EXAMPLES OF EQUIVALENCE RELATIONS

"There's glory for you!"
"I don't know what you mean by 'glory,'" Alice said.
Humpty Dumpty smiled contemptuously. "Of course you don't–till I tell you. I meant 'there's a nice knock-down argument for you'!"
"But 'glory' doesn't mean 'a nice knock-down argument,'" Alice objected.
"When I use a word," Humpty Dumpty said in a rather a scornful tone, "it means just what I choose it to mean–neither more nor less."
"The question is," said Alice, "whether you can make words mean different things."
"The question is," said Humpty Dumpty, "which is to be master–that's all."
Alice was too much puzzled to say anything, so after a minute Humpty Dumpty began again. "They've a temper, some of them—particularly verbs, they're the proudest—adjectives you can do anything with, but not verbs–however, I can manage the whole lot! Impenetrability! That's what I say!"
"Would you tell me, please," said Alice, "what that means?"
"Now you talk like a reasonable child," said Humpty Dumpty, looking very much pleased. "I meant by 'impenetrability' that we've had enough of that subject, and it would be just as well if you'd mention what you meant to do next, as I suppose you don't intend to stop here all the rest of your life."
"That's a great deal to make one word mean," Alice said in a thoughtful tone.
"When I make a word do a lot of work like that," said Humpty Dumpty, "I always pay it extra."

EXAMPLE 1. In how many ways can we express the number 13 as the sum of two natural numbers?

The answer probably seems obvious to you. Here are all the ways to do it:

$$13 = 0 + 13\,,\quad 13 = 1 + 12\,,\quad 13 = 2 + 11\,,\quad 13 = 3 + 10\,,$$

$$13 = 4 + 9\,,\quad 13 = 5 + 8\,,\quad 13 = 6 + 7\,.$$

So there are seven ways, and that's it. Right? (This seems to be what most students believed when this problem was first discussed in class.)

Now, suppose I were to tell you that there are other solutions, such as, for example $13 = 9+4$. You would probably argue that is "the same" as $13 = 4+9$. But then I would ask you **exactly what do you mean by "the same"?**

The answer is that

The words "the same" mean whatever we make them mean (within reason), and we can choose to make them mean different things depending on what is most convenient to us.

In the specific case of our problem of counting the solutions of $13 = x + y$, $x \in \mathbb{N}$, $y \in \mathbb{N}$, we have the following two options:

OPTION 1. We define a *solution* of $13 = x+y$, $x \in \mathbb{N}$, $y \in \mathbb{N}$, to be an ordered pair $(x, y) \in \mathbb{N} \times \mathbb{N}$ such that $x + y = 13$.

OPTION 2. We let A be the set of all ordered pairs $(x, y) \in \mathbb{N} \times \mathbb{N}$ such that $x + y = 13$, and define a *solution* of $13 = x + y$, $x \in \mathbb{N}$, $y \in \mathbb{N}$, to be an R-equivalence class, relative to the equivalence relation R on A such that, if $(x, y) \in A \times A$ and $(x', y') \in A \times A$, then $(x, y)R(x', y')$ if and only if $(x = x' \wedge y = y') \vee (x = y' \wedge y = x')$. (In other words, a "solution" in this sense is the same as a "solution" in the sense of Option 1, except that now we are treating a pair (x', y') obtained from a pair (x, y) by switching the order of the components as being "the same solution" as (x, y). For example, the pair $(9, 4)$ is "the same solution" as the pair $(4, 9)$ in the sense of Option 2, but not in the sense of Option 1.

WHAT IS THE "RIGHT" ANSWER, THEN? The answer is:

1. If you define "solution" as in Option 1, then the problem of expressing 13 as the sum of two natural numbers has 14 solutions.

2. If you define "solution" as in Option 2, then the problem of expressing 13 as the sum of two natural numbers has 7 solutions.

Which of the two is "right"? I do not understand the question. Most students seemed to think that the answer was "obviously" Option 2. If you believe that, then PLEASE THINK IT OVER AGAIN. And let me give you one example showing that Option 1 may be reasonable in some situations. Suppose you and a friend have lots of one-dollar bills and no coins, and you have to pay a \$13 check. In how many ways can you divide up the \$13 between the two of you? Obviously, the solution where you pay \$4 and your friend pays \$9 is **not** "the same" as the solution where you pay \$9 and your friend pays \$4!

EXAMPLE 2. Suppose we want to deal with integers, but when we consider an integer n we are only interested in the remainder of dividing n by 7. That is, as far as we are concerned, the integers 4, 11, 18, -3, -10 are all "the same," in the sense that when we divide them by 7 we get the same remainder.

We cannot really write "$4 = 11$," because 4 and 11 are not at all the same number. The precise mathematical way to do what we want is to introduce an equivalence relation, called "congruence modulo 7."

DEFINITION. Let a, b be integers. We say that a is *congruent to b modulo* 7, and write "$a \equiv_7 b$", or "$a = b \bmod 7$", if $a - b$ is divisible by 7. $\diamondsuit$

FORMAL VERSION OF THE DEFINITION:

$\forall a \in \mathbb{Z}, \forall b \in \mathbb{Z}, (a \equiv_7 b \longleftrightarrow 7 | a - b)$

or, if you prefer

$\forall a \in \mathbb{Z}, \forall b \in \mathbb{Z}, (a \equiv_7 b \longleftrightarrow \exists k \in \mathbb{Z}, a - b = 7.k)$.

PROBLEM. Prove that $\equiv_7$ is an equivalence relation.

PROBLEM. Determine the equivalence classes for $\equiv_7$.

