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ENVELOPES, HIGHER-ORDER OPTIMALITY CONDITIONS, AND LIE BRACKETS[+]

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ABSTRACT

We describe a generalization to Optimal Control of the theory of envelopes of the classical Calculus of Variations. This generalization extends our previous work, by allowing the use of "quasi-extremal" trajectories, i.e. trajectories that satisfy all the conditions of the Pontryagin Maximum Principle except for the fact that the sign of the constant λ_0 that appears in the Hamiltonian multiplying the Lagrangian is not restricted, and by making it possible to prove nonoptimality of some "abnormal" trajectories. We illustrate the use of the envelope technique by presenting an alternative proof of a theorem of Agrachev, Gamkrelidze and Schättler on the number of switchings of time-optimal bang-bang trajectories for two-vector-field systems in three dimensions.

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§1. Introduction .

In the classical Calculus of Variations, the theory of envelopes provides an elegant geometric approach for proving, in some cases, that certain extremal trajectories are not in fact optimal. We announced in [3] an extension of this theory to Optimal Control problems, and showed how the theory could be used, in combination with other geometric considerations and Lie bracket calculations, to obtain information about optimal trajectories, e.g. to show that in some cases a bang-bang extremal with too many switchings cannot be optimal. The purpose of this partly expository note is to give a brief review of the theory and present an improved version (which, in particular, allows for the treatment of some "abnormal" trajectories), to give precise formulations of some definitions that were omitted in [3], and to illustrate the use of the method by proving the following result:

Theorem 1.1. *Let f and g be smooth vector fields on an open subset Ω of $\mathbb{R}^3$. Let $\bar{p}$ be a point of Ω such that each of the two triples $(g,[f,g],[f-g,[f,g]])$, $(g,[f,g],[f+g,[f,g]])$, is linearly independent at $\bar{p}$. Then there exists a neighborhood U of $\bar{p}$ and a time $T>0$ such that, if γ is any time-optimal bang-bang trajectory of the system*

$$\dot{x} = f(x) + ug(x) \;\;,\;\; x \in \Omega \;\;,\;\; |u| \le 1\,, \tag{1.1}$$

which is entirely contained in U and has duration not greater than T, then γ has at most two switchings.

(In the above statement, and throughout the paper, "$[\ldots,\ldots]$" is the Lie bracket. The word "smooth" means "of class C^∞." A *trajectory* of a control system such as (1.1) is an absolutely continuous curve $\gamma:[a,b]\to\Omega$ that satisfies, for almost every t, the equation $\dot{\gamma}(t)=f(\gamma(t))+u(t)g(\gamma(t))$ for some measurable $[-1,1]$-valued function $u(\cdot)$. The *duration* of γ is the number $b-a$, γ is *bang-bang* if $u(\cdot)$ can be chosen to take values in the set $\{-1,1\}$, and a *switching* is a time t such that it is not the case that, for some $\varepsilon>0$, γ is, on the interval $[t-\varepsilon,t+\varepsilon]\cap[a,b]$, an integral curve of $f-g$ or of $f+g$.)

Theorem 1.1 has been proved by other methods by H. Schättler in [2] and by A. Agrachev and R. Gamkrelidze in [1], after an earlier, weaker result had been established by us in [3] using the envelopes method. (The result of [3] differed from Theorem 1.1 in that it required the much stronger hypothesis that the triple $(f,g,[f,g])$ be linearly independent at $\bar{p}$, thereby excluding, for instance, the case when $\bar{p}$ is an equilibrium point.) Our purpose here is to show how the envelopes approach can give the full result by means of a very simple calculation. A subsidiary purpose is to show the importance and usefulness of working with the appropriate algebraic formalism arising from differential geometry, especially Lie brackets and alternating tensors.

The envelope theorems stated here are strengthened versions of those announced in [3], in that they apply to families of "quasi-extremals" (defined below) that need not be true extremals. This is important because —as will be illustrated in our proof of Theorem 1.1— in some cases an "abnormal" extremal may be embeddable in a family of quasi-extremals but not in a familiy of extremals. Also, we give precise statements of the differentiability conditions that were only sketchily referred to in [3].

§2. Envelopes .

We consider a general control system of the form

$$\dot{x} = f(x,u)\ ,\ x \in M\ ,\ u \in U\ , \tag{2.1}$$

where we assume that M is a smooth manifold, U is a set, and each map $f_u : x \to f(x,u)$ is a smooth vector field on M. A *control* is a U-valued function on some compact interval $[a,b]$. The control $u(\cdot)$ is *f-admissible* on a subset S of M if the time-varying vector-field $(t,x) \to f(x,u(t))$ satisfies local Lipschitz-Carathéodory conditions on S, i.e. if $f(x,u(t))$ is jointly measurable in x and t, and for every coordinate patch $(\Omega,(x_1,\ldots,x_n))$ and every compact subset K of Ω there exists an integrable function $\varphi : [a,b] \to \mathbb{R}$ such that the bound

$$\Big\| f\big(x,u(t)\big) \Big\| + \sum_{i=1}^{n} \Big\| \frac{\partial f}{\partial x_i}\big(x,u(t)\big) \Big\| \le \varphi(t) \tag{2.2}$$

holds for all $x \in K \cap S$, $t \in [a,b]$. A set $\mathcal{U}$ of controls is *uniformly admissible* on S if, for each Ω, K as above, there is an integrable φ such that (2.2) holds for all $u(\cdot) \in \mathcal{U}$, $x \in K \cap S$, $t \in [a,b]$. A control is *f-admissible* if it is f-admissible on M. A *trajectory* for a control $u(\cdot) : [a,b] \to U$ is an absolutely continuous curve $\gamma : [a,b] \to M$ that satisfies $\dot{\gamma}(t) = f(\gamma(t),u(t))$ for almost every t. In that case, the numbers a, b will also be denoted by $T_-(\gamma)$, $T_+(\gamma)$, and referred to as the *initial and terminal times* of γ. Also, we use $x_-(\gamma)$, $x_+(\gamma)$ to denote, respectively, the *initial and terminal states* of γ, i.e. $x_\pm(\gamma) \stackrel{\text{def}}{=} \gamma(T_\pm(\gamma))$.

An *f-admissible pair* is a pair $(\gamma, u(\cdot))$ consisting of a trajectory γ and a corresponding control $u(\cdot)$, such that $u(\cdot)$ is f-admissible on some open set containing γ. The set of all such pairs is denoted by $\mathrm{Adm}(f)$. A set $\mathcal{A}$ of f-admissible pairs is *uniformly f-admissible* if there exists an open set S, containing all the curves γ for all the pairs in $\mathcal{A}$, such that the set of all controls occurring in $\mathcal{A}$ is uniformly f-admissible on S.

If A is a topological space and $\mathcal{A} = \{(\gamma^\alpha, u^\alpha(\cdot))\}_{\alpha\in A}$ is a parametrized family of members of $\mathrm{Adm}(f)$, then $\mathcal{A}$ is *f-continuous* if (a) the *initial and terminal time maps* $\alpha \to T_\pm(\gamma^\alpha)$ and the map $(t,\alpha) \to \gamma^\alpha(t)$ are continuous, and (b) for every $\bar\alpha \in A$ there is a neighborhood $N(\bar\alpha)$ of $\bar\alpha$ such that the set $\{(\gamma^\alpha, u^\alpha(\cdot)) : \alpha \in N(\bar\alpha)\}$ is uniformly f-admissible.

If $A \subseteq \mathbb{R}^m$, then we want to define what it means for $\mathcal{A}$ to be *f-differentiable.* For simplicity, we limit ourselves to the case when M is an open subset of $\mathbb{R}^n$ rather than a general manifold. We first introduce the notation $\mathrm{Bor}(J, \mathbb{R}^n)$ to denote the space of all finite $\mathbb{R}^n$-valued Borel measures on the compact interval J. This space is the dual of $C(J, \mathbb{R}^n)$, the space of continuous $\mathbb{R}^n$-valued functions on J. So $\mathrm{Bor}(J, \mathbb{R}^n)$ can be given the weak* topology, i.e. the weakest topology with respect to which all the maps $E_\varphi : \mathrm{Bor}(J, \mathbb{R}^n) \ni \mu \to \int_J \varphi\, d\mu$, for $\varphi \in C(J, \mathbb{R}^n)$, are continuous. (Here $\varphi\, d\mu$ means $\sum_i \varphi_i\, d\mu_i$.) In particular, if $\mu : A \ni \alpha \to \mu^\alpha \in \mathrm{Bor}(J, \mathbb{R}^n)$ is a map, we will call μ *differentiable* at a point $\bar\alpha$ if it is differentiable at $\bar\alpha$ as a map into $\mathrm{Bor}(J, \mathbb{R}^n)$ with the weak* topology, i.e. if the map $\alpha \to E_\varphi(\mu^\alpha)$ is differentiable at $\bar\alpha$ for all $\varphi \in C(J, \mathbb{R}^n)$.

