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Continuous dependence with respect to the input of trajectories of control-affine systems*

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Abstract. We study the continuous dependence on the input of trajectories of control-affine systems belonging to the class $C^0(m)$ of all systems Σ of the form

$$\Sigma: \qquad \dot{x} = f_0(x) + \sum_{i=1}^{m} u_i(t) f_i(x),$$

where $f_0, \ldots, f_m$ are continuous vector fields on some open subset of $\mathbb{R}^n$ and the control functions belong to $L^1([0,T], \mathbb{R}^m)$. We give a simple necessary and sufficient condition for a control sequence $\{u^j\}_{j=1}^{\infty}$ to "$\mathcal{T}^0$-converge" to a control u^∞, i.e., to be such that, for every system Σ in $C^0(m)$, the trajectories generated by the u^j converge as $j \to \infty$ to the trajectories generated by u^∞. We also characterize $\mathcal{T}^k$-convergence (the concept of control convergence that arises when we use, instead of $C^0(m)$, the class $C^k(m)$ of systems Σ where the f_i are of class C^k) for $k \geq 1$ in the scalar input case, and we explain how the analogous characterization for the multi-input case fails to be true, unless one restricts oneself to the class $C^k_{comm}(m)$ of systems for which the vector fields $f_1, \ldots, f_m$ commute. As a preliminary, we define a "topology of trajectory convergence" (or "T-convergence") on the set of all time-varying vector fields $\Omega \times I \ni (x,t) \mapsto f(x,t) \in \mathbb{R}^n$, where Ω is an open subset of $\mathbb{R}^n$ and I is an interval, and we study some of its properties. This enables us to make the definition of $\mathcal{T}^k$-convergence precise for sequences and, more generally, for nets, by saying that a net $\{u^\alpha\}_{\alpha \in A}$ in $L^1([0,T], \mathbb{R}^m)$ $\mathcal{T}^k$-converges to a limit u^∞ if for every system Σ in $C^k(m)$ the time-varying vector fields $(x,t) \mapsto f_0(x) + \sum_{i=1}^m u_i^\alpha(t) f_i(x)$ $\mathcal{T}^k$-converge to $(x,t) \mapsto f_0(x) + \sum_{i=1}^m u_i^\infty(t) f_i(x)$.

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1 Introduction

In the control theory literature, the property of continuous dependence of the trajectories with respect to the control has received wide attention (cf. [1], [2], [3], [5], [7], [9], [12], [19], [20]), because of its fundamental role in proofs of closedness of reachable sets, existence of optimal controls, and continuity or lower semicontinuity of value functions. It is clear that continuous dependence always holds for trivial reasons if the space of input functions is given a sufficiently strong topology, but continuous dependence results are useful only when the topology involved is reasonably weak. (For example, many existence theorems for optimal controls depend on the fact that the space of inputs is compact, or has many compact subsets.) Thus it is of interest to determine concepts of convergence of inputs that lead to continuous dependence of trajectories and are as weak as possible.

In this paper we characterize the weakest possible concept of convergence of a sequence of input functions such that the trajectories depend continuously on the input for *all* control-affine systems of the form

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i f_i(x) \,, \quad x \in \Omega \,, \tag{1}$$

with a right-hand side continuous with respect to the state variable x, and we prove some results on the corresponding problem for more restricted classes of systems. These questions are in fact special cases of the following more general situation: we are given

(a) a set $\mathcal{U}$ of input functions $t \mapsto u(t)$ defined on an interval I,

(b) a class $\mathcal{C}$ of control systems Σ of the form $\dot{x} = g(x,u)$, all of which admit the members of $\mathcal{U}$ as inputs.

We then seek to characterize the *weakest* topology $\mathcal{T}$ on $\mathcal{U}$ with the property that, if $\mathcal{U}$ is equipped with $\mathcal{T}$, then

(CD) the trajectories of Σ depend continuously on the input $u \in \mathcal{U}$ for all systems Σ in $\mathcal{C}$.