SOLUTION. By the Division theorem (book, page 255, Theorem 31.1)), given any integer a there exist $q, r \in \mathbb{Z}$ such that $a = 7q + r$ and $0 \leq r <$. This means that $a \equiv_7 r$. So by Proposition 12.10 (page 83), $[a]_{\equiv_7} = [r]_{\equiv_7}$. It follows that if a is an arbitrary integer then the equivalence class $[a]_{\equiv_7} = [r]_{\equiv_7}$, where r is one of the integers $0, 1, 2, 3, 4, 5, 6$, So every equivalence class is one of the following seven classes: $[0]_{\equiv_7}$, $[1]_{\equiv_7}$, $[2]_{\equiv_7}$, $[3]_{\equiv_7}$, $[4]_{\equiv_7}$, $[5]_{\equiv_7}$, $[6]_{\equiv_7}$. Furthermore, these seven classes are all different. (Proof: suppose $r \in \mathbb{Z}$, $s \in \mathbb{Z}$, $0 \leq r < 7$ $0 \leq s < 7$, and $r \neq s$. Suppose $[r]_{\equiv_7} = [s]_{\equiv_7}$. Proposition 12.9 implies that $r \in [r]_{\equiv_7}$and $s \in [s]_{\equiv_7}$. Since $[r]_{\equiv_7} = [s]_{\equiv_7}$, we have $r \in [r]_{\equiv_7}$ and $s \in [r]_{\equiv_7}$. Then Proposition 12.11 tells us that $r \equiv_7 s$, so $7|r - s|$. But this is impossible, because $0 < |r - s| < 7$.) Hence **there are seven equivalence classes, namely, $[0]_{\equiv_7}$, $[1]_{\equiv_7}$, $[2]_{\equiv_7}$, $[3]_{\equiv_7}$, $[4]_{\equiv_7}$, $[5]_{\equiv_7}$, and $[6]_{\equiv_7}$.**

EXAMPLE 3. *(A generalization of Example 2.)*

DEFINITION. Fix a positive integer n. Let a, b be integers. We say that a is *congruent to b modulo n*, and write "$a \equiv_n b$", or "$a = b \bmod n$", if $a - b$ is divisible by n. ♢

FORMAL VERSION OF THE DEFINITION:

$\forall n \in \mathbb{Z}, (n > 0 \longrightarrow (\forall a \in \mathbb{Z}, \forall b \in \mathbb{Z}, (a \equiv_n b \longleftrightarrow n|a - b)))$

or, if you prefer

$\forall n \in \mathbb{Z}, (n > 0 \longrightarrow (\forall a \in \mathbb{Z}, \forall b \in \mathbb{Z}, (a \equiv_n b \longleftrightarrow \exists k \in \mathbb{Z}, a - b = n.k)))$.

PROBLEM. Prove that for every positive integer n $\equiv_n$ is an equivalence relation.

PROBLEM. Determine the equivalence classes for $\equiv_n$.

EXAMPLE 4. *Dominoes* are rectangles divided into two squares, on each one of which is written an integer between 0 and 6. So, mathematically, we can think of a "domino" as an ordered pair (a, b) such that $a \in \{0, 1, 2, 3, 4, 5, 6\}$ and $b \in \{0, 1, 2, 3, 4, 5, 6\}$. However, we do not care which of the two numbers is first, so we want to treat the pairs $(3, 5)$ and $(5, 3)$, for example, as being "the same."

We can do all this in a precise way by first defining a *pre-domino* to be an ordered pair (a, b) such that $a \in \{0, 1, 2, 3, 4, 5, 6\}$ and $b \in \{0, 1, 2, 3, 4, 5, 6\}$. (Therefore, if we use PD to denote the set of all pre-dominoes, then

$$PD = \{0, 1, 2, 3, 4, 5, 6\} \times \{0, 1, 2, 3, 4, 5, 6\} .$$

In particular, $|PD| = 49$. That is, there are 49 pre-dominoes.)

We then define an equivalence relation R_{dom} on the set PD of pre-dominoes by declaring that two pre-dominoes (a, b), (c, d) satisfy $(a, b)R_{dom}(c, d)$ if either ($a = b$ and $c = d$) or ($a = d$ and $b = c$).

The equivalence classes determined by this relation are the *dominoes*.

PROBLEM. Prove that R_{dom} is an equivalence relation.

PROBLEM. How many dominoes are there?

EXAMPLE 5. *(A generalization of Example 4.)* We would like to do for a general positive integer n the same thing that we did in Example 4 for $n = 6$.

DEFINITION. Fix a positive integer n. A *n-pre-domino* is an ordered pair $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ such that $0 \leq a \leq n$ and $0 \leq b \leq n$.

We then define an equivalence relation—called $R_{dom,n}$—on the set PD_n of n-pre-dominoes by stipulating that two n-pre-dominoes (a, b), (c, d) satisfy $(a, b)R_{dom,n}(c, d)$ if either ($a = b$ and $c = d$) or ($a = d$ and $b = c$).

The equivalence classes determined by this relation are the *n-dominoes*.

PROBLEM. Prove that $R_{dom,n}$ is an equivalence relation.

PROBLEM. How many n-dominoes are there?

EXAMPLE 6. Consider an ordinary chessboard. The chessboard has 64 squares, forming an 8×8 square array. Let Sq be the set of squares, so $|Sq| = 64$. Mathematically, we can think of each square as an ordered pair of integers between 1 and 8, so the set Sq of squares is the Cartesian product $\{1, 2, 3, 4, 5, 6, 7, 8\} \times \{1, 2, 3, 4, 5, 6, 7, 8\}$. (The (i, j) pair corresponds to the square in the i-row, from left to right, and the j-th column, from bottom to top.)

Two squares α, β are *bishop-equivalent* if one can go from α to β by one or several steps in each of which one moves diagonally.

PROBLEM. Prove that bishop-equivalence is an equivalence relation.

PROBLEM. Determine the equivalence classes. How many equivalence classes are there?

EXAMPLE 7. Consider an ordinary chessboard, as in Example 6. A *knight move* consists of moving from a square up or down two squares and then left or right one square, or moving left or right two squares and then up or down one square.

Two squares α, β are *knight-equivalent* if one can go from α to β by one or several steps in each of which one executes a knight move.

PROBLEM. Prove that knight-equivalence is an equivalence relation.

PROBLEM. Determine the equivalence classes. How many equivalence classes are there?

EXAMPLE 8. (A generalization of Example 7.) Fix three positive integers n, k, ℓ. Consider a chessboard, as in Examples 6, 7, except that now the board is $n \times n$. A *k-ℓ-knight move* consists of moving from a square up or down k squares and then left or right ℓ squares, or moving left or right k squares and then up or down ℓ squares.