We are now ready to define differentiability of a family $\mathcal{A}$ as above. We will call $\mathcal{A}$ *differentiable* at a point $\alpha \in A$ if three requirements are satisfied. First, we require $\mathcal{A}$ to be f-continuous on some neighborhood of α. Given this, it follows in particular that there exists an interval $I = [a, b]$ and a neighborhood N of α in $\mathbb{R}^m$ such that, for all $\beta \in N \cap A$, the interval $I(\beta) = [T_-(\gamma^\beta), T_+(\gamma^\beta)]$ is contained in the interior of I. Now consider, for $\beta \in N \cap A$, the $\mathbb{R}^n$-valued function m_α^β and the $\mathbb{R}^{n\times n}$-valued function M_α^β given by $m_\alpha^\beta(t) = f(\gamma^\alpha(t), u^\beta(t))$ and $M_\alpha^\beta(t) = \frac{\partial f}{\partial x}\big(\gamma^\alpha(t), u^\beta(t)\big)$ for $t \in I$. (In principle, these functions are only defined on $I(\alpha) \cap I(\beta)$, but we extend them to all of I as follows. First, we extend γ^α to I by making it constant on $[a, T_-(\alpha)]$ and on $[T_+(\alpha), b]$. This makes the functions m_α^β and M_α^β well defined on $I(\alpha)$, and we then extend them to I by defining them to be zero outside of $I(\alpha)$.) In view of the continuity of $\mathcal{A}$, these functions are integrable if N is small enough. Our second requirement is then that the function M_α^β should converge to M_α^α in $L^1(I, \mathbb{R}^{n\times n})$ as $\beta \to \alpha$. Finally, the third requirement is that the map $\beta \to \mu_\alpha^\beta$ be differentiable at α as a map into $\mathrm{Bor}(I, \mathbb{R}^n)$, where μ_α^β is the absolutely continuous Borel vector measure on I with density m_α^β.

Remark 2.1. With this definition, if the differentiability condition holds on an interval I, it does not necessarily follow that it holds on any proper subinterval of I. (To see this, consider the system $\dot{x} = u$, $x \in \mathbb{R}$, $u \in \mathbb{R}$, let $u^0(t) \equiv 0$ and, for $h > 0$, let $u^h : [0,1] \to [-1,1]$ equal 0 up to time $\theta(h) - h$, then 1 for $\theta(h) - h < t < \theta(h)$, then -1 for $\theta(h) < t < \theta(h) + h$, and then 0 up to time 1, where $\theta(h) = h + (\frac{1}{2} - h)(1 + \sin(\frac{1}{h}))$.) It is for this reason that the definition on manifolds is more difficult to formulate, since one cannot just proceed locally and work on intervals small enough to be contained in the domain of a chart. $\square$

The preceding —rather technical— definition implies that $x_+(\gamma^\alpha)$ is a differentiable function of α if $x_-(\gamma^\alpha)$ is, and the differential of the map $\alpha \to x_+(\gamma^\alpha)$ can be obtained by solving the *variational equation*, as we now explain. If $\mu \in \mathrm{Bor}([a,b], \mathbb{R}^n)$, define $F_\mu : [a,b] \to \mathbb{R}^n$ by $F_\mu(t) = \mu([a,t])$. Then F_μ is a right-continuous vector-valued function of bounded variation on $[a,b]$ and, if φ is continuous, then the integral $\int_{[a,b]} \varphi\, d\mu$ is the same as the Riemann-Stieltjes integral $\int_{[a,b]} \varphi\, dF_\mu$. If $R : [a,b] \to \mathbb{R}^{n\times n}$ is integrable, then we can consider the equation

$$d\eta(t) = R(t)\eta(t)dt + d\mu \quad . \tag{2.3}$$

A *solution of* (2.3) *with initial condition* $\eta(a-) = \eta_0$ is a function $\eta : [a,b] \to \mathbb{R}^n$ of bounded variation such that $\eta(t) = \eta_0 + \int_0^t R(s)\eta(s)\, ds + F_\mu(t)$ for all $t \in [a,b]$. (Notice that the actual value $\eta(a)$ will differ from η_0 if $\{a\}$ is an atom of μ.) It can be shown that this is equivalent to the requirement that $\eta(t) = \Phi(t,a)\eta_0 + \int_{[a,t]} \Phi(t,s)\, d\mu(s)$ for all t, where Φ is the fundamental solution of $\dot{X} = R(t)X$. This implies, in particular, the *existence and uniqueness of solutions*, for every $R(\cdot)$, μ, η_0, of the initial value problem consisting of (2.3) and the condition $\eta(a-) = \eta_0$. If we now take $R(t) = \frac{\partial f}{\partial x}(\gamma^\alpha(t), u^\alpha(t))$, and μ to be the weak* limit of the measures $(\frac{1}{h})(\mu_\alpha^{\alpha+hv} - \mu_\alpha^\alpha)$ as $h \to 0$, then the resulting equation is the *variational equation at* α *in the direction of* v. Let δ_v be the solution of this equation with initial condition $\delta_v(a-) = \delta_v^*$, where δ_v^* is the derivative of $x_-(\gamma^\beta)$ at α in the direction of v. Then the directional derivative of $x_+(\gamma^\beta)$ at α in the direction of v is equal to $\delta_v(b)$.

Now suppose that we are given a system (2.1) as above and, in addition, a *Lagrangian* L, i.e. a function $M \times U \ni (x,u) \to L(x,u) \in \mathbb{R}$ which is smooth in x for each fixed $u \in U$. We can then consider the *augmented system*

$$\dot{\xi} = F(\xi,u) \ , \ \xi \in M \times \mathbb{R} \ , \tag{2.4}$$

where $\xi = (x,y)$ and $F(\xi,u) = (f(x,u), L(x,u))$. A control $u(\cdot)$ is (f,L)-*admissible* on a subset S of M if it is admissible on $S \times \mathbb{R}$ for the augmented system. We call a pair $(\gamma, u(\cdot))$ (f,L)-*admissible* if $u(\cdot)$ is a control, γ is a trajectory of (2.1) corresponding to $u(\cdot)$, and the *augmented pair* $(\gamma^{\mathrm{aug}}, u(\cdot))$ is F-admissible. (Here $\gamma^{\mathrm{aug}}(t) = (\gamma(t), \kappa(t))$, and κ is the *running cost*, i.e. $\kappa(t) = \int_{T_-(\gamma)}^{t} L(\gamma(s),u(s))\,ds$.) We write $\mathrm{Adm}(f,L)$ to denote the class of all (f,L)-admissible pairs. A subset of $\mathrm{Adm}(f,L)$ (resp. a family of members of $\mathrm{Adm}(f,L)$ parametrized by $\alpha \in A$, with A a topological space) is said to be (f,L)-*uniformly admissible* (resp. (f,L)-*continuous*) if the corresponding set (resp. parametrized family) of augmented pairs is uniformly F-admissible (resp. F-continuous). Similarly, if $A \subseteq \mathbb{R}^m$ then we call a family parametrized by A (f,L)-*differentiable* if the corresponding set of augmented pairs is F-differentiable.

