false. Since "$(\forall x)(P(x) \vee Q(x))$" is true, as we have alreayd shown, it follows that the implication "$(\forall x)(P(x) \vee Q(x)) \Rightarrow ((\forall x)P(x) \vee (\forall x)Q(x))$" is false.

Remark. Notice that I did ***not*** say that "this cannot be proved because it isn't true." I said that "this cannot be proved because it need not be true," which is quite different. Whether or not a sentence such as (0.0.1) is true depends very much on which specific predicates you plug in for $P(x)$ and $Q(x)$. For example, you could take $P(x)$ to be any one-variable predicate you want (say, "x is a frog", or "$x > 32$") and then take $Q(x)$ to be the same as $P(x)$. Then (0.0.1) is true. (If you don't like this example, here is another one: take $P(x)$ to be "x is a frog", and $Q(x)$ to be "x is a Gila monster". Take the universe of discourse—i.e., the range of the variable x—to be the set of all animals. Then "$(\forall x)(P(x) \vee Q(x))$" says that "every animal is a frog or a Gila monster", which is obviously false, as can be proved by giving a counterexample, e.g., my dog Rex[1]. On the other hand, "$(\forall x)P(x)$" says that "every animal is a frog", which is false, and "$(\forall x)Q(x)$" says that "every animal is a Gila monster", which is also false. Hence the disjunction "$(\forall x)P(x) \vee (\forall x)Q(x)$" is false. Since both "$(\forall x)(P(x) \vee Q(x))$" and "$(\forall x)P(x) \vee (\forall x)Q(x)$" are false, the implication "$(\forall x)(P(x) \vee Q(x)) \Rightarrow (\forall x)P(x) \vee (\forall x)Q(x)$" is true.

Example 5. Suppose $P(x)$ and $Q(x)$ are one-variable predicates. Prove the following:

$$((\forall x)P(x) \vee (\forall x)Q(x)) \Rightarrow (\forall x)(P(x) \vee Q(x)) \ . \tag{0.0.2}$$

SOLUTION. Here is a proof.

Step 1.	Assume $(\forall x)P(x) \vee (\forall x)Q(x)$	[Assumption]
Step 2.	Assume $(\forall x)P(x)$	[Assumption]
Step 3.	Let a be arbitrary	[Declaration]
Step 4.	$P(a)$	[Rule $\forall_{use}$, from 2 & 3]
Step 5.	$P(a) \vee Q(a)$	[Rule $\vee_{get}$, from 3]
Step 6.	$(\forall x)(P(x) \vee Q(x))$	[Rule $\forall_{get}$, from 3 & 5]
Step 7.	$(\forall x)P(x) \Rightarrow (\forall x)(P(x) \vee Q(x))$	[Rule $\Rightarrow_{get}$, from 2 & 6]
Step 8.	Assume $(\forall x)Q(x)$	[Assumption]
Step 9.	Let a be arbitrary	[Declaration]
Step 10.	$Q(a)$	[Rule $\forall_{use}$, from 8 & 9]
Step 11.	$P(a) \vee Q(a)$	[Rule $\vee_{get}$, from 10]
Step 12.	$(\forall x)(P(x) \vee Q(x))$	[Rule $\forall_{get}$, from 9 & 11]
Step 13.	$(\forall x)Q(x) \Rightarrow (\forall x)(P(x) \vee Q(x))$	[Rule $\Rightarrow_{get}$, from 8 & 12]
Step 14.	$(\forall x)(P(x) \vee Q(x))$	[Rule $\vee_{use}$, from 1, 7 & 13]
Step 15.	$((\forall x)P(x) \vee (\forall x)Q(x)) \Rightarrow (\forall x)(P(x) \vee Q(x))$	[Rule $\Rightarrow_{get}$, from 1 & 14]

THE END

[1] I am going through this to stress an important point. ***A counterexample has to be concrete and precise.*** For example, if you are trying to disprove the assertion that "every integer is even", and you say "well, pick any odd number," then I don't like that. I would very much prefer that you say "the number 3 is an integer but is not even". Similarly, if you said "pick any animal you want, say a cow or a giraffe," then I am not happy. I want a concrete, specific animal.

Example 6. Suppose $P(x)$ and $Q(x)$ are one-variable predicates. Prove the following:

$$(\exists x)(P(x) \vee Q(x)) \Leftrightarrow ((\exists x)P(x) \vee (\exists x)Q(x)) \ . \tag{0.0.3}$$

SOLUTION. Here is a proof.