Two squares α, β are *k-ℓ-knight-equivalent* if one can go from α to β by one or several steps in each of which one executes a k-ℓ-knight move.

PROBLEM. Prove that k-ℓ-knight-equivalence is an equivalence relation.

PROBLEM. Determine the equivalence classes. How many equivalence classes are there? (If you find it too difficult to do this for completely general n, k, ℓ, try at least a few examples, showing in particular that the answer isn't always as simple as that of Example 6.)

EXAMPLE 9. *Constructing (or inventing, if you prefer) the rational numbers.* Suppose we know how to talk about sets, and about integers, but we have not yet discovered (or "invented," or "created") the rational numbers. How could we do it?

We could try to define a "rational number" to be an ordered pair (m, n) of integers such that $n \neq 0$. Then we could say that we will use " $\frac{m}{n}$ " as an alternative notation for "(m, n)", and we we would have a set of objects that are of the desired form. The trouble with this definition is that it would imply that, for example, $\frac{4}{3}$ and $\frac{8}{6}$ are different objects, whereas we want to be able to write

$$\frac{4}{3} = \frac{8}{6},$$

i.e., to treat $\frac{4}{3}$ and $\frac{8}{6}$ as being "the same number."

The correct way to do this is to introduce an equivalence relation. To begin with, we define a *pre-rational number* to be an ordered pair (m, n) of integers such that $n \neq 0$. We use $Pre\mathbb{Q}$ to denote the set of all pre-rational numbers. Then

$$Pre\mathbb{Q} = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : n \neq 0\}.$$

We define a relation $R_{\mathbb{Q}}$ on $Pre\mathbb{Q}$ by letting

$$(m, n) R_{\mathbb{Q}} (m', n') \longleftrightarrow mn' = m'n\,, \qquad \text{if} \quad (m, n), (m', n') \in Pre_{\mathbb{Q}}\,.$$

THEOREM. $R_{\mathbb{Q}}$ is an equivalence relation.

PROOF. We prove that $R_{\mathbb{Q}}$ is transitive. We leave the proofs of the other two properties (reflexitivity and transitivity) to you.
We want to prove that

$$\forall x, \forall y, \forall z, ((x \in Pre\mathbb{Q} \wedge y \in Pre\mathbb{Q} \wedge z \in Pre\mathbb{Q} \wedge xR_{\mathbb{Q}}y \wedge yR_{\mathbb{Q}}z) \longrightarrow xR_{\mathbb{Q}}z)\,.$$

Let x, y, z be arbitrary.
Assume that $x \in Pre\mathbb{Q}$, $y \in Pre\mathbb{Q}$, $z \in Pre_{\mathbb{Q}}$, $xR_{\mathbb{Q}}y$ and $yR_{\mathbb{Q}}z$.
Then we can pick $m, n, m', n', m'', n'' \in \mathbb{Z}$ such that $x = (m, n)$, $y = (m', n')$, $z = (m'', n'')$, $n \neq 0$, $n' \neq 0$, $n'' \neq 0$.
Then $mn' = m'n$ and $m'n'' = m''n'$.
Since $mn' = m'n$, we have $mn'n'' = m'nn''$. So $mn'n'' = m'n''n$. Since $m'n'' = m''n'$, we can conclude that $mn'n'' = m''n'n$. Since $n' \neq 0$, we can cancel n' in the previous equation, and get $mn'' = m''n$. But this says that $xR_{\mathbb{Q}}z$.
Therefore $(x \in Pre\mathbb{Q} \wedge y \in Pre\mathbb{Q} \wedge z \in Pre_{\mathbb{Q}} \wedge xR_{\mathbb{Q}}y \wedge yR_{\mathbb{Q}}z) \longrightarrow xR_{\mathbb{Q}}z$.
So $\forall x, \forall y, \forall z, ((x \in Pre\mathbb{Q} \wedge y \in Pre\mathbb{Q} \wedge z \in Pre\mathbb{Q} \wedge xR_{\mathbb{Q}}y \wedge yR_{\mathbb{Q}}z) \longrightarrow xR_{\mathbb{Q}}z)$.
Therefore $R_{\mathbb{Q}}$ is transitive. ◊

DEFINITION. A *rational number* is an equivalence class of the relation $R_{\mathbb{Q}}$ on $Pre\mathbb{Q}$. ♢

NOTATION. If $m, n \in \mathbb{Z}$ and $n \neq 0$, we use $\frac{m}{n}$ to denote the $R_{\mathbb{Q}}$-equivalence class of the pair (m, n). That is,

$$\frac{m}{n} = [(m, n)]_{R_{\mathbb{Q}}}\,.$$

NOTATION. We use $\mathbb{Q}$ to denote the set of all rational numbers.

With these notations we can write, for example,

$$\frac{4}{3} = \frac{8}{6}\,,$$

because $(4, 3)R_{\mathbb{Q}}(8, 6)$ (since $4 \times 6 = 3 \times 8$), so $[(4, 3)]_{R_{\mathbb{Q}}} = [(8, 6)]_{R_{\mathbb{Q}}}$, i.e., $\frac{4}{3} = \frac{8}{6}$.

PROBLEM. Complete the proof that $R_{\mathbb{Q}}$ is an equivalence relation, by proving that $R_{\mathbb{Q}}$ is reflexive and symmetric.

THE ARITHMETICAL OPERATIONS FOR RATIONAL NUMBERS. We would like to define what is meant by the sum and the product of two rational numbers. We will do it for the sum, and leave it up to you to define the product and to justify the definition.