Let us write $\mathcal{T}(\mathcal{U},\mathcal{C})$ to denote this topology, and assume temporarily that the definition of $\mathcal{T}(\mathcal{U},\mathcal{C})$ has been justified, by

(J1) assigning a mathematically precise meaning to (CD)

(J2) proving the existence of a weakest topology on $\mathcal{U}$ for which (CD) holds.

Then the problems to be studied here are those of characterizing the topology $\mathcal{T}(\mathcal{U},\mathcal{C})$ in two cases, namely, (i) when $\mathcal{C}$ is the class $C^0(m)$ of all systems of the form (1), with $f_0,\ldots,f_m$ continuous vector fields on some open subset Ω of some Euclidean space $\mathbb{R}^n$, and (ii) when $\mathcal{C}$ is an interesting subclass of $C^0(m)$, such as the set $C^1(m)$ of all systems (1) with C^1 vector fields. The input space $\mathcal{U}$ will be $L^1([0,T],\mathbb{R}^m)$, the set of all Lebesgue-integrable functions $t\mapsto u(t)=(u_1(t),\ldots,u_m(t))$ on some fixed interval $[0,T]$. Convergence with respect to $\mathcal{T}(L^1([0,T],\mathbb{R}^m),C^0(m))$ will be called "$\mathcal{T}^0$-convergence." For case (i), we will provide a complete and rather simple characterization of sequential $\mathcal{T}^0$-convergence by showing that a sequence $\{u^j\}$ $\mathcal{T}^0$-converges to a limit u^∞ if and only if the indefinite integrals $t\mapsto\int_0^t u^j(s)ds$ converge uniformly on $[0,T]$ to $t\mapsto\int_0^t u^\infty(s)ds$ and in addition $\sup_j\|u^j\|_{L^1}<\infty$. (Naturally, in order to provide a complete characterization of the topology $\mathcal{T}(L^1([0,T],\mathbb{R}^m),C^0(m))$, we should not limit our analysis to sequences, and should seek instead to characterize $\mathcal{T}^0$-convergence of nets. This, however, appears to be a much more difficult question, as explained below, on which we will only be able to present some partial results.) For case (ii), we provide an even simpler characterization of $\mathcal{T}(L^1([0,T],\mathbb{R}^1),C^1(1))$-convergence of general nets $\{u^\alpha\}_{\alpha\in A}$ to a limit u^∞, by showing that it is equivalent to uniform convergence on $[0,T]$ of the indefinite integrals $t\mapsto\int_0^t u^\alpha(s)ds$ to the limit $t\mapsto\int_0^t u^\infty(s)ds$. This result is, however, valid only for the single-input case. To explain why, we analyze the m-input situation and prove, in Sections 5 and 6, two results (Theorems 4 and 5), according to which (a) a characterization similar to that for $m=1$ is true for general m on the class $C^1_{comm}(m)$ of systems (1) with C^1 vector fields such that all the Lie brackets $[f_j,f_k]$ for $j,k\in\{1,\ldots,m\}$ vanish, and (b) the result becomes false for any class $\mathcal{C}$ that contains at least one system in $C^1(m)\backslash C^1_{comm}(m)$. This shows that Lie brackets are intimately related to input convergence, and ought to play a decisive role in any effort to achieve a better understanding of $\mathcal{T}(L^1([0,T],\mathbb{R}^m),C^1(m))$-convergence for $m>1$.

So far, we have assumed that steps (J1) and (J2) had been carried out. We explain how this is done in Section 2, where we define, on the set $TVVF(\Omega,I)$ of all time-varying vector fields $f:\Omega\times I\mapsto\mathbb{R}^n$ (where Ω is an open subset of $\mathbb{R}^n$ and I is a subinterval of $\mathbb{R}$), a topology $\mathcal{T}_T(\Omega,I)$—called the *topology of trajectory convergence*, or *T-convergence*—and argue that this topology captures the concept of "continuous dependence with respect to f of the solutions of the ordinary differential equation $\dot{x}=f(x,t)$." This topology takes into account the possibility of explosions and nonuniqueness of solutions—a fact of paramount importance for us, since, typically, the time-varying vector fields that arise from the systems (1) are continuous only with respect to x—and we will show in Section 2 that it has all the right properties, at least when restricted to the set $TVVF_{Car,LIB}(\Omega, I)$ of those f's that satisfy the Carathéodory condition (i.e., are such that f is measurable in t for each x and continuous in x for each t) and are "locally integrably bounded," i.e., that satisfy, for every compact subset K of Ω, an integral bound