We define the *cost* $J(\gamma,u(\cdot))$ of a pair $(\gamma,u(\cdot))$ belonging to $\mathrm{Adm}(f,L)$ to be the integral $\int_a^b L(\gamma(t),u(t))\,dt$, where $[a,b]$ is the domain of definition of $u(\cdot)$. We call $(\gamma,u(\cdot))$ *optimal* if there does not exist another pair $(\tilde{\gamma},\tilde{u}(\cdot)) \in \mathrm{Adm}(f,L)$, such that $x_-(\tilde{\gamma}) = x_-(\gamma)$, $x_+(\tilde{\gamma}) = x_+(\gamma)$, and $J(\tilde{\gamma},\tilde{u}(\cdot)) < J(\gamma,u(\cdot))$. A *quasi-extremal* is a triple $\Gamma = (\zeta, u(\cdot), \lambda_0)$, where $u(\cdot)$ is an (f,L)-admissible control defined on some interval $[a,b]$, λ_0 is a real number, and ζ is an absolutely continuous curve, defined on $[a,b]$ and with values in the cotangent bundle T^*M of M, such that, if we write $\zeta(t) = (\gamma(t),\lambda(t))$ (so that $\lambda(t)$ is, for each t, a covector at $\gamma(t)$, i.e. a member of the cotangent space $T^*_{\gamma(t)}(M)$), then the *nontriviality condition* $(\lambda(t),\lambda_0) \neq (0,0)$ is satisfied for all t, and the equations $\dot{\gamma}(t) = f\big(\gamma(t),u(t)\big)$,

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x}\Big(\gamma(t),\lambda(t),\lambda_0,u(t)\Big) \ , \tag{2.5}$$

$$\begin{aligned} 0 &= H\big(\gamma(t),\lambda(t),\lambda_0,u(t)\big) \\ &= \min_{v\in U}\left\{H\big(\gamma(t),\lambda(t),\lambda_0,v\big) : v \in U\right\} \ , \end{aligned} \tag{2.6}$$

hold for almost all $t \in [a,b]$. (Here $H : T^*M \times \mathbb{R} \times U \to \mathbb{R}$ is the *Hamiltonian*, given by $H(x,\lambda,\lambda_0,u) = \langle \lambda, f(x,u)\rangle + \lambda_0 L(x,u)$ if $\lambda \in T^*_x M$. The first equation simply says that γ is a trajectory for $u(\cdot)$. Formulas (2.5) and (2.6) are known, respectively, as the *adjoint equation* and the *null minimizaton condition.* Although (2.5) appears to depend on the choice of local coordinates, it is well known that in fact it is intrinsic as long as the $\lambda(t)$'s are regarded as covectors.) If in addition $\lambda_0 \geq 0$, then Γ is said to be an *extremal.* The quasi-extremal is *normal* if $\lambda_0 \neq 0$, and *abnormal* if $\lambda_0 = 0$. The *projection* of $\Gamma = (\zeta, u(\cdot), \lambda_0)$ is the pair $\pi(\Gamma) = (\gamma, u(\cdot))$. The *Pontryagin Maximum Principle* says that, if $(\gamma, u(\cdot)) \in \mathrm{Adm}(f, L)$ is optimal, then there exists an extremal Γ such that $\pi(\Gamma) = (\gamma, u(\cdot))$.

A *family of quasi-extremals*, depending on a parameter α, with values in a set A, is a collection $\mathbf{\Gamma} = \{\Gamma^\alpha : \alpha \in A\}$ of quasi-extremals. If $\mathbf{\Gamma}$ is such a family, and $\Gamma^\alpha = (\zeta^\alpha, u^\alpha(\cdot), \lambda_0^\alpha)$, with the domain of definition of u^α being an interval $[a^\alpha, b^\alpha]$, then, if the parameter set A is a topological space, we call $\mathbf{\Gamma}$ *continuous* if the corresponding family of projections is (f, L)-continuous. If, in addition, A is a subset of a Euclidean space $\mathbb{R}^m$, then we call $\mathbf{\Gamma}$ *differentiable* if the family of associated projections is (f, L)-differentiable. We emphasize that the definitions of continuity and differentiability are formulated in terms of the projections, so they impose no continuity or differentiability requirement on λ^α as a function of α.

Now suppose that $\mathbf{\Gamma}$ is a *one-parameter family of quasi-extremals with fixed initial points and fixed initial times* (henceforth abbreviated as f.i.p.f.i.t.1p.f.q.e.), i.e. that the set A is a compact interval $I = [c, d]$, the initial times a^α all coincide, and the initial points $\gamma^\alpha(a)$ are all equal. We can then write $\zeta^\alpha(t) = (\gamma^\alpha(t), \lambda^\alpha(t))$, and define $\delta(\alpha) = \gamma^\alpha(b^\alpha)$. We say that $\mathbf{\Gamma}$ satisfies the *envelope condition* if

1. $\mathbf{\Gamma}$ is differentiable on I,
2. the cost function $\alpha \to J(\gamma^\alpha, u^\alpha(\cdot))$ is locally Lipschitz,
3. there exists an (f, L)-admissible control $I \ni \alpha \to v(\alpha) \in U$ such that δ is a trajectory for $v(\cdot)$, and the equation

$$H\big(\delta(\alpha), \lambda^\alpha(b^\alpha), \lambda_0^\alpha, v^\alpha\big) = 0 \tag{2.7}$$

holds for almost all $\alpha \in I$.

We then define the *envelope* of $\mathbf{\Gamma}$ to be the admissible pair $\Delta = (\delta, v(\cdot))$. We call $\mathbf{\Gamma}$ *normal* if all the Γ^α are normal, and *almost normal* if Γ^α is normal for almost every α.

The main result on envelopes is then the following *envelope theorem*:

Theorem 2.1. *Suppose the f.i.p.f.i.t.1p.f.q.e.* $\mathbf{\Gamma}$ *is almost normal, and let* $[c,d]$ *be its domain. Assume that* $\mathbf{\Gamma}$ *satisfies the envelope condition, and let* Δ *be the envelope of* $\mathbf{\Gamma}$*. Then*

$$J\big(\gamma^c,u^c(\cdot)\big)+J(\Delta)=J\big(\gamma^d,u^d(\cdot)\big)\ . \tag{2.8}$$

PROOF. We use a differentiation argument. We want to show that for each point $\alpha\in[c,d]$ the equality

$$J\big(\gamma^c,u^c(\cdot)\big)+\int_c^\alpha L\big(\delta(t),v(t)\big)\,dt=J\big(\gamma^\alpha,u^\alpha(\cdot)\big) \tag{2.9}$$

holds. The left side is an absolutely continuous function of α, and the right side is locally Lipschitz. Moreover, both sides clearly agree if $\alpha=c$. Hence our result will follow if we prove that the derivatives of both sides agree for almost every α. So it suffices to prove that the derivatives agree at those points α where (a) (2.7) holds, (b) $\lambda_0^\alpha\neq 0$, (c) the derivative of the integral in (2.9) is equal to $L(\delta(\alpha),v(\alpha))$, (d) the derivative of $\beta\to\delta(\beta)$ is equal to $f(\delta(\alpha),v(\alpha))$, and (e) $c<\alpha<d$.

Fix such an α. As in the definition of differentiability given above, we fix an interval $[a,b]$ that contains all the intervals $[a^\beta,b^\beta]$ for all β in some neighborhood of α, and we extend the functions $t\to f(\gamma^\alpha(t),u^\beta(t))$, $t\to L(\gamma^\alpha(t),u^\beta(t))$ to this interval.

The variational equation for the augmented system is a system $d\eta=\frac{\partial f}{\partial x}\eta\,dt+d\hat\mu$, $d\kappa=\frac{\partial L}{\partial x}\kappa\,dt+d\hat\nu$, where both partial derivatives are evaluated at $(\gamma^\alpha(t),u^\alpha(t))$, and $\hat\mu$, $\hat\nu$ are, respectively, the vector- and scalar Borel measures that are the limits as $h\to 0$ of the measures $\mu_\alpha^{\alpha+h}$, $\nu_\alpha^{\alpha+h}$, with densities $\frac{1}{h}(m_\alpha^{\alpha+h}-m_\alpha^\alpha)$ and $\frac{1}{h}(n_\alpha^{\alpha+h}-n_\alpha^\alpha)$. (Here $m_\alpha^\beta(t)=f(\gamma^\alpha(t),u^\beta(t))$ and $n_\alpha^\beta(t)=L(\gamma^\alpha(t),u^\beta(t))$.) If $(\bar\eta,\bar\kappa)$ is the solution of this system with initial conditions $\bar\eta(a-)=0$, $\bar\kappa(a-)=0$, then $\bar\kappa(b)$ is the derivative of $J(\gamma^\alpha,u^\alpha(\cdot))$ with respect to α. So it suffices to show that $\bar\kappa(b)=L(\delta(\alpha),v(\alpha))$. Now $\lambda_0^\alpha\bar\kappa(b)=\lambda_0^\alpha\int_a^b\frac{\partial L}{\partial x}\bar\eta\,dt+\lambda_0^\alpha F_{\hat\nu}(b)=\int_a^b(-\dot\lambda^\alpha-\lambda^\alpha\frac{\partial f}{\partial x})\bar\eta\,dt+\lambda_0^\alpha F_{\hat\nu}(b)$. Since $\lambda(\cdot)$ is absolutely continuous and $\bar\eta$ is of bounded variation, we can integrate by parts and get $\lambda_0^\alpha\bar\kappa(b)=-\lambda^\alpha(b)\bar\eta(b)+\int_a^b(\lambda^\alpha\,d\bar\eta-\lambda^\alpha\frac{\partial f}{\partial x}\bar\eta\,dt)+\lambda_0^\alpha F_{\hat\nu}(b)$. But the integrand is just $\lambda^\alpha d\hat\mu$. Also, $\bar\eta(b)$ is the derivative of $x_+(\gamma^\alpha)$, i.e. $\bar\eta(b)=f(\delta(\alpha),v(\alpha))$. Then (2.7) implies that $-\lambda^\alpha(b)\bar\eta(b)=\lambda_0^\alpha L(\delta(\alpha),v(\alpha))$. So we have shown that

$$\lambda_0^\alpha\bar\kappa(b)=\lambda_0^\alpha L(\delta(\alpha),v(\alpha))+\int_{[a,b]}[\lambda^\alpha\,d\hat\mu+\lambda_0^\alpha\,d\hat\nu]\ . \tag{2.10}$$