To define addition of rational numbers, we let r, s be two fixed but arbitrary rational numbers and seek to define $r + s$. Since r and s are rational numbers, each of of them is an $R_{\mathbb{Q}}$-equivalence class. So we can pick integers m, n, m', n' such that $n \neq 0 \neq n'$, $r = [(m, n)]_{R_q q}$ and $s = [(m', n')]_{R_q q}$ (that is, equivaently, $r = \frac{m}{n}$ and $s = \frac{m'}{n'}$). We would like to define the sum $r + s$ to be given by the expression that we are all used to from high-school arithmetic, where we learned that

$$\frac{m}{n} + \frac{m'}{n'} = \frac{mn' + m'n}{nn'}\,.$$

The trouble with this is the following: r is the equivalence class of lots of different pairs (m, n), and s is the equivalence class of lots of different pairs (m', n'). Suppose that we choose two different pairs (m_1, n_1), (m_2, n_2) such that

$$r = [(m_1, n_1)]_{R_{\mathbb{Q}}} = [(m_2, n_2)]_{R_{\mathbb{Q}}}\,,$$

and two different pairs (m_1', n_1'), (m_2', n_2') such that

$$s = [(m_1', n_1')]_{R_{\mathbb{Q}}} = [(m_2', n_2')]_{R_{\mathbb{Q}}}\,.$$

Then, using (m_1, n_1) and (m_1', n_1'), we would conclude that $r + s$ is the equivalence class of the pair $(m_1 n_1' + m_1' n_1, n_1 n_1')$. On the other hand, using (m_2, n_2) and (m_2', n_2'), we would conclude that $r + s$ is the equivalence class of the pair $(m_2 n_2' + m_2' n_2, n_2 n_2')$. Suppose it turned out that these two equivalence classes are not the same, i.e., that

$$\neg\Big((m_1 n_1' + m_1' n_1, n_1 n_1')R_{\mathbb{Q}}(m_2 n_2' + m_2' n_2, n_2 n_2')\Big)\,,$$

or, if you prefer, that

$$\Big[(m_1n_1' + m_1'n_1, n_1n_1')\Big]_{R_{\mathbb{Q}}} \neq \Big[(m_2n_2' + m_2'n_2, n_2n_2')\Big]_{R_{\mathbb{Q}}},$$

or

$$\frac{m_1n_1' + m_1'n_1}{n_1n_1'} \neq \frac{m_2n_2' + m_2'n_2}{n_2n_2'}.$$

This would mean that "the rational number $r+s$" defined using (m_1, n_1) and (m_1', n_1') is not the same as "the rational number $r+s$" defined using (m_2, n_2) and (m_2', n_2'). In that case, we would not have a unique, well defined sum $r+s$.

Fortunately, this does not happen, because the following theorem is true:

THEOREM. Let r, s be rational numbers. Let the pairs $(m_1, n_1) \in Pre\mathbb{Q}$, $(m_2, n_2) \in Pre\mathbb{Q}$ be such that $r = [(m_1, n_1)]_{R_{\mathbb{Q}}} = [(m_2, n_2)]_{R_{\mathbb{Q}}}$, and let $(m_1', n_1') \in Pre\mathbb{Q}$, $(m_2', n_2') \in Pre\mathbb{Q}$ be such that $s = [(m_1', n_1')]_{R_{\mathbb{Q}}} = [(m_2', n_2')]_{R_{\mathbb{Q}}}$. Then

$$(m_1n_1' + m_1'n_1, n_1n_1')R_{\mathbb{Q}}(m_2n_2' + m_2'n_2, n_2n_2').$$

PROOF. The fact that $[m_1, n_1]_{R_{\mathbb{Q}}} = [m_2, n_2]_{R_{\mathbb{Q}}}$ implies that

$$m_1n_2 = m_2n_1.$$

The fact that $[m_1', n_1']_{R_{\mathbb{Q}}} = [m_2', n_2']_{R_{\mathbb{Q}}}$ implies that

$$m_1'n_2' = m_2'n_1'.$$

We want to prove that $(m_1n_1' + m_1'n_1, n_1n_1')R_{\mathbb{Q}}(m_2n_2' + m_2'n_2, n_2n_2')$, i.e., that

$$(m_1n_1' + m_1'n_1)n_2n_2' = (m_2n_2' + m_2'n_2)n_1n_1'. \qquad (0.0.1)$$

But

$$\begin{aligned} (m_1n_1' + m_1'n_1)n_2n_2' &= m_1n_1'n_2n_2' + m_1'n_1n_2n_2' \\ &= (m_1n_2)n_1'n_2' + (m_1'n_2')n_1n_2 \\ &= (m_2n_1)n_1'n_2' + (m_2'n_1')n_1n_2 \\ &= m_2n_2'n_1n_1' + m_2'n_2n_1n_1' \\ &= (m_2n_2' + m_2'n_2)n_1n_1', \end{aligned}$$

so (0.0.1) holds, and our proof is complete. ♢

DEFINITION. Let r, s be rational numbers. Then the *sum* of r and s is the rational number $r+s$ given by $r+s = \frac{mn'+m'n}{nn'}$, if $(m,n) \in Pre\mathbb{Q}$, $(m', n') \in Pre\mathbb{Q}$ are such that $r = \frac{m}{n}$ and $s = \frac{m}{n}$. ♢

The definition is justified because the previous theorem tells us that, no matter how we choose the members (m,n), $(m'n')$ of the equivalence classes r, s, the resulting rational number $\frac{mn'+m'n}{nn'}$ will be the same.

PROBLEM. Use $\mathbf{0}$ to denote the rational number $\frac{0}{1}$, i.e., the equivalence class $[(0,1)]_{R_{\mathbb{Q}}}$. Prove that addition of rational numbers, as defined above, has the following properties:

$$\forall r \in \mathbb{Q}, \forall s \in \mathbb{Q}, r+s=s+r$$

(commutative law of addition),

$$\forall r \in \mathbb{Q}, \forall s \in \mathbb{Q}, \forall t \in \mathbb{Q}, (r+s)+t=r+(s+t)\,,$$

(associative law of addition),

$$\forall r \in \mathbb{Q}, r+\mathbf{0}=r\,,$$

and

$$\forall r \in \mathbb{Q}, \exists s \in \mathbb{Q}, r+s=\mathbf{0}$$

(existence of additive inverses).

PROBLEM. Define *multiplication of rational numb*ers. That is, follow the model of the definition of addition given earlier, to define the operation of multiplication. This will require that you state and prove a theorem, similar to the one we stated and proved before we defined the sum of two rational numbers, needed to justify your definition by showing that the rational number $r.s$ that you defined does not depend on how you choose members (m,n), (m',n') of the equivalence classes r, s.