$$||f(x,t)|| \leq \varphi_K(t) \quad \text{for} \quad (x,t) \in K \times I\,,$$

with a locally integrable function $\varphi_K : I \mapsto \mathbb{R}$. Moreover, the topology induced by $\mathcal{T}_T(\Omega, I)$ on the subset $TVVF_{Car,LIB}(\Omega, I)$ of $TVVF(\Omega, I)$ is the weakest topology on $TVVF_{Car,LIB}(\Omega, I)$ that gives rise to joint continuous dependence of the solutions on f and the initial condition.

Having defined $\mathcal{T}_T(\Omega, I)$, it is clear how to interpret (CD) and how to define $\mathcal{T}(\mathcal{U}, \mathcal{C})$: if $\Sigma \in \mathcal{C}$ is of the form $\dot{x} = g^\Sigma(x,u)$, $x \in \Omega^\Sigma$, then each input $t \mapsto u(t)$ belonging to $\mathcal{U}$ gives rise to a time-varying vector field $f^\Sigma(u) \in TVVF(\Omega^\Sigma, I)$, defined by

$$f^\Sigma(u) \stackrel{\text{def}}{=} \text{the map } \ (x,t) \mapsto g^\Sigma(x, u(t))\,. \tag{2}$$

Then f^Σ is the *input-to-vector-field map* associated with Σ. If $\mathcal{T}$ is a topology on $\mathcal{U}$, then the statement that "the trajectories of Σ depend continuously on the input with respect to $\mathcal{T}$" can be translated as "f^Σ is continuous as a map from $\mathcal{U}$, equipped with $\mathcal{T}$, to $TVVF(\Omega^\Sigma, I)$, equipped with $\mathcal{T}_T(\Omega^\Sigma, I)$." Then the formal definition of $\mathcal{T}(\mathcal{U}, \mathcal{C})$ is simply the following:

> *$\mathcal{T}(\mathcal{U}, \mathcal{C})$ is the weakest topology on $\mathcal{U}$ that renders all the input-to-vector-field maps $f^\Sigma : \mathcal{U} \mapsto TVVF(\Omega^\Sigma, I)$ continuous, for all $\Sigma \in \mathcal{C}$, when each $TVVF(\Omega^\Sigma, I)$ is given the topology $\mathcal{T}_T(\Omega^\Sigma, I)$ of trajectory convergence.*

To analyze in more detail the special case when $\mathcal{U} = L^1([0,T], \mathbb{R}^m)$ and $\mathcal{C}$ is the class $C^0(m)$, let us agree to use f_i^Σ to denote the vector fields f_i corresponding to a given system $\Sigma \in C^0(m)$ of the form (1), and let us go on using Ω^Σ to denote the state space of Σ. Write $\mathcal{U}_T^m \stackrel{\text{def}}{=} L^1([0,T], \mathbb{R}^m)$. As before, use "$\mathcal{T}^0$-convergence" for "convergence with respect to $\mathcal{T}(\mathcal{U}_T^m, C^0(m))$." Then our problem is to determine necessary and sufficient conditions for a sequence $\{u^j\}_{j=1}^\infty$ of functions belonging to the input space $\mathcal{U}_T^m$ to $\mathcal{T}^0$-converge to an input $u^\infty \in \mathcal{U}_T^m$, i.e., for the following property to hold:

(TC) for every possible choice of the system $\Sigma \in C^0(m)$, the time-varying vector fields $f^\Sigma(u^j)$ T-converge to $f^\Sigma(u^\infty)$.