The integral in (2.10) is equal to the *two-sided* limit, as $h\to 0$, of $(1/h)\rho(h)$, where $\rho(h)=\int_a^b H\big(\gamma^\alpha(t),\lambda^\alpha(t),\lambda_0^\alpha,u^{\alpha+h}(t)\big)\,dt$. The null minimization condition implies that $\rho(h)\geq 0$

for all h. But then the two-sided limit must be equal to zero. So $\lambda_0^\alpha \bar{\kappa}(b) = \lambda_0^\alpha L(\delta(\alpha), v(\alpha))$ and, since $\lambda_0^\alpha \neq 0$, we get the equality $\bar{\kappa}(b) = L(\delta(\alpha), v(\alpha))$. □

The almost normality hypothesis is sometimes too restrictive, so we now establish a corollary of Theorem 2.1 that makes it possible to handle *abnormal* envelopes. We let $\mathbf{\Gamma} = \{\Gamma^\alpha : \alpha \in A\}$, where A is open in $\mathbb{R}^m$, be a f.i.p.f.i.t. but not necessarily one-parameter family of quasi-extremals. As before, we call $\mathbf{\Gamma}$ *almost normal* if Γ^α is normal for almost every $\alpha \in A$. We assume that we are given a smooth nowhere vanishing vector field E on A. We say that the pair $(\mathbf{\Gamma}, E)$ satisfies the *envelope condition* if $\mathbf{\Gamma}$ is continuous, and for every integral curve $\theta : [c, d] \to A$ of E, the resulting f.i.p.f.i.t.1p.f.q.e. $\mathbf{\Gamma}^\theta = \{\Gamma^{\theta(s)} : s \in [c, d]\}$ satisfies the envelope condition. We then have

Theorem 2.2. *Suppose $\mathbf{\Gamma}$ is an almost normal f.i.p.f.i.t family of quasi-extremals, with domain A. Assume that E is a smooth nowhere vanishing vector field on A such that the pair $(\mathbf{\Gamma}, E)$ satisfies the envelope condition. Let $\theta : [c, d] \to A$ be an integral curve of E, and let Δ^θ be the envelope of the one-parameter family $\mathbf{\Gamma}^\theta$. Then*

$$J\left(\gamma^{\theta(c)}, u^{\theta(c)}(\cdot)\right) + J(\Delta^\theta) = J\left(\gamma^{\theta(d)}, u^{\theta(d)}(\cdot)\right) . \tag{2.11}$$

Theorem 2.2 is not just a particular case of Theorem 2.1, because the individual f.i.p.f.i.t.1p.f.q.e.'s $\mathbf{\Gamma}^\theta$ may fail to be almost normal, and therefore one cannot directly apply Theorem 2.1 to them. The proof of Theorem 2.2 is, however, quite simple. It clearly suffices to show that Formula (2.11) holds for the restriction of θ to sufficiently small subintervals of $[c, d]$. So we may assume, in particular, that θ is a one-to-one map. If we pick a small smooth connected embedded hypersurface S that goes through $\theta(c)$ and is such that E is not tangent to S, and let $\mathcal{E}$ denote the flow of E (i.e. $s \to \mathcal{E}(q, s)$ is the integral curve of E which goes through q at time $s = 0$), then we may assume that, for some $\varepsilon > 0$, $\mathcal{E}$ maps the product $\Sigma = S \times (c - \varepsilon, d + \varepsilon)$ diffeomorphically onto an open subset U of A. For each $q \in S$, we let θ_q be the curve $s \to \mathcal{E}(q, s)$, $c \le s \le d$. If we prove that

$$J\left(\gamma^{\theta_q(c)}, u^{\theta_q(c)}(\cdot)\right) + J(\Delta^{\theta_q}) = J\left(\gamma^{\theta_q(d)}, u^{\theta_q(d)}(\cdot)\right) \tag{2.12}$$

for all $q \in S$, then our result will follow by taking $q = \theta(c)$. On the other hand, (2.12) will follow for all $q \in S$ if we prove it for a dense set of q's, since both sides of (2.12) depend continuously on q. So it suffices to prove (2.12) for almost all $q \in S$. The almost normality of $\mathbf{\Gamma}$ implies that the quasi-extremal $\Gamma^{\mathcal{E}(q,s)}$ is normal for all $(q,s) \in \Sigma$, except possibly on the complement of a subset B of Σ of Lebesgue measure zero. By Fubini's Theorem, the set $B(q) = \{s : (q,s) \in B\}$ has measure zero for almost all $q \in S$. Hence the f.i.p.f.i.t.1p.f.q.e. $\mathbf{\Gamma}^{\theta_q}$ is almost normal for almost all $q \in S$. Therefore we can apply Theorem 2.1 and conclude that (2.12) holds for almost all $q \in S$. As explained before, it then follows that (2.12) actually holds for all $q \in S$, and in particular for $q = \theta(c)$. $\square$

§3. Proof of Theorem 1.1 .

We now show how to use Theorem 2.2 to prove Theorem 1.1. We let f, g, Ω, $\bar{p}$ be as §1. We write $X = f - g$, $Y = f + g$. We let $V(p)$ denote, for $p \in \Omega$, the tangent space to Ω at p. (Naturally, all the $V(p)$ can be canonically identified with $\mathbb{R}^3$, but it is better *not* to do that, and to force ourselves to think in "differential geometric" terms. In particular, we will *never* carry out algebraic operations such as addition or exterior multiplication unless all the vectors involved are tangent vectors at the same point p.) We use $V^k(p)$ to denote the k-th exterior power of $V(p)$, so that $V^1(p) = V(p)$, and $V^2(p)$, $V^3(p)$ are linear spaces of dimensions 3 and 1, respectively. (In particular, if ω_1 and ω_2 are two elements of $V^3(p)$ such that $\omega_2 \neq 0$, then the quotient ω_1/ω_2 is a well defined real number.) Naturally, V^k denotes the corresponding bundle, and a section of V^k is called a *field of k-vectors.*

Vector fields are regarded as first-order differential operators which act on smooth functions on the left. Points can be identified with particular Schwartz distributions (Dirac delta functions) so, by duality, we will have vector fields act on points on the right, and write, for instance, pZ instead of $Z(p)$ for the value of the vector field Z at the point p. (We will not be strict, however, and will occasionally revert to the "old" notation $Z(p)$. Also, to be completely consistent we should write $p\varphi$ for the value at p of the function φ, but we will also use the more familiar notation $\varphi(p)$. Then, if φ is a function, the expression $pZ\varphi$ can either be read as $(pZ)\varphi$, i.e. the derivative of φ at p in the direction of Z, or as $Z\varphi(p)$, i.e. the value of $Z\varphi$ at p. Of course both readings give the same value.) Tangent vectors are also Schwarz distributions. (For instance, if v is the tangent vector at $t = 0$ to the curve γ, then $v = \lim_{h\to 0} h^{-1}(\delta_{\gamma(h)} - \delta_{\gamma(0)})$, where δ_p is the Delta function at p.) Diffeomorphisms act on functions on the left. (If Φ is a diffeomorphism and φ a function, then $\Phi\varphi$ is the function usually known as the pullback of φ by Φ, i.e. $\Phi\varphi(p) = \varphi(\Phi(p))$. So, by duality, diffeomorphisms act on distributions on the right. In particular, we write $p\Phi$, $v\Phi$ if p, v are, respectively, a point and a tangent vector. (The vector $v\Phi$ is the object usually known as "the image of v under the differential of Φ," and often denoted by $\Phi_*(v)$ or $d\Phi(v)$.) The diffeomorphisms of the flow of a vector field Z are denoted by e^{tZ}. So $t \to p\mathrm{e}^{tZ}$ is the integral curve of Z that goes through p at time 0, and the formula $\frac{d}{dt}(p\mathrm{e}^{tZ}) = pZ\mathrm{e}^{tZ} = p\mathrm{e}^{tZ}Z$ holds. Products of vector fields are well defined as diferential operators. In particular, the formula