PROBLEM. Use $\mathbf{1}$ to denote the rational number $\frac{1}{1}$, i.e., the equivalence class $[(1,1)]_{R_{\mathbb{Q}}}$. Using the definition of multiplication of rational numbers given in the previous problem, prove that multiplication has the following properties:

$$\forall r \in \mathbb{Q}, \forall s \in \mathbb{Q}, r.s=s.r$$

(commutative law of multiplication),

$$\forall r \in \mathbb{Q}, \forall s \in \mathbb{Q}, \forall t \in \mathbb{Q}, (r.s).t=r.(s.t)\,,$$

(associative law of multiplication),

$$\forall r \in \mathbb{Q}, r.\mathbf{1}=r\,,$$

$$\forall r \in \mathbb{Q}, (x \neq \mathbf{0} \longrightarrow \exists s \in \mathbb{Q}, r+s=\mathbf{0})$$

(existence of multiplicative inverses), and

$$\forall r \in \mathbb{Q}, \forall s \in \mathbb{Q}, \forall t \in \mathbb{Q}, (r+s).t=r.t+s.t\,,$$

(distributive law of multiplication with respect to addition).

EXAMPLE 10. What is a "curve"? Let us talk about curves in the plane $\mathbb{R}^2$, or in three-dimensional space $\mathbb{R}^3$.

In your Calculus courses, you have been exposed to "curves" in many ways. Sometimes a "curve" is given by an equation; for example, the equation $x^2 + y^2 = R^2$ describes a circle C_R of radius R and center at $(0,0)$. Sometimes a "curve" is given by "parametric equations"; for example, the equations $x = R\cos t$, $y = R\sin t$, $0 \leq t \leq 2\pi$, are parametric equations of C_R. Sometimes a "curve" is "oriented," whatever that means; for example, the circle C_R oriented counterclockwise and the circle C_R oriented clockwise are not "the same" as oriented curves. The distinction between the two "oriented versions" of C_R matters. For example, if $\mathbf{F}$ is the vector field $-y\,\mathbf{i} + x\,\mathbf{j}$, then the integral $\int_{C_R} \mathbf{F} \cdot d\mathbf{r}$ is equal to $2\pi R^2$ if C_R is oriented counterclockwise, and to $-2\pi R^2$ if C_R is oriented clockwise. So one should really be talking about *integrals along oriented curves*, not just along curves.

To make all this mathematically precise, we look first at the set $PC(\mathbb{R}^2)$ of *parametrized curves in* $\mathbb{R}^2$. Various mathematically interesting concepts of "curve in $\mathbb{R}^2$" can be obtained by considering different equivalence relations on $PC(\mathbb{R}^2)$. (A similar thing can be done for curves in $\mathbb{R}^3$.) Here, we will just consider two such concepts, giving rise to the notions of *oriented curve* and *unoriented curve*.

DEFINITION. A *parametrized curve* in $\mathbb{R}^2$ is a continuous $\mathbb{R}^2$-valued function $\mathbf{r}$ defined on some closed bounded interval $[a, b]$. ♢

In other words, a parametrized curve is exactly a pair of parametric equations $x = f(t)$, $y = g(t)$, where f and g are continuous real functions on an interval $[a, b]$. In vector form, we can write these parametric equations as

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}\,, \quad a \leq t \leq b\,.$$

We use $PC(\mathbb{R}^2)$ to denote the set of all parametrized curves in $\mathbb{R}^2$.

Now suppose that $\mathbf{r}_1 : [a_1, b_1] \mapsto \mathbb{R}^2$ and $\mathbf{r}_2 : [a_2, b_2] \mapsto \mathbb{R}^2$ are two parametrized curves. We will want to say that $\mathbf{r}_1$ and $\mathbf{r}_2$ are "equivalent up to an orientation-preserving reparametrization" if we can go from $\mathbf{r}_1$ to $\mathbf{r}_2$, and viceversa, by just "changing time without changing the direction of time." Precisely,

DEFINITION. Let $\mathbf{r}_1 : [a_1, b_1] \mapsto \mathbb{R}^2$ and $\mathbf{r}_2 : [a_2, b_2] \mapsto \mathbb{R}^2$ be two parametrized curves in $\mathbb{R}^2$. We say that $\mathbf{r}_1$ is *equivalent to* $\mathbf{r}_2$ *up to an orientation-preserving reparametrization* if there exist continuous functions $\varphi : [a_1, b_1] \mapsto [a_2, b_2]$ and $\psi : [a_2, b_2] \mapsto [a_1, b_1]$, such that $\varphi(\psi(t)) = t$ whenever $t \in [a_2, b_2]$, $\psi(\varphi(t)) = t$ whenever $t \in [a_1, b_1]$, $\mathbf{r}_1(\psi(t)) = \mathbf{r}_2(t)$ for all $t \in [a_2, b_2]$, $\varphi(a_1) = a_2$, and $\varphi(b_1) = b_2$. ♢

THEOREM. Equivalence up to an orientation-preserving reparametrization is an equivalence relation on $PC(\mathbb{R}^2)$.

PROOF. I will prove that the relation is symmetric, and leave it up to you to do the proofs of reflexitvity and transitivity.

Suppose that $\mathbf{r}_1 : [a_1, b_1] \mapsto \mathbb{R}^2$ is equivalent to $\mathbf{r}_2 : [a_2, b_2] \mapsto \mathbb{R}^2$ up to an orientation-preserving reparametrization. We want to prove that $\mathbf{r}_2$ is equivalent to $\mathbf{r}_1$ up to an orientation-preserving reparametrization.

To prove this, what we need is to construct a pair of continuous functions $\Phi : [a_2, b_2] \mapsto [a_1, b_1]$, $\Psi : [a_1, b_1] \mapsto [a_2, b_2]$, such that $\Phi(\Psi(t)) = t$ whenever $t \in [a_1, b_1]$, $\Psi(\Phi(t)) = t$ whenever $t \in [a_2, b_2]$, $\mathbf{r}_2(\Psi(t)) = \mathbf{r}_1(t)$ for all $t \in [a_1, b_1]$, $\Phi(a_2) = a_1$, and $\Phi(b_2) = b_1$.