As will be explained in Section 2, one may substitute for (TC) either one of the following conditions, both of which turn out in fact to be equivalent to (TC):

(TC') For every possible choice of the system $\Sigma \in C^0(m)$, if (i) ξ^j are maximal trajectories of (1) corresponding to the u^j and satisfying the initial conditions $\xi^j(0) = \bar{x}^j \in \Omega^\Sigma$, (ii) $\bar{x}^j \to \bar{x} \in \Omega^\Sigma$, and (iii) for the limiting initial value problem

$$\dot{x} = f^\Sigma(u^\infty)(x,t) \stackrel{\mathrm{def}}{=} f_0^\Sigma(x) + \sum_{i=1}^m u_i^\infty(t) f_i^\Sigma(x)\,,\ \ x(0) = \bar{x} \tag{3}$$

there is uniqueness of solutions as well as global existence on $[0,T]$, then the ξ^j are defined on $[0,T]$ for j large enough and converge uniformly on $[0,T]$ to the unique maximal solution ξ^∞ of (3).

(TC") For every possible choice of the system $\Sigma \in C^0(m)$, if (i) $\bar{x} \in \Omega^\Sigma$, (ii) ξ^j are maximal trajectories of (1) corresponding to the u^j and satisfying $\xi^j(0) = \bar{x}$, and (iii) the initial value problem (3) has uniqueness of solutions as well as global existence on $[0,T]$, then the ξ^j are defined on $[0,T]$ for j large enough and converge uniformly on $[0,T]$ to the unique maximal solution ξ^∞ of (3).

Notice that (TC') asserts "joint continuous dependence on the input and the initial condition," whereas (TC") makes the weaker assertion of "continuous dependence on the input for each fixed initial condition," so the implication (TC')⇒(TC") is trivial, but the converse (TC")⇒(TC') is not at all obvious. (Example A.3 in the appendix shows that for a general sequence $\{f^j\}$ of time-varying vector fields it need not be true that convergence for each fixed initial condition of the trajectories of f^j to those of an $f \in TVVF(\Omega, I)$ implies convergence of trajectories for convergent initial conditions.)

Our characterization of sequential $\mathcal{T}^0$-convergence will be formulated in terms of a type of convergence somewhat weaker than other concepts of weak convergence previously considered in the literature. Precisely, let us say that a net $\{u^\alpha\}_{\alpha\in A} \subseteq \mathcal{U}_T^m$ based on a directed set $(A, \preceq_A)$ *converges in the integral sense* (or, for short, *I-converges*) to a $u^\infty \in \mathcal{U}_T^m$, if the following holds:

(IC) The indefinite integrals $U^\alpha(t) = \int_0^t u^\alpha(s)ds$ converge uniformly with respect to $t \in [0,T]$ to $U^\infty(t) = \int_0^t u^\infty(s)ds$.

Also, let us say that $\{u^\alpha\}_{\alpha\in A}$ satisfies the *uniform boundedness* condition if

(UB) There exist a constant $C > 0$ and an $\alpha_0 \in A$ such that $\|u^\alpha\|_{L^1} \leq C$ whenever $\alpha_0 \preceq_A \alpha$.

Our main result (Theorem 1) then says that a sequence $\{u^j\}_{j=1}^\infty$ in $\mathcal{U}_T^m$ $\mathcal{T}^0$-converges to a function $u^\infty \in \mathcal{U}_T^m$ if and only if $\{u^j\}_{j=1}^\infty$ satisfies (UB) and I-converges to u^∞. An alternative formulation is in terms of *uniform weak convergence*. We say that a net $\{u^\alpha\}_{\alpha\in A}$ *UW-converges* to a limit u^∞ if the following holds.

(UWC) For every continuous function $\varphi : [0,T] \mapsto \mathbb{R}$, if we define $\mathbb{R}^m$-valued functions U_φ^α by $U_\varphi^\alpha(t) = \int_0^t u^\alpha(s)\varphi(s)ds$ for $\alpha \in A \cup \{\infty\}$, then the U_φ^α converge to U_φ^∞ uniformly on $[0,T]$.

For sequences, it is easily seen (cf. Section 4) that the conjunction (IC)$\wedge$(UB) is equivalent to (UWC), so Theorem 1 says in fact that a sequence $\{u^j\}_{j=1}^\infty$ $\mathcal{T}^0$-converges to a limit u^∞ if and only if it UW-converges to u^∞.