$$\frac{\partial^{\nu_1+\ldots+\nu_n}}{\partial t_1^{\nu_1}\ldots t_n^{\nu_n}}(p\mathrm{e}^{t_1Z_1}\ldots\mathrm{e}^{t_nZ_n}) = pZ_1^{\nu_1}\mathrm{e}^{t_1Z_1}\ldots Z_n^{\nu_n}\mathrm{e}^{t_nZ_n} \tag{3.1}$$

holds. The *Lie bracket* of two vector fields Z, W is the vector field $[Z,W]\stackrel{\mathrm{def}}{=}ZW - WZ$. We use ad_Z to denote the operator $W \to [Z,W]$, acting on vector fields. The formula

$$\frac{d}{dt}p\mathrm{e}^{tZ}W\mathrm{e}^{-tZ} = p\mathrm{e}^{tZ}[Z,W]\mathrm{e}^{-tZ} \tag{3.2}$$

justifies using the notation $\mathrm{e}^{t\,\mathrm{ad}_Z}(W)$ for $\mathrm{e}^{tZ}W\mathrm{e}^{-tZ}$.

We use ω_X, ω_Y to denote, respectively, the fields of 3-vectors $g \wedge [f,g] \wedge [X,[f,g]]$ and $g \wedge [f,g] \wedge [Y,[f,g]]$. Our hypothesis is that $\omega_X(\bar{p})$ and $\omega_Y(\bar{p})$ are $\neq 0$. Hence we may assume, by shrinking Ω if necessary, that ω_X and ω_Y are $\neq 0$ throughout Ω. From now on we assume that this is the case, and also that Ω is connected, so the quotient $\varphi \stackrel{\mathrm{def}}{=} -\omega_Y/\omega_X$ is a well defined nowhere vanishing smooth function on Ω. We are also interested in the field of 2-vectors ξ given by $\xi = g \wedge [f,g]$. Since $\xi \wedge [X,[f,g]] \neq 0$, it is clear that ξ never vanishes on Ω.

In the remaining of this proof we will be introducing various positive numbers ε_1, ε_2, ..., each one strictly smaller than the preceding one, and associating to each ε_i the neighborhood Ω_i of $\bar{p}$, defined to be the open ball of radius ε_i and center $\bar{p}$. We begin by picking ε_1 such that the closure of Ω_1 is a subset of Ω. We then pick a constant $C > 0$ such that $\|X(p)\| \leq C$ and $\|Y(p)\| \leq C$ for all $p \in \Omega_1$. We pick ε_2 arbitrarily (but subject, of course, to the condition $0 < \varepsilon_2 < \varepsilon_1$). From now on, every time a new ε_i has been picked, we will also pick a time $T_i > 0$ such that $\varepsilon_i + CT_i < \varepsilon_{i-1}$. Moreover, if $i > 2$ we will require that $T_i < T_{i-1}$. It will then follow that every trajectory of (1.1) that goes through a point of Ω_i and has duration $\leq T_i$ is entirely contained in Ω_{i-1}.

For $p \in \Omega_2$, $|t| \leq T_2$, define $\eta(p,t)$ by $\eta(p,t)\stackrel{\mathrm{def}}{=}p(g \wedge \mathrm{e}^{t\,\mathrm{ad}_Y}(g))$, so that, for each fixed t, η is a field of 2-vectors on a domain that contains Ω_2. It is easy to see that $\eta(p,0) = 0$, and $\frac{\partial}{\partial t}\eta(p,0) = p\xi$, so $\eta = t(\xi + t\hat{\eta})$, where $\hat{\eta}$ is a smooth 2-vector-valued function of p and t. Now pick ε_3, T_3 subject to the usual requirement that $0 < T_3 < T_2$ and $\varepsilon_3 + CT_3 < \varepsilon_2$, and the additonal condition that $\xi(p) + t\hat{\eta}(p,t)$ never vanishes when $p \in \Omega_3$ and $|t| \leq T_3$. It then follows that $\hat{\eta}(p,t) \neq 0$ whenever $p \in \Omega_3$ and $0 < |t| < T_3$. We then pick ε_4 and T_4 such that $\varepsilon_4 + CT_4 < \varepsilon_3$, and $T_4 < T_3$.

We also need to consider the 3-vector $\rho(p,s,t) = p\big(g \wedge e^{s\,\mathrm{ad}_Y}(g) \wedge e^{t\,\mathrm{ad}_Y}(g)\big)$. This 3-vector vanishes when $s = 0$ or $t = 0$ or $t - s = 0$. Therefore ρ is equal to the product of $st(t-s)$ times a smooth function $\hat{\rho}(p,s,t)$. A simple computation then shows that $\hat{\rho}(p,0,0) = \frac{1}{2}p(g \wedge [f,g] \wedge [Y,[f,g]])$, so that $\hat{\rho}(\bar{p},0,0) \neq 0$. Hence, by making ε_4 and T_4 smaller, if necessary, we may assume that $\hat{\rho}(p,s,t) \neq 0$ whenever $p \in \Omega_4$ and $|t| \leq T_4$, $|s| \leq T_4$. It then follows that, under the same conditions, $\rho(p,s,t) \neq 0$ unless one of the quantities s, t, $t-s$ vanishes. In particular, $\rho(p,s,t) \neq 0$ if $p \in \Omega_4$ and $0 < s < t \leq T_4$. Also, we define $\rho^*(p,t) = p(g \wedge e^{t\,\mathrm{ad}_Y}(g) \wedge [f,g])$. Then

$$\rho^*(p,t) = -\frac{t^2}{2}\Big(\big(pg \wedge p[f,g] \wedge [Y,[f,g]]\big) + t\hat{\rho}^*(p,t)\Big)\ , \tag{3.3}$$

where $\hat{\rho}^*$ is smooth. So, by making ε_4 and T_4 even smaller, we may also assume that $\hat{\rho}^*(p,t) \neq 0$ whenever $p \in \Omega_4$ and $|t| \leq T_4$, and therefore $\rho^*(p,t) \neq 0$ under these conditions if, in addition, $t \neq 0$.

We now consider $XYXY$ *trajectories*, i.e. bang-bang trajectories γ obtained as follows: we start at a point p of Ω_4, follow an integral curve of X during a time t_0, then an integral curve of Y during time t_1, then an integral curve of X during time t_2 and, finally, an integral curve of X during time t_3. We require that all the times t_0, t_1, t_2, t_3 be > 0, and that the total time $t_0 + t_1 + t_2 + t_3$ be $\leq T_4$. We let G be the collection of all such curves. Notice that, for any choice of $p \in \Omega_4$ and the t_i as above, the curve γ is well defined and entirely contained in Ω_3. For any such curve, we write $p_0 = p$, $p_1 = p_0 e^{t_0 X}$, $p_2 = p_1 e^{t_1 Y}$, $p_3 = p_2 e^{t_2 X}$, $p_4 = p_3 e^{t_3 Y}$. Since $p_1 \in \Omega_3$ and $0 < t_1 < T_3$, we have $\eta(p_1,t_1) \neq 0$, so that *the vector fields g and $e^{t_1 \mathrm{ad}_Y}(g)$ are linearly independent at p_1.*

Let $I = [0, t_0 + t_1 + t_2 + t_3]$ be the domain of definition of γ, let $u(\cdot)$ be the corresponding control, and let $\tilde{\gamma}$ be the pair $(\gamma, u(\cdot))$. If $\tilde{\gamma}$ is time-optimal, then it has to be the projection of an extremal $\Gamma = (\gamma, \lambda(\cdot), u(\cdot), \lambda_0)$. Then the covectors $\lambda(t_i)$ must satisfy $\langle \lambda(t_i), p_i g\rangle = 0$ for $i = 1,2,3$, because the p_i are switching points. Equivalently, we have

$$\begin{aligned} 0 = \langle \lambda(t_1), p_1 g\rangle &= \langle \lambda(t_1), p_1 e^{t_1 \mathrm{ad}_Y}(g)\rangle \\ &= \langle \lambda(t_1), p_1 e^{t_1 \mathrm{ad}_Y} e^{t_2 \mathrm{ad}_X}(g)\rangle\,. \end{aligned} \tag{3.4}$$

On the other hand, $\lambda(t_1)$ cannot vanish because , if it did, then the equality $\langle\lambda(t_1), f(p_1)\rangle + \lambda_0 = 0$, would imply $\lambda_0 = 0$, which would contradict the nontriviality condition. (Note that it is possible to have $\langle\lambda(t_1), f(p_1)\rangle = \lambda_0 = 0$, i.e. Γ could be an abnormal extremal.) Since $\eta(p_1, t_1) \neq 0$, the vectors $p_1 g$ and $p_1 e^{t_1 \mathrm{ad}_Y}(g)$ are linearly independent, so their linear span S is two-dimensional. Therefore the covector $\lambda(t_1)$ is uniquely determined up to a positive multiplicative constant by the first two of the three conditions of (3.4). The extremal Γ will be abnormal iff $f(p_1) \in S$.