We know that there exist continuous functions $\varphi : [a_1, b_1] \mapsto [a_2, b_2]$ and $\psi : [a_2, b_2] \mapsto [a_1, b_1]$, such that $\varphi(\psi(t)) = t$ whenever $t \in [a_2, b_2]$, $\psi(\varphi(t)) = t$ whenever $t \in [a_1, b_1]$, $\mathbf{r}_1(\psi(t)) = \mathbf{r}_2(t)$ for all $t \in [a_2, b_2]$, $\varphi(a_1) = a_2$, and $\varphi(b_1) = b_2$.

Take $\Psi = \varphi$ and $\Phi = \psi$. Then $\Psi(\Phi(t)) = t$ whenever $t \in [a_2, b_2]$ (because $\varphi(\psi(t)) = t$ whenever $t \in [a_2, b_2]$), and $\Phi(\Psi(t)) = t$ whenever $t \in [a_1, b_1]$ (because $\psi(\varphi(t)) = t$ whenever $t \in [a_1, b_1]$). Let us prove that $\mathbf{r}_2(\Psi(t)) = \mathbf{r}_1(t)$ for all $t \in [a_1, b_1]$. If $t \in [a_1, b_1]$, then $t = \psi(\varphi(t)$, and then it follows that $\mathbf{r}_1(t) = \mathbf{r}_1(\psi(\varphi(t))) = \mathbf{r}_2(\varphi(t))$, using the fact that $\mathbf{r}_1(\psi(s)) = \mathbf{r}_2(s)$ for all $s \in [a_2, b_2]$, with $s = \varphi(t)$. But then $\mathbf{r}_1(t) = \mathbf{r}_2(\Psi(t))$, which is exactly what we wanted to prove. Finally, let us prove that $\Phi(a_2) = a_1$ and $\Phi(b_2) = b_1$. We have $\Phi(a_2) = \psi(a_2) = \psi(\varphi(a_1)) = a_1$, so $\Phi(a_2) = a_1$. Similarly, $\Phi(b_2) = b_1$. So Φ and Ψ have all the required properties, and then $\mathbf{r}_2$ is equivalent to $\mathbf{r}_1$ up to an orientation-preserving reparametrization. This concludes the proof that our relation is symmetric. $\diamondsuit$

DEFINITION. An *oriented curve in* $\mathbb{R}^2$ is an equivalence class of the equivalence relation "equivalence up to an orientation-preserving reparametrization." $\diamondsuit$

DEFINITION. Let $\mathbf{r}_1 : [a_1, b_1] \mapsto \mathbb{R}^2$ and $\mathbf{r}_2 : [a_2, b_2] \mapsto \mathbb{R}^2$ be two parametrized curves in $\mathbb{R}^2$. We say that $\mathbf{r}_1$ is *equivalent to* $\mathbf{r}_2$ *up to a reparametrization* if there exist continuous functions $\varphi : [a_1, b_1] \mapsto [a_2, b_2]$, $\psi : [a_2, b_2] \mapsto [a_1, b_1]$, such that $\varphi(\psi(t)) = t$ whenever $t \in [a_2, b_2]$, $\psi(\varphi(t)) = t$ whenever $t \in [a_1, b_1]$, $\mathbf{r}_1(\psi(t)) = \mathbf{r}_2(t)$ for all $t \in [a_2, b_2]$. and either $\varphi(a_1) = a_2$ and $\varphi(b_1) = b_2$ or $\varphi(a_1) = b_2$ and $\varphi(b_1) = a_2$. $\diamondsuit$

THEOREM. Equivalence up to reparametrization is an equivalence relation on $PC(\mathbb{R}^2)$.

PROOF. Exercise for the reader.

DEFINITION. An *unoriented curve in* $\mathbb{R}^2$ is an equivalence class of the equivalence relation "equivalence up to reparametrization." $\diamondsuit$

If this sounds too abstract to you, here is an example. Consider the circle C_R with equation $x^2 + y^2 = R^2$. Here are two parametrizations of C_R:

1. $\mathbf{r}_1$ is the parametrization given by $\mathbf{r}_1(t) = R\cos t\,\mathbf{i} + R\sin t\,\mathbf{j}$, $0 \le t \le 2\pi$.

2. $\mathbf{r}_2$ is the parametrization given by $\mathbf{r}_2(t) = R\cos t\,\mathbf{i} - R\sin t\,\mathbf{j}$, $0 \le t \le 2\pi$.

Let us show that $\mathbf{r}_1$ and $\mathbf{r}_2$ are equivalent up to reparametrization. For this purpose, we must construct a pair of continuous functions $\varphi : [0, 2\pi] \mapsto [0, 2\pi]$, $\psi : [0, 2\pi] \mapsto [0, 2\pi]$, such that $\varphi(\psi(t)) = t$ whenever $t \in [0, 2\pi]$, $\psi(\varphi(t)) = t$ whenever $t \in [0, 2\pi]$, $\mathbf{r}_1(\psi(t)) = \mathbf{r}_2(t)$ for all $t \in [0, 2\pi]$, and either $\varphi(0) = 0$ and $\varphi(2\pi) = 2\pi$ or $\varphi(0) = 2\pi$ and $\varphi(2\pi) = 0$.

Let us define $\varphi(t) = 2\pi - t$, $\psi(t) = 2\pi - t$, for $0 \le t \le 2\pi$. Then

$$\varphi(\psi(t)) = 2\pi - \psi(t) = 2\pi - (2\pi - t) = t$$

whenever $t \in [0, 2\pi]$, and

$$\psi(\varphi(t)) = 2\pi - \varphi(t) = 2\pi - (2\pi - t) = t$$

whenever $t \in [0, 2\pi]$. Furthermore, if $0 \leq t \leq 2\pi$, then

$$\begin{aligned}
\mathbf{r}_1(\psi(t)) &= R\cos\psi(t)\,\mathbf{i} + R\sin\psi(t)\,\mathbf{j} \\
&= R\cos(2\pi - t)\,\mathbf{i} + R\sin(2\pi - t)\,\mathbf{j} \\
&= R\cos(-t)\,\mathbf{i} + R\sin(-t)\,\mathbf{j} \\
&= R\cos t\,\mathbf{i} - R\sin t\,\mathbf{j} \\
&= \mathbf{r}_2(t)\,.
\end{aligned}$$

Finally, the condition that "either $\varphi(0) = 0$ and $\varphi(2\pi) = 2\pi$ or $\varphi(0) = 2\pi$ and $\varphi(2\pi) = 0$" is clearly true, because $\varphi(0) = 2\pi$ and $\varphi(2\pi) = 0$.