For more general nets, the condition (IC)$\wedge$(UB) still implies $\mathcal{T}^0$-convergence, which in turn implies UW-convergence, but the reverse implications no longer hold. (This is shown in the appendix by means of two examples.) Thus, although Theorem 1 provides a complete characterization of sequential $\mathcal{T}^0$-convergence in $\mathcal{U}_T^m$, it does not give a full characterization of the topology itself. To get such a characterization, one would have to understand $\mathcal{T}^0$-convergence of nets, and we have not yet been able to find a satisfactory characterization of $\mathcal{T}^0$-convergence of nets, comparable in simplicity to our results for sequences.

For sequences, the condition (IC)$\wedge$(UB) or, equivalently, (UWC), is strictly weaker than weak convergence in $L^1([0,T],\mathbb{R}^m)$, as explained in Section 7. In addition, the characterization of sequential $\mathcal{T}^0$-convergence in $\mathcal{U}_T^m$ depends crucially on the fact that we are working with the collection of *all* systems in $C^0(m)$. We could equally well have defined a concept of $\mathcal{T}^k$-convergence for any integer $k \geq 0$ by just taking $\mathcal{C}$ to be the class $C^k(m)$ of all systems of the form (1) with vector fields f_i of class C^k. But then it is no longer obvious that condition (UB) is still necessary for sequential $\mathcal{T}^k$-convergence for $k \geq 1$. (The proof that (UB) is necessary for sequential $\mathcal{T}^0$-convergence in $\mathcal{U}_T^m$ depends on using the uniform boundedness theorem and requires that we admit systems with an arbitrary continuous right-hand side. On the other hand, the necessity of (IC) depends only on the fact that the systems $\dot{x} = u_i$, $i = 1,\ldots,m$ are in our class, so (IC) is still necessary for trajectory convergence for practically any reasonable class of systems.) In fact, we show in Section 5 that a sequence $\{u^j\}_{j=1}^\infty \subseteq \mathcal{U}_T^1$ may $\mathcal{T}^k$-converge to a $u^\infty \in \mathcal{U}_T^1$ even though (UB) fails.

To our knowledge, the problem of characterizing sequential $\mathcal{T}^k$-convergence in $\mathcal{U}_T^m$ for $k \geq 1$ is still open for $m > 1$, and appears to be quite difficult. The case $m = 1$ is special, as explained in Section 5, where we give a complete characterization (in Theorem 3) of the topology $\mathcal{T}(\mathcal{U}_T^1, C^1(1))$ by showing that a net $\{u^\alpha\}_{\alpha\in A} \subseteq \mathcal{U}_T^1$ $\mathcal{T}(\mathcal{U}_T^1, C^1(1))$-converges to a limit u^∞ iff the u^α I-converge to u^∞. The profound reason why this characterization is possible for $m = 1$ but not for $m > 1$ is provided by Theorems 4 and 5: Theorem 4 shows that the condition of Theorem 3 is necessary and sufficient, for general m, for $\mathcal{T}(\mathcal{U}_T^m, \mathcal{C})$-convergence of the u^α to u^∞, if we take $\mathcal{C} = C^1_{comm}(m)$, the class of systems (1) where the f_i are of class C^1 and $f_1, \ldots, f_m$ *commute* (i.e., the Lie brackets $[f_i, f_j]$ vanish for $1 \leq i, j \leq m$), whereas Theorem 5 shows that for any class $\mathcal{C}$ such that $C^1_{comm}(m) \subseteq \mathcal{C} \subseteq C^1(m)$ and $\mathcal{C} \neq C^1_{comm}(m)$, the condition no longer suffices for $\mathcal{T}(\mathcal{U}_T^m, \mathcal{C})$-convergence, because there exists a sequence $\{u^j\}_{j=1}^\infty$ such that

(1) $\int_0^t u^j(s)ds \to 0$ uniformly;

(2) for every $\Sigma \in \mathcal{C}$ there is a time-varying vector field $h^\Sigma \in TVVF(\Omega^\Sigma, [0,T])$ such that the $\{f^\Sigma(u^j)\}$ T-converge to h^Σ;

(3) it is not true that $h^\Sigma = f^\Sigma(0)$ for all $\Sigma \in \mathcal{C}$.