Step 1.	Assume $(\exists x)(P(x) \vee Q(x))$.	[Assumption]
Step 2.	Pick a such that $P(a) \vee Q(a)$.	[Rule $\exists_{use}$, from 1]
Step 3.	Assume $P(a)$	[Assumption]
Step 4.	$(\exists x)P(x)$	[Rule $\exists_{get}$, from 3]
Step 5.	$(\exists x)P(x) \vee (\exists x)Q(x)$	[Rule $\vee_{get}$, from 4]
Step 6.	$P(a) \Rightarrow ((\exists x)P(x) \vee (\exists x)Q(x))$	[Rule $\Rightarrow_{get}$, from 3 & 5]
Step 7.	Assume $Q(a)$	[Assumption]
Step 8.	$(\exists x)Q(x)$	[Rule $\exists_{get}$, from 7]
Step 9.	$(\exists x)P(x) \vee (\exists x)Q(x)$	[Rule $\vee_{get}$, from 8]
Step 10.	$Q(a) \Rightarrow ((\exists x)P(x) \vee (\exists x)Q(x))$	[Rule $\Rightarrow_{get}$, from 7 & 9]
Step 11.	$(\exists x)P(x) \vee (\exists x)Q(x)$	[Rule $\vee_{use}$, from 2, 6,& 10]
Step 12.	$(\exists x)P(x) \vee (\exists x)Q(x)$	[Rule $\exists_{use}$, from 2 & 11]
Step 13.	$(\exists x)(P(x) \vee Q(x)) \Rightarrow ((\exists x)P(x) \vee (\exists x)Q(x))$	[Rule $\Rightarrow_{get}$, from 1 & 12]
Step 14.	Assume $(\exists x)P(x) \vee (\exists x)Q(x)$	[Assumption]
Step 15.	Assume $(\exists x)P(x)$	[Assumption]
Step 16.	Pick a such that $P(a)$.	[Rule $\exists_{use}$, from 15]
Step 17.	$P(a) \vee Q(a)$.	[Rule $\vee_{get}$, from 16]
Step 18.	$(\exists x)(P(x) \vee Q(x))$	[Rule $\exists_{get}$, from 17]
Step 19.	$(\exists x)P(x) \Rightarrow (\exists x)(P(x) \vee Q(x))$	[Rule $\Rightarrow_{get}$, from 15 & 18]
Step 20.	Assume $(\exists x)Q(x)$	[Assumption]
Step 21.	Pick a such that $Q(a)$.	[Rule $\exists_{use}$, from 20]
Step 22.	$P(a) \vee Q(a)$.	[Rule $\vee_{get}$, from 21]
Step 23.	$(\exists x)(P(x) \vee Q(x))$	[Rule $\exists_{get}$, from 22]
Step 24.	$(\exists x)Q(x) \Rightarrow (\exists x)(P(x) \vee Q(x))$	[Rule $\Rightarrow_{get}$, from 20 & 23]
Step 25.	$(\exists x)(P(x) \vee Q(x))$	[Rule $\vee_{use}$, from 14, 19 & 24]
Step 26.	$((\exists x)P(x) \vee (\exists x)Q(x)) \Rightarrow (\exists x)(P(x) \vee Q(x))$	[Rule $\Rightarrow_{get}$, from 14 & 25]
Step 27.	$((\exists x)P(x) \vee (\exists x)Q(x)) \Leftrightarrow (\exists x)(P(x) \vee Q(x))$	[Rule $\Leftrightarrow_{get}$, from 13 & 26]

THE END

Example 7. Suppose $P(x)$ and $Q(x)$ are one-variable predicates. Prove the following:

$$(\exists x)(P(x) \wedge Q(x)) \Leftrightarrow ((\exists x)P(x) \wedge (\exists x)Q(x)) \ . \tag{0.0.4}$$

SOLUTION. This cannot be proved because it need not be true. For example, suppose we take the universe of discourse to be the set of all U.S. senators.

Suppose $P(x)$ stands for "x is a Democrat", and $Q(x)$ stands for "x is a Republican". Then "$(\exists x)(P(x) \wedge Q(x))$" says that "some senators are both Democrat and Republican", which is false, whereas "$(\exists x)P(x)$" says that "some senators are Democrats", which is true, and "$(\exists x)Q(x)$" says that "some senators are Republicans," which is also true. Hence the conjunction "$(\exists x)P(x) \wedge (\exists x)Q(x)$" is true. Since "$(\exists x)(P(x) \wedge Q(x))$" is false, it follows that the biconditional "$(\exists x)(P(x) \wedge Q(x)) \Leftrightarrow ((\exists x)P(x) \wedge (\exists x)Q(x))$" is false.