So $\mathbf{r}_1$ and $\mathbf{r}_2$ are equivalent under reparametrization. In other words, **$\mathbf{r}_1$ and $\mathbf{r}_2$ define the same unoriented curve.**

Are $\mathbf{r}_1$ and $\mathbf{r}_2$ equivalent under orientation-preserving reparametrization? In other words, do **$\mathbf{r}_1$ and $\mathbf{r}_2$ define the same oriented curve?** The answer is "no," but this is not completely trivial.

First of all, notice that the φ and ψ that we have defined above do not count for showing that $\mathbf{r}_1$ and $\mathbf{r}_2$ are equivalent under orientation-preserving reparametrization, because in the definition of "equivalence under orientation-preserving reparametrization" the function φ has to satisfy $\varphi(0) = 0$, and $\varphi(2\pi) = 2\pi$, whereas the φ that we have constructed satisfies $\varphi(0) = 2\pi$, and $\varphi(2\pi) = 0$. (That is, this φ "changes the direction of time from forward to backward.")

This observation, however, does not suffice to prove that the parametrized curves $\mathbf{r}_1$ and $\mathbf{r}_2$ are not equivalent under orientation-preserving reparametrization. Just because one particular choice of φ and ψ does not work, it does not mean that there is not other choice that works. To prove that $\mathbf{r}_1$ and $\mathbf{r}_2$ are not equivalent under orientation-preserving reparametrization we have to show that there do not exist φ, ψ that satisfy the conditions of the definition of "equivalence under orientation-preserving reparametrization." That is, we have to show that

(*) *there does not exist a pair of continuous functions $\varphi : [0, 2\pi] \mapsto [0, 2\pi]$, $\psi : [0, 2\pi] \mapsto [0, 2\pi]$, such that*

$$\varphi(\psi(t)) = t \quad \text{whenever} \quad t \in [0, 2\pi]\,,$$

$$\psi(\varphi(t)) = t \quad \text{whenever} \quad t \in [0, 2\pi]\,,$$

$$\mathbf{r}_1(\psi(t)) = \mathbf{r}_2(t) \quad \text{for all} \quad t \in [0, 2\pi]\,,$$

$$\varphi(0) = 0\,,$$

and

$$\varphi(2\pi) = 2\pi\,.$$

If we want to prove (*), we find ourselves in the familiar "problematic" situation that we have encountered earlier many times. **Proving that an object of a certain kind exists is easy, because all we have to do is exhibit one**

such object. But proving that an object of a certain kind does not exist is hard, because if we try to find one and fail, that is not enough to conclude that the object does not exist. The reason for our failure to find it might just have been that we weren't smart enough to look for it in the right place.

So **we have to think!!!** Let us think, then. We want to prove that a certain thing (in our case, a certain pair of functions) does not exist. The most obvious way to get started is to imagine that the thing does exist, and explore that possibility, hoping that we will get to something impossible. In other words, we should try to do a proof by contradiction.

So let us suppose that there exists a pair (φ, ψ) satisfying all the conditions of (*). Pick one such pair (φ, ψ).

Then the condition that $\varphi(0) = 0$, together with the fact that $\psi(\varphi(t)) = t$, implies $\psi(0) = \psi(\varphi(0)) = 0$. So $\psi(0) = 0$. Then $\psi(t)$ is close to 0 if t is close to 0. (This can be made precise using the "ε-δ definition of limit" and the definition of continuous function.) But, if s is close to 0 and $s \in [0, 2\pi]$, then $\mathbf{r}_1(s) = R\cos s\,\mathbf{i} + R\sin s\,\mathbf{j}$, so the $\mathbf{j}$-component of $\mathbf{r}_1(s)$ is $R\sin s$, which is positive if s is positive and close to 0. In particular, since $\psi(t)$ is close to 0 if t is close to 0, and $\psi(t)$ is positive, we can conclude that *the* $\mathbf{j}$*-component of* $\mathbf{r}_1(\psi(t))$ *is positive if* t *is positive and close to* 0. On the other hand, $\mathbf{r}_1(\psi(t)) = \mathbf{r}_2(t) = R\cos t\,\mathbf{i} - R\sin t\,\mathbf{j}$. So *the* $\mathbf{j}$*-component of* $\mathbf{r}_1(\psi(t))$ *is negative if* t *is positive and close to* 0. We have thus reached a contradiction, and this shows that the pair (φ, ψ) does not exist.

Summarizing:

1. $\mathbf{r}_1$ **and** $\mathbf{r}_2$ **define the same unoriented curve.**

2. $\mathbf{r}_1$ **and** $\mathbf{r}_2$ **do not define the same oriented curve.**

EXAMPLE 11. A curve in the plane divides the plane into several pieces. Consider, for example, the following three curves C_1, C_2, C_3:

Clearly, C_1 divides the plane into two pieces, C_2 divides it into three pieces, and C_3 divides it into five pieces. (In case you want names and equations, C_1 is a *circle*, given by $x^2 + y^2 = 1$, C_2 is a *lemniscate*, given by $x^4 - x^2 + y^2 = 0$, and C_3 is a *four-petal rose*, given in polar coordinates by $r = \cos 2\varphi$ and in Cartesian coordinates by $(x^2 + y^2)^3 = (x^2 - y^2)^2$.)

When we say that "a curve divides the plane into a certain number of pieces," what does this mean, precisely? What is a "piece." The answer, as you have probably guessed by now, is that a "piece" is an equivalence class of some equivalence relation. Here are the precise definitions:

DEFINITION. Let A be a subset of the plane $\mathbb{R}^2$. (We could equally well consider sets of points of $\mathbb{R}^3$, or of $\mathbb{R}^4$, or of $\mathbb{R}^n$ for any n, but here we will just stick to the case $n = 2$.) We say that a point $P \in A$ is *connected to a point $Q \in A$ by a path in A* if there exists a continuous curve $\xi : [0,1] \mapsto \mathbb{R}^2$ such that $\xi(0) = P$, $\xi(1) = Q$, and $\xi(t) \in A$ for every $t \in [0,1]$. $\diamondsuit$

THEOREM. Let A be a subset of the plane $\mathbb{R}^2$. Let R be the relation on A given by

$$R = \{(P,Q) \in A \times A : P \text{ is connected to } Q \text{ by a path in } A\}\,.$$

Then R is an equivalence relation.