Now define

$$\sigma(p, t_0, t_1, t_2) = p e^{t_0 \mathrm{ad}_X}\left(g \wedge e^{t_1 \mathrm{ad}_Y}(g) \wedge e^{t_1 \mathrm{ad}_Y} e^{t_2 \mathrm{ad}_X}(g)\right). \tag{3.5}$$

Then, for each choice of the numberst_0, t_1, t_2, the map $p \to \sigma(p, t_0, t_1, t_2)$ is a field of 3-vectors. Also, $\sigma(p, t_0, t_1, t_2) = p e^{t_0 \mathrm{ad}_X}\left(\hat{\sigma}(p_1, t_1, t_2)\right)$, where $\hat{\sigma}(p_1, t_1, t_2) = p_1\big(g \wedge e^{t_1 \mathrm{ad}_Y}(g) \wedge e^{t_1 \mathrm{ad}_Y} e^{t_2 \mathrm{ad}_X}(g)\big)$. It is clear that σ and $\hat{\sigma}$ vanish when $t_1 = 0$ or $t_2 = 0$, so they are products of $t_1 t_2$ times smooth functions. Write $\hat{\sigma}(p_1, t_1, t_2) = t_1 t_2 \hat{\sigma}^*(p_1, t_1, t_2)$. Then

$$\hat{\sigma}^*(p_1, t_1, t_2) = p_1\left(g \wedge \frac{e^{t_1 \mathrm{ad}_Y} - 1}{t_1}(g) \wedge e^{t_1 \mathrm{ad}_Y} \frac{e^{t_2 \mathrm{ad}_X} - 1}{t_2}(g)\right) . \tag{3.6}$$

A Taylor expansion about $t_1 = t_2 = 0$ gives $\hat{\sigma}^*(p_1, t_1, t_2) = p_1(g \wedge G_1 \wedge G_2)$, where $G_1 = [f, g] + \frac{t_1}{2}[Y, [f, g]] + \ldots$ and $G_2 = [f, g] + \frac{t_2}{2}[X, [f, g]] + t_1[Y, [f, g]] + \ldots$. Therefore $\hat{\sigma}^*(p_1, 0, 0) = 0$, and

$$\frac{\partial \hat{\sigma}^*}{\partial t_1}(p_1, 0, 0) = \frac{1}{2}\omega_Y(p_1) , \tag{3.7}$$

$$\frac{\partial \hat{\sigma}^*}{\partial t_2}(p_1, 0, 0) = \frac{1}{2}\omega_X(p_1) . \tag{3.8}$$

Then the Implicit Function Theorem implies that, near $p_1 = \bar{p}$, $t_1 = t_2 = 0$, we can solve the equation $\hat{\sigma}^*(p_1, t_1, t_2) = 0$ in a unique way for t_2 as a function of t_1 and p_1. More precisely, there exist ε_5, T_5, $\hat{T}_5$ such that $0 < \varepsilon_5 < \varepsilon_4$, $T_5 < \hat{T}_5 < T_4$, and a smooth function $(p_1, t_1) \to \tau(p_1, t_1)$, defined for $p_1 \in \Omega_5$, $|t_1| < T_5$, with the property that, if (p_1, t_1) belongs to the domain of τ, then $\hat{\sigma}^*(p_1, t_1, \tau(p_1, t_1)) = 0$ and, if $\hat{\sigma}^*(p_1, t_1, t) = 0$, and $|t| < \hat{T}_5$, then $t = \tau(p_1, t_1)$. In particular, $\tau(p_1, 0) = 0$, so that the function τ is the product of t_1 times a smooth function $\hat{\tau}(p_1, t_1)$. The derivative of τ with respect to t_1 can be computed by implicit differentiation, and the result is

$$\frac{\partial \tau}{\partial t_1}(p_1, t_1) = -\frac{\frac{\partial \hat{\sigma}^*}{\partial t_1}(p_1, t_1, \tau(p_1, t_1))}{\frac{\partial \hat{\sigma}^*}{\partial t_2}(p_1, t_1, \tau(p_1, t_1))} . \tag{3.9}$$

The value of this derivative for $t_1 = 0$ is therefore equal to $-\frac{\omega_Y(p_1)}{\omega_X(p_1)}$, i.e. to $\varphi(p_1)$. On the other hand, this derivative is equal to $\hat{\tau}(p_1, 0)$. So we have obtained the expansion

$$\tau(p_1, t_1) = t_1(\varphi(p_1) + t_1\hat{\varphi}(p_1, t_1)) , \tag{3.10}$$

where $\hat{\varphi}$ is a smooth function of its arguments.

We now pick ε_6 and T_6 such that, whenever $p_1 \in \Omega_6$ and $|t_1| < T_6$, then $\varphi(p_1) + t_1\hat{\varphi}(p_1, t_1) \neq 0$. (This is possible because $\varphi(\bar{p}) \neq 0$.) It then follows that the value of $\tau(p_1, t_1)$ never vanishes when $p_1 \in \Omega_6$ and $0 < t_1 < T_6$. We then pick ε_7 and T_7, and restrict our trajectories γ by considering only those for which $p \in \Omega_7$ and the total duration is $\leq T_7$. Let G_7 be the resulting class of trajectories. Notice that, for any $\gamma \in G_7$, the point p_1 must belong to Ω_6. If there exists *some* time-optimal trajectory γ in G_7, then for such a γ the equation $\hat{\sigma}^*(p_1, t_1, t_2) = 0$ must hold. Therefore $t_2 = \tau(p_1, t_1)$. This implies in particular that $\hat{\tau}(p_1, t_1) > 0$ for at least one value of $p_1 \in \Omega_6$ and t_1 such that $|t_1| < T_6$. Therefore the inequality $\hat{\tau}(p_1, t_1) > 0$ holds for *all* $p_1 \in \Omega_6$ and $t_1 \in [-T_6, T_6]$. In particular, we have $\varphi(p_1) > 0$ for $p_1 \in \Omega_6$. From now on we assume that these conditions hold, for otherwise the class of time-optimal trajectories in G_7 would be void, and the desired conclusion would follow vacuously.

Now pick a $\gamma \in G_7$. Assume that γ is time-optimal. It then follows that $t_2 = \tau(p_1, t_1)$. If S is, as before, the linear span of $p_1 g$ and $p_1 \mathrm{e}^{t_1 \mathrm{ad}_Y}(g)$, then the fact that the three vectors $p_1 g$, $p_1 \mathrm{e}^{t_1 \mathrm{ad}_Y}(g)$ and $p_1 \mathrm{e}^{t_1 \mathrm{ad}_Y} \mathrm{e}^{t_2 \mathrm{ad}_X}(g)$ are linearly dependent, while the first two are independent, implies that there are numbers a and b such that

$$p_1 \mathrm{e}^{t_1 \mathrm{ad}_Y} \mathrm{e}^{t_2 \mathrm{ad}_X}(g) = a p_1 g + b p_1 \mathrm{e}^{t_1 \mathrm{ad}_Y}(g) . \tag{3.11}$$

It is clear that a and b depend on p_1 and t_1. We want to compute them explicitly. To compute a, we take the exterior product of both sides of (3.11) with $p_1\big(\mathrm{e}^{t_1 \mathrm{ad}_Y}(g) \wedge [Y, [f, g]]\big)$, and use the expansion (3.10). An easy calculation yields

$$a(p_1, t_1) = -\hat{\tau}(p, t_1) + t_1 \hat{a}(p_1, t_1) , \tag{3.12}$$

where $\hat{a}$ is a smooth function. One can also compute b, and the result is

$$b(p_1, t_1) = 1 + \hat{\tau}(p, t_1) + t_1 \hat{b}(p_1, t_1) , \tag{3.13}$$

where $\hat{b}$ is also a smooth function.