Example 8. Suppose $P(x)$ is a one-variable predicate. Prove the following:

$$(\exists x)P(x) \Leftrightarrow (\sim (\forall x) \sim P(x))\,.$$

SOLUTION: Here is a proof:

Step 1.	Assume $(\exists x)P(x)$	[Assumption]
Step 2.	Assume $(\forall x) \sim P(x)$	[Assumption]
Step 3.	Pick a such that $P(a)$	[Rule $\exists_{use}$ from 1]
Step 4.	$\sim P(a)$	[Rule $\forall_{use}$ from 2]
Step 5.	$P(a) \wedge \sim P(a)$	[Rule $\wedge_{get}$ from 3 & 4]
Step 6.	$(P(a) \wedge \sim P(a)) \Rightarrow ((\sim (\forall x)P(x)) \wedge (\forall x)P(x))$	[Instance of tautology]
Step 7.	$(\sim (\forall x)P(x)) \wedge (\forall x)P(x)$	[Rule $\Rightarrow_{get}$ from 5 & 6]
Step 8.	$(\sim (\forall x)P(x)) \wedge \sim (\forall x)P(x)$ (*contradiction*)	[Rule $\exists_{get}$ from 2, 3 & 7]
Step 9.	$\sim (\forall x) \sim P(x)$.	[Rule 2 from 2 & 8]
Step 10.	$(\exists x)P(x) \Rightarrow (\sim (\forall x) \sim P(x))$.	[Rule $\Rightarrow_{get}$ from 1 & 9]
Step 11.	Assume $\sim (\forall x) \sim P(x)$.	[Assumption]
Step 12.	Assume $\sim (\exists x)P(x)$.	[Assumption]
Step 13.	Let a be arbitrary.	[Declaration]
Step 14.	Assume $P(a)$.	[Assumption]
Step 15.	$(\exists x)P(x)$.	[Rule $\exists_{get}$ from 14]
Step 16.	$(\exists x)P(x) \wedge \sim (\exists x)P(x)$ (*contradiction*)	[Rule $\wedge_{get}$ from 12 & 15]
Step 17.	$\sim P(a)$	[Rule 2, from 14 & 16]
Step 18.	$(\forall x) \sim P(x)$	[Rule $\forall_{get}$ from 13 & 17]
Step 19.	$(\forall x) \sim P(x) \wedge (\sim (\forall x) \sim P(x))$ (*contradiction*)	[Rule $\wedge_{get}$ from 11 & 18]
Step 20.	$(\exists x)P(x)$	[Rule 2, from 12 & 19]
Step 21.	$(\sim (\forall x) \sim P(x)) \Rightarrow (\exists x)P(x)$	[Rule $\Rightarrow_{get}$ from 11 & 20]
Step 22.	$(\exists x)P(x) \Leftrightarrow (\sim (\forall x) \sim P(x))$	[Rule $\Leftrightarrow_{get}$ from 10 & 21]

THE END

Example 9. Prove the following:

$$(\forall x)(\forall y)(\forall z)((x = y \wedge y = z) \Rightarrow x = z)\ .$$

SOLUTION: Here is a proof:

Step 1.	Let a be arbitrary	[Declaration]
Step 2.	Let b be arbitrary	[Declaration]
Step 3.	Let c be arbitrary	[Declaration]
Step 4.	Assume $a = b \wedge b = c$	[Assumption]
Step 5.	$a = b$	[Rule $\wedge_{use}$, from 4]
Step 6.	$b = c$	[Rule $\wedge_{use}$, from 4]
Step 7.	$a = c$	[Rule SEE, from 5 & 6]
Step 8.	$(a = b \wedge b = c) \Rightarrow a = c$	[Rule $\Rightarrow_{get}$, from 4 & 7]
Step 9.	$(\forall z)((a = b \wedge b = z) \Rightarrow a = z)$	[Rule $\forall_{get}$, from 3 & 8]
Step 10.	$(\forall y)(\forall z)((a = y \wedge y = z) \Rightarrow a = z)$	[Rule $\forall_{get}$, from 2 & 9]
Step 11.	$(\forall x)(\forall y)(\forall z)((x = y \wedge y = z) \Rightarrow x = z)$	[Rule $\forall_{get}$, from 1 & 10]

THE END