(What really happens is that the sequence $\{u^j\}$ $\mathcal{T}(\mathcal{U}_T^m, \mathcal{C})$-converges to a "generalized input" that is no longer an ordinary input. A more detailed discussion of these issues is provided in §6.)

The paper is organized as follows. In Section 2 we present the definition and basic properties of the topology of T-convergence. We then give, in Section 3, our precise definition of $\mathcal{T}^0$-convergence. The main theorem (Theorem 1) of the paper is stated and proved in Section 4. In Section 5 we discuss how the situation changes when $\mathcal{T}^k$-convergence for $k \geq 1$ is considered instead of $\mathcal{T}^0$-convergence. In Section 6 we briefly discuss some of the phenomena that can cause a sequence $\{f^\Sigma(u^j)\}$ to T-converge to something other than its limit $f^\Sigma(u^\infty)$, in particular the occurrence of Lie bracket terms. Section 7 discusses how our continuous dependence conditions relate to those of Buttazzo-Conti, Kurzweil-Vorel and Neustadt, and shows that our $\mathcal{T}^0$-convergence conditions for sequences in $L^1([0,T], \mathbb{R}^m)$ are weaker than weak convergence in L^1. Finally, we include an appendix that presents three examples, showing (i) that UW-convergence of nets does not imply $\mathcal{T}^0$-convergence, (ii) that $\mathcal{T}^0$-convergence of nets does not imply the uniform boundedness condition (UB), and (iii) that, for general time-varying vector fields, "continuous dependence of trajectories for every fixed initial condition" does not imply "joint continuous dependence of trajectories on the vector field and the initial condition."

2 The topology of trajectory convergence

We begin by giving a precise meaning to the "T-convergence" condition that appears in (TC).

In what follows, the word "interval" means "connected, nonempty subset of $\mathbb{R}$." An interval is *nontrivial* if it is not reduced to a single point.

If Ω is a metric space, we use d_Ω to denote the corresponding distance function. A *curve* in Ω is a continuous map ξ from an interval $I \subseteq \mathbb{R}$ to Ω. If I is compact, then ξ will be called an *arc* in Ω. We use $ARC(\Omega)$ to denote the set of all arcs in Ω. For every $-\infty < a \leq b < \infty$, we let $C^0([a,b],\Omega)$ denote the set of all arcs in Ω whose domain is $[a,b]$. Then

$$ARC(\Omega) = \bigcup_{-\infty<a\leq b<\infty} C^0([a,b],\Omega)\,. \tag{4}$$

For each $\xi \in ARC(\Omega)$, we let $a(\xi)$, $b(\xi)$ be the unique a, b such that ξ belongs to $C^0([a,b],\Omega)$. If $\xi \in ARC(\Omega)$, let $\tilde{\xi}$ be the continuous curve $\tilde{\xi} : \mathbb{R} \mapsto \Omega$ obtained by extending ξ to the whole real line by just letting $\tilde{\xi}$ be constant on $]-\infty, a(\xi)]$ and on $[b(\xi), +\infty[$. We topologize $ARC(\Omega)$ by means of the metric $d_{ARC} : ARC(\Omega) \times ARC(\Omega) \mapsto \mathbb{R}$ given by

$$d_{ARC}(\xi,\eta) = |a(\xi) - a(\eta)| + |b(\xi) - b(\eta)| + \sup\{d_\Omega(\tilde{\xi}(t), \tilde{\eta}(t)) \,:\, t \in \mathbb{R}\}\,.$$

Then a sequence $\{\xi^j\}$ of arcs, possibly defined on different time intervals $[a^j, b^j]$, converges to an arc $\xi : [a,b] \mapsto \Omega$, if (i) $a^j \to a$, (ii) $b^j \to b$, and (iii) $\xi^j \to \xi$ uniformly, i.e., $\xi^j(t^j) \to \xi(t)$ whenever $t^j \in [a^j, b^j]$, $t^j \to t$.