PROOF. I will prove that R is symmetric, and leave it up to you to prove that R is reflexive and transitive.

Suppose $P \in A$, $Q \in A$, and P is connected to Q by a path in A. We want to prove that Q is connected to P by a path in A. The idea is to do this by picking a path in A that goes from P to Q and following it backwards, thereby obtaining a path in A that goes from Q to P.

To do this precisely, pick a continuous curve $\xi : [0,1] \mapsto \mathbb{R}^2$ such that $\xi(0) = P$, $\xi(1) = Q$, and $\xi(t) \in A$ for every $t \in [0,1]$. Define another curve $\eta : [0,1] \mapsto \mathbb{R}^2$ by letting $\eta(t) = \xi(1-t)$ for $0 \le t \le 1$. Then η is continuous, $\eta(0) = \xi(1) = Q$, $\eta(1) = \xi(0) = P$, and $\eta(t) \in A$ for all $t \in [0,1]$, because if $t \in [0,1]$ then $\eta(t) = \xi(1-t)$, and $\xi(1-t) \in A$.

So Q is connected to P by a path in A, and our proof that R is symmetric is complete. $\diamondsuit$

DEFINITION. Let A be a subset of the plane $\mathbb{R}^2$. Let R be the equivalence relation on A of the previous theorem. Then the equivalence classes of R are the *connected components* of A. $\diamondsuit$

COMMENT. The notion of "connected component" is the precise mathematical translation of "piece" that we were looking for. To apply it to the three curves considered earlier, we look in each case at the plane $\mathbb{R}^2$ minus the curve. So we let $A_1 = \mathbb{R}^2 - C_1$, $A_2 = \mathbb{R}^2 - C_2$, $A_3 = \mathbb{R}^2 - C_3$. Now that we have a precise definition of "piece," we can prove rigorously that A_1 has two "pieces," A_2 has three, and A_3 has five.

PROBLEM. Prove that A_1 has exactly two connected components.

PROBLEM. Prove that A_2 has exactly three connected components.

PROBLEM. Prove that A_3 has exactly five connected components.

EXAMPLE 12. Some surfaces in three-dimensional space have two "sides." Other surfaces have only one "side." An example of a surface with only one side is the **Möbius strip**. You can build a Möbius strip as follows: cut a rectangular band of paper with two very long sides and two very short sides; label the vertices a, b, c, d as shown in the picture; then glue the ab side to the cd side after twisting the band, in such a way that a is glued to d and b to c.

When we say that "a surface has a certain number of sides," what does this mean, precisely? What is a "side" of a surface?

If you have guessed that a "side" of a surface is an equivalence class of some equivalence relation, then you have guessed right! Here are the precise definitions:

DEFINITION. Let S be a surface in $\mathbb{R}^3$. A *unit normal* to S is an ordered pair $(P, \mathbf{v})$ consisting of (a) a point P of S, and (b) a vector $\mathbf{v}$ such that $|\mathbf{v}| = 1$ and $\mathbf{v}$ is perpendicular to the surface at P. $\diamondsuit$

NOTATION. Let S be a surface in $\mathbb{R}^3$. We use $N(S)$ to denote the set of all unit normals to S. (In mathematical jargon, $N(S)$ is called the "unit normal bundle" of S.) $\diamondsuit$

DEFINITION. Let S be a surface in $\mathbb{R}^3$. A unit normal $(P, \mathbf{v}) \in N(S)$ is said to be *connected in $N(S)$ to a unit normal $(Q, \mathbf{w}) \in N(S)$ by a path in $N(S)$* if there exists a continuous curve $\xi : [0, 1] \mapsto \mathbb{R}^6$ such that $\xi(0) = (P, \mathbf{v})$, $\xi(1) = (Q, \mathbf{w})$, and $\xi(t) \in N(S)$ for every $t \in [0, 1]$. $\diamondsuit$

COMMENT. Why $\mathbb{R}^6$? That's because the unit normals to S are pairs $(P, \mathbf{v})$, and P and $\mathbf{v}$ have three components each, so the pair $(P, \mathbf{v})$ has six components.

THEOREM. Let S be a surface in $\mathbb{R}^3$. Let R be the relation on $N(S)$ given by

$$R = \left\{ \Big((P, \mathbf{v}), (Q, \mathbf{w}) \Big) : (P, \mathbf{v}) \text{ is connected to } (Q, \mathbf{w}) \text{ by a path in } N(S) \right\}.$$

Then R is an equivalence relation on S.

DOES THIS THEOREM NEED A PROOF? Not really, because this is just a special case of the theorem of Example 11. (In Example 11 we only talked about subsets of $\mathbb{R}^2$, but we could equally well have worked with subsets of $\mathbb{R}^3$ or $\mathbb{R}^4$ or even $\mathbb{R}^6$.)

DEFINITION. Let S be a surface in $\mathbb{R}^3$. A *side* of S is an equivalence class of the relation R of the previous theorem. $\diamondsuit$

COMMENT. Using the terminology of Example 11, **the sides of S are the connected components of** $N(S)$.

ANOTHER COMMENT. How should we think of unit normals and the relation defined above? You should visualize this as follows: a unit normal is just a little person (of height equal to 1) standing on the surface, perpendicularly to it. A path in the set of unit normals is what you get when our little person walks on the surface. Two unit normals are connected by a path if our little person can walk from one to the other. A surface such as a Möbius strip has only one side because our little person can start walking at some point on the surface and after a long walk she can come back to the point where she started, but "upside down." A spherical surface has two sides because if our little person starts walking at some point on the surface and is on the "outside" of the surface, then no matter how she walks she can never return to the same position but finding herself now on the "inside."