In order to prove that γ is not optimal, it suffices to prove that the restriction of γ to some subinterval is not optimal. In particular, we may make the first X part shorter, if necessary, and assume that $t_0 < -\tau(p_2, -t_1)$. (Recall that τ is also defined for negative values of t_1, and in that case its values are themselves negative.) *From now on we make this extra assumption.*

We now let $\bar{\gamma}$ be the curve γ restricted to the interval $[0, t_0 + t_1 + t_2]$, i.e. the piece of γ until the second switching point. We pick a small open disc A in $\mathbb{R}^2$, centered at the point $(t_0, t_1) \in \mathbb{R}^2$, and such that, if $(s_0, s_1) \in A$, then it follows that $s_0 > 0$, $s_1 > 0$, $s_0 + s_1 < T_7$ (so that $\tau(\tilde{p}, s_1)$ is defined and positive), and $s_0 + s_1 + \tau(\tilde{p}, s_1) < T_7$. (Here, and in what follows, we write $\tilde{p}$ for $pe^{s_0 X}$.) Moreover, we require that the inequality $s_0 < -\tau(\tilde{p}e^{s_1 Y}, -s_1)$, which has been assumed to hold for $s_0 = t_0$, $s_1 = t_1$, actually hold for all $(s_1, s_2) \in A$.

We let $\mathbf{\Gamma}$ be the family of quasi-extremals Γ^α obtained as follows: for $\alpha = (s_0, s_1) \in A$, we consider the trajectory γ^α that starts at p, is an integral curve of X up to time s_0, then an integral curve of Y up to time s_1, and then an integral curve of X up to time s_2, where $s_2 = \tau(\tilde{p}, s_1)$. This γ^α is then seen to be the projection of a quasi-extremal Γ^α by the following argument. The vectors $\tilde{p}g$ and $\tilde{p}e^{s_1 \mathrm{ad}_Y}(g)$ are linearly independent, and so there exists a covector Λ^α at $\tilde{p}$ that satisfies $\langle \Lambda^\alpha, \tilde{p}g \rangle = 0$ and $\langle \Lambda^\alpha, \tilde{p}e^{s_1 \mathrm{ad}_Y}(g) \rangle = 0$. This covector is uniquely determined up to a sign if we choose a normalization, e.g. by requiring that $\|\Lambda^\alpha\| = 1$. (The remaining sign ambiguity will be removed below.) We then define $\lambda^\alpha(t)$, for t in the domain of γ^α, by solving the adjoint equation with the condition $\lambda^\alpha(s_0) = \Lambda^\alpha$. We claim that, possibly after changing its sign, the resulting covector function $\lambda^\alpha(t)$ will satisfy the Hamiltonian minimization condition, i.e. the switching function $r(t) = \langle \lambda^\alpha(t), g(\gamma^\alpha(t)) \rangle$ will be ≥ 0 for $0 \leq t \leq s_0$ and $s_0 + s_1 \leq t \leq s_0 + s_1 + s_2$, and ≤ 0 for $s_0 \leq t \leq s_0 + s_1$. Since $\rho(q, s, t) \neq 0$ for $q \in \Omega_4$ and $0 < s < t \leq T_4$, we can in particular take $q = \tilde{p}$, $t = s_1$, and $0 < s < t$, and conclude that the linear span of the vectors $\tilde{p}g$ and $\tilde{p}e^{s_1 \mathrm{ad}_Y}(g)$ does not contain the vector $\tilde{p}e^{sY}(g)$ for any s such that $0 < s < s_1$. Equivalently, $r(t)$ never vanishes on the open interval $(s_0, s_0 + s_1)$ and so, by changing the sign of Λ^α, if necessary, we may assume that r is actually *negative* on this interval. Also, since $s_2 = \tau(\tilde{p}, s_1)$, the vectors $\tilde{p}g$, $\tilde{p}e^{s_1 \mathrm{ad}_Y}(g)$ and $\tilde{p}e^{s_1 \mathrm{ad}_Y}e^{s \mathrm{ad}_X}(g)$ are independent whenever $0 < s < s_2$, so r never vanishes on $(s_0 + s_1, s_0 + s_1 + s_2)$. Similarly, the vectors $pe^{sX}ge^{(s-s_0)X}$, $\tilde{p}g$ and $\tilde{p}e^{s_1 \mathrm{ad}_Y}(g)$ are also linearly independent if $0 \leq s < s_0$, because $pe^{sX}ge^{(s_0-s)X} \wedge \tilde{p}g \wedge \tilde{p}e^{s_1 \mathrm{ad}_Y}(g) = -\sigma(\tilde{p}e^{s_1 Y}, -s_1, -(s_0 - s))$, which is $\neq 0$ since $s_0 - s < -\tau(\tilde{p}e^{s_1 Y}, -s_1)$. So r does not vanish on $[0, s_0)$ either. To show that r has the correct sign everywhere it suffices to show that r changes sign at s_0 and $s_0 + s_1$. Since the derivative of r at t is just $\langle \lambda^\alpha(t), [f,g](\gamma^\alpha(t))\rangle$, it suffices to show that the vectors $\tilde{p}g$, $\tilde{p}[f,g]$ and $\tilde{p}e^{s_1 \mathrm{ad}_Y}(g)$, as well as the vectors $\tilde{p}e^{s_1 Y}g$, $\tilde{p}e^{s_1 Y}[f,g]$ and $\tilde{p}ge^{s_1 Y}$, are independent. But this is equivalent to showing that $\rho^*(\tilde{p}, s_1) \neq 0$ and $\rho^*(\tilde{p}e^{s_1 Y}, -s_1) \neq 0$, and these inequalities follow since $0 < s_1 < T_4$ and the points $\tilde{p}$, $\tilde{p}e^{s_1 Y}$ are in Ω_4.

Having shown that the Hamiltonian minimization condition is satisfied, it follows easily that the minimized Hamiltonian is constant and therefore, if its constant value is defined to be $-\lambda_0^\alpha$, we obtain a quasi-extremal. Notice that *this quasi-extremal may fail to be a true extremal*, because there is no reason why, once we have fixed the sign of Λ^α by requiring that r be negative on $(s_0, s_0 + s_1)$, i.e. that $\langle \Lambda^\alpha, \tilde{p}[f,g]\rangle < 0$, the inequality $\langle \Lambda^\alpha, \tilde{p}f\rangle \leq 0$ should follow. (If γ itself is a *normal* extremal, then this last inequality will hold strictly for $s_0 = t_0$, $s_1 = t_1$, and therefore it will also hold for neighboring s_0, s_1. However, if γ is abnormal, the inequality will not hold strictly. If, as s_0, s_1 vary near t_0, t_1, the vector $\tilde{p}f$ passes from one side of the linear span of $\tilde{p}g$ and $\tilde{p}e^{s_1 \mathrm{ad}_Y}(g)$ to the other, then the inequality will go in the wrong direction for some s_0, s_1.)

We have now completed the construction of our family of quasi-extremals $\mathbf{\Gamma}$. Our next task is to construct a nowhere vanishing smooth vector field E on A which satisfies the envelope condition. We take E to be of the form $\psi_0 \frac{\partial}{\partial t_0} + \psi_1 \frac{\partial}{\partial t_1}$, and seek to determine the smooth functions ψ_i so that, if Φ is the map $(s_0, s_1) \to \tilde{p}e^{s_1 Y}e^{s_2 X}$, where $s_2 = \tau(\tilde{p}, s_1)$, then the image of every trajectory of E is a trajectory of (1.1). Equivalently, if $d\Phi$ denotes the differential of Φ, then we want $d\Phi(E)(s_0, s_1)$ to be a convex combination of $X(\Phi(s_0, s_1))$ and $Y(\Phi(s_0, s_1))$.