If $\xi : J \mapsto \Omega$ is a curve, the *graph* of ξ is the set $G(\xi) = \{(\xi(t), t) : t \in J\}$.

If $\xi_i : [a_i, b_i] \mapsto \Omega$ for $i = 1, 2$ are arcs in Ω, then the *concatenation* $\xi_1 \# \xi_2$ is defined if and only if $b_1 = a_2$ and $\xi_1(b_1) = \xi_2(a_2)$, and is the arc whose graph is $G(\xi_1) \cup G(\xi_2)$. Concatenation is an associative partially defined binary operation on $ARC(\Omega)$.

A subset $\mathcal{A} \subseteq ARC(\Omega)$ is *bounded* if there is a compact subset K of $\Omega \times \mathbb{R}$ such that $G(\xi) \subseteq K$ for all $\xi \in \mathcal{A}$. We call $\mathcal{A}$ *equicontinuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_\Omega(\xi(t), \xi(s)) < \varepsilon$ whenever $\xi \in \mathcal{A}$, $t, s \in [a(\xi), b(\xi)]$, and $|t - s| \leq \delta$. It is then clear that a subset $\mathcal{A}$ of $ARC(\Omega)$ is relatively compact iff $\mathcal{A}$ is bounded and equicontinuous.

An *arc system in* Ω *with time domain* I is a subset $\mathcal{A}$ of $ARC(\Omega)$ such that

(AS.1) $G(\xi) \subseteq \Omega \times I$ for every $\xi \in \mathcal{A}$,

(AS.2) $\mathcal{A}$ contains every arc in Ω whose domain is of the form $\{t\}$ for $t \in I$, (AS.3) $\mathcal{A}$ is closed under concatenations (that is, if ξ_1, ξ_2 are in $\mathcal{A}$ and $\xi_1\#\xi_2$ is defined, then $\xi_1\#\xi_2 \in \mathcal{A}$),

(AS.4) $\mathcal{A}$ is closed under restrictions (that is, if $\xi \in \mathcal{A}$ and $a(\xi) \leq \alpha \leq \beta \leq b(\xi)$, then the restriction $\xi\lceil[\alpha,\beta]$ of ξ to $[\alpha,\beta]$ is in $\mathcal{A}$).

We use $AS(\Omega, I)$ to denote the set of all arc systems in Ω with time domain I. If $\mathcal{A} \in AS(\Omega, I)$ and $K \subseteq \Omega \times I$, we write

$$\mathcal{A}(K) \stackrel{\text{def}}{=} \{\xi \in \mathcal{A} : G(\xi) \subseteq K\}\,. \tag{5}$$

If $\mathcal{A} \in AS(\Omega, I)$, then it follows easily that, if (i) the arcs $\xi_1, \ldots, \xi_m$ belong to $\mathcal{A}$, (ii) $[a(\xi_1), b(\xi_1)] \cup \cdots \cup [a(\xi_m), b(\xi_m)]$ is an interval, (iii) $\xi_i\lceil I_i \cap I_j = \xi_j\lceil I_i \cap I_j$ for all $i, j \in \{1, \ldots, m\}$, where we write $I_\ell = [a(\xi_\ell), b(\xi_\ell)]$ for $\ell \in \{1, \ldots m\}$, and (iv) we let $\xi_1 \vee \cdots \vee \xi_m$ be the arc whose graph is $G(\xi_1) \cup \ldots \cup G(\xi_m)$, then $\xi_1 \vee \ldots \vee \xi_m$ is also in $\mathcal{A}$.