Our first task is to compute $d\Phi(E)$ at a point (s_0, s_1). We have $d\Phi(E)(s_0, s_1) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \Phi(s_0 + \psi_0\varepsilon, s_1 + \psi_1\varepsilon)$. Therefore $\Phi(s_0 + \psi_0\varepsilon, s_1 + \psi_1\varepsilon) = pe^{(s_0+\psi_0\varepsilon)X}e^{(s_1+\psi_1\varepsilon)Y}e^{s_2(\varepsilon)X}$, where $s_2(\varepsilon) = \tau(pe^{(s_0+\psi_0\varepsilon)X}, s_1 + \psi_1\varepsilon)$. Hence

$$d\Phi(E)(s_0, s_1) = \tilde{p}\Big(\psi_0 X + \psi_1 Y + \kappa e^{s_1 \mathrm{ad}_Y}(X)\Big)e^{s_1 Y}e^{s_2 X}\,, \tag{3.14}$$

where $\kappa = \kappa_0\psi_0 + \kappa_1\psi_1$ and $\kappa_i = \frac{d\tau}{ds_i}(\tilde{p}, s_1)$ for $i = 0, 1$. In order to write $d\Phi(E)(s_0, s_1)$ as a convex combination $(AX + BY)(\Phi(s_0, s_1))$, we first seek to write it as a *linear* combination, without worrying about the sign of A and B. We seek to find A and B such that $\hat{C}(\psi_0, \psi_1) = C(A, B)$, where $\hat{C}(\psi_0, \psi_1)$, $C(A, B)$ are, respectively, abbreviations for $\tilde{p}\Big(\psi_0 X + \psi_1 Y + \kappa e^{s_1 \mathrm{ad}_Y}(X)\Big)$ and $\tilde{p}\Big(Ae^{s_1 \mathrm{ad}_Y}e^{s_2 \mathrm{ad}_X}(X) + Be^{s_1 \mathrm{ad}_Y}e^{s_2 \mathrm{ad}_X}(Y)\Big)$. If we use $Y =$

$X+2g$ and $\mathrm{e}^{s_2\mathrm{ad}_X}(X)=X$, we have $C(A,B)=\tilde{p}\Big((A+B)\mathrm{e}^{s_1\mathrm{ad}_Y}(X)+2B\mathrm{e}^{s_1\mathrm{ad}_Y}\mathrm{e}^{s_2\mathrm{ad}_X}(g)\Big)$. Next use the fact that $\tilde{p}\mathrm{e}^{s_1Y}\mathrm{e}^{s_2X}(g)=a(\tilde{p},s_1)\tilde{p}g+b(\tilde{p},s_1)\tilde{p}\mathrm{e}^{s_1\mathrm{ad}_Y}(g)$, together with $X=Y-2g$ and $\mathrm{e}^{s_1\mathrm{ad}_Y}(Y)=Y$, and conclude that $C(A,B)=\tilde{p}\Big(-BaX+B(a+b)Y+(A+B-Bb)\mathrm{e}^{s_1\mathrm{ad}_Y}(X)\Big)$ where, of course, a and b are evaluated at $(\tilde{p},s_1)$. In order to make $\hat{C}(\psi_0,\psi_1)$ equal $C(A,B)$, we must satisfy the conditions $\psi_0=-Ba$, $\psi_1=B(a+b)$, and $A+B-Bb=\kappa$. Naturally, we can achieve this by taking $B=1$, then choosing ψ_0 to be $-a$ and ψ_1 to be $a+b$. Then the identity $\kappa=\kappa_0\psi_0+\kappa_1\psi_1$ determines κ, and the last equation will be satisfied by letting $A=\kappa+b-1$, i.e. $A=Q-1$, where $Q\stackrel{\mathrm{def}}{=}-a\kappa_0+(a+b)\kappa_1+b$. Assume for the moment that $Q>1$. Then we will have written $\hat{C}(\psi_0,\psi_1)$ as a combination of the desired kind, with strictly positive coefficients. To make this into a *convex* combination, all we need is to divide A, B, ψ_0 and ψ_1 by $A+B$, i.e. by Q. This leads to the formulas $\psi_1=\frac{a+b}{Q}$, $\psi_0=\frac{-a}{Q}$, which define the vector field E. Therefore everything we need will follow if we prove that $Q>1$. In view of (3.12) and (3.13), b equals $1+\hat{\tau}$ plus lower order terms, and $a=-\hat{\tau}$ plus lower order terms. Also, κ_1 is equal to $\hat{\tau}$ plus lower order terms. Finally, the function $\tau(\tilde{p},s_1)$ vanishes when $s_1=0$, and so its partial derivative with respect to s_0 is equal to zero when $s_0=0$. So κ_0 is equal to s_1 times a smooth function. So $Q-1$ is equal to $2\hat{\tau}$ plus lower order terms. This means that, when p is close to $\bar{p}$ and t_0, t_1 are small (so that s_0, s_1 are also small), the inequality $Q>1$ will indeed hold. Therefore, by making ε_7 and T_7 smaller, if necessary, we can conclude that the family $\mathbf{\Gamma}$ of quasi-extremals we have constructed, together with the vector field E, satisfies the envelope condition.

Next we have to show that $\mathbf{\Gamma}$ is almost normal. If we fix s_0, then the derivative with respect to s_1 of $V=p\mathrm{e}^{s_0X}(f\wedge g\wedge \mathrm{e}^{s_1\mathrm{ad}_Y}(g))$ is $W=p\mathrm{e}^{s_0X}(f\wedge g\wedge \mathrm{e}^{s_1\mathrm{ad}_Y}([f,g]))$. We already showed that g, $\mathrm{e}^{s_1\mathrm{ad}_Y}(g)$ and $\mathrm{e}^{s_1\mathrm{ad}_Y}([f,g])$ are linearly independent at $p\mathrm{e}^{s_0X}$. So V and W can vanish simultaneously only if $p\mathrm{e}^{s_0X}f$ is a multiple of $p\mathrm{e}^{s_0X}g$, i.e. only if $p\mathrm{e}^{s_0X}(X\wedge g)=0$ Call an s_0 for which this happens "bad." If s_0 is not bad, then the set $B(s_0)$ of values of s_1 for which $V=0$ is discrete. Now, the derivative with respect to s_0 of $h=p\mathrm{e}^{s_0\mathrm{ad}_X}(X\wedge g)$ is $h'=p\mathrm{e}^{s_0\mathrm{ad}_X}(X\wedge[f,g])$, and the derivative of that is $h''=p\mathrm{e}^{s_0\mathrm{ad}_X}(X\wedge[X,[f,g]])$. Since $g\wedge[f,g]\wedge[X,[f,g]]\neq 0$, the equality $h=h'=h''$ can only hold if $p\mathrm{e}^{s_0X}X=0$, in which case $p\mathrm{e}^{s_0X}=p$ and $X(p)=0$. So, if $X(p)\neq 0$, the set of bad values of s_0 is discrete. Now, the quasi-extremal γ^α that corresponds to $\alpha=(s_0,s_1)$ is abnormal iff $V=0$ i.e. iff $s_1\in B(s_0)$. If $X(p)\neq 0$, then we have shown that $B(s_0)$ has measure zero for almost all s_0. So the set of abnormal α's has measure zero, and the family $\mathbf{\Gamma}$ is almost normal. So we have shown how to construct an almost normal $\mathbf{\Gamma}$ and an E such that $(\mathbf{\Gamma}, E)$ satisfy the envelope condition, provided that $X(p) \neq 0$.

We now conclude the argument and prove nonoptimality of our trajectory γ. Assume γ was optimal. Then, to begin with, $X(p)$ must be $\neq 0$, because $t_0 > 0$. Let $\mathbf{\Gamma}$, A, E, Φ be as above. Let $\theta : [0, \delta] \to A$ be an integral curve of E such that $\theta(\delta) = (t_0, t_1)$. Let $\tilde{\theta}(s) = \Phi(\theta(s))$. Let $\tilde{\gamma}^* = \gamma^{\theta(0)}$. Let $\hat{\gamma}$ be the concatenation of $\tilde{\gamma}$ and $\tilde{\theta}$. Then Theorem 2.2 implies that both γ^* and $\hat{\gamma}$ have exactly the same duration. If γ is time-optimal, then γ^* is time-optimal as well, and therefore $\hat{\gamma}$ is optimal. But $\hat{\gamma}$ is a concatenation of a bang-bang trajectory with two switchings and a singular arc. (The arc $\tilde{\theta}$ is singular, because the coefficients A, B constructed above are strictly positive.) As shown above, the adjoint variable for such a trajectory is completely determined, up to a normalization factor, by the location of the first two switchings and then, when we get to the third switching point, the switching function has a nonvanishing derivative. Therefore a junction with a singular arc is impossible at this third switching point. So $\hat{\gamma}$ cannot be time-optimal, and therefore γ is not time-optimal either. So we have showed that γ cannot be optimal, if γ is an $XYXY$ trajectory. Clearly, a similar argument works for $YXYX$ trajectories. This concludes the proof of Theorem 1.1. $\square$

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