If $\mathcal{A} \in AS(\Omega, I)$, we define $\mathcal{A}^\#$ to be the set of all curves $\xi : J \mapsto \Omega$ such that J is a subinterval of I and $\xi\lceil L \in \mathcal{A}$ for every compact subinterval L of J. The members of $\mathcal{A}^\#$ will be called *$\mathcal{A}$-curves*. An $\mathcal{A}$-curve $\xi \in \mathcal{A}^\#$ is *maximal* if there does not exist $\eta \in \mathcal{A}^\#$ such that $G(\xi)$ is a proper subset of $G(\eta)$. We use $\mathcal{A}^\#_{max}$ to denote the set of all maximal elements of $\mathcal{A}^\#$. A trivial application of Zorn's lemma shows that, if $\mathcal{A} \in AS(\Omega, I)$, then every $\xi \in \mathcal{A}^\#$ can be extended to an $\eta \in \mathcal{A}^\#_{max}$. This implies, in particular, that through every point of $\Omega \times I$ there passes the graph of a $\xi \in \mathcal{A}^\#_{max}$.

If $\mathcal{A} \in AS(\Omega, I)$, we say that $\mathcal{A}$ has the *compactness property* if for every compact subset $K \subseteq \Omega \times I$ the set $\mathcal{A}(K)$ is compact in $ARC(\Omega)$. We say that $\mathcal{A}$ has the *local existence property* if for every $(x, t) \in \Omega \times I$ there exists $\xi \in \mathcal{A}$ such that t belongs to the interior of $[a(\xi), b(\xi)]$ relative to I and $\xi(t) = x$. We use $AS_{comp,le}(\Omega, I)$ to denote the set of all $\mathcal{A} \in AS(\Omega, I)$ that have the compactness property and the local existence property.

Proposition 1 *Let Ω be a locally compact metric space, and let $I \subseteq \mathbb{R}$ be an interval. Assume that $\mathcal{A} \in AS_{comp,le}(\Omega, I)$. Let $\xi : J \mapsto \Omega$ be a curve in $\mathcal{A}^\#$. Then $\xi \in \mathcal{A}^\#_{max}$ if and only if* (i) *J is relatively open in I and* (ii) *for every compact subset K of $\Omega \times I$ the set $\{t \in J : (\xi(t), t) \in K\}$ is compact.*

PROOF. Let $t_+ = \sup J$, $t_- = \inf J$.

Suppose first that (i) and (ii) hold, but ξ is not maximal. Let $\eta \in \mathcal{A}^\#$ be an extension of ξ to a strictly larger interval J'. Pick $s \in J' \backslash J$. Then either $s \geq t_+$ or $s \leq t_-$. Assume that $s \geq t_+$. (The other case is similar.) Then $t_+ \in I$, because $t_+ \leq s \in I$ and $t_+ \geq t$ for some $t \in J \subseteq I$. Therefore $t_+ \notin J$, because J

is relatively open in I by (i). Pick $t \in J$. Let K_0 be the set $\{\eta(\tau) : t \leq \tau \leq t_+\}$. Let $K = K_0 \times [t, t_+]$. Then K is compact, so the set $L = \{\tau \in J : (\xi(\tau), \tau) \in K\}$ is compact by (ii). But this is a contradiction, because $[t, t_+[\subseteq L$ but $t_+ \notin L$.

Now assume that ξ is maximal. Suppose that J is not relatively open in I. Then either $t_+ \in J$ and $t_+ < \sup I$ or $t_- \in J$ and $t_- > \inf I$. Assume that $t_+ \in J$ and $t_+ < \sup I$, the other case being similar. By the local existence property, there is an $\eta \in \mathcal{A}$ for which $\eta(t_+) = \xi(t_+)$ and whose domain Q contains t in its interior relative to I. By the restriction property, we may assume that the domain of η is of the form $[t_+, t_+ + \varepsilon]$ for some $\varepsilon > 0$. But then $G(\xi) \cup G(\eta)$ is the graph of an arc in $\mathcal{A}^{\#}$, contradicting the maximality of ξ. So J is relatively open in I, as stated.

Let $K \subseteq \Omega \times I$ be compact, and assume that $L = \{t \in J : (\xi(t), t) \in K\}$ is not compact. Then in particular $L \neq \emptyset$. Let $\pi : \Omega \times I \mapsto I$ be the projection map $(x, t) \mapsto t$. Let $b = \sup L$, $a = \inf L$. Then $a \in I$ and $b \in I$, because $L \subseteq \pi(K) \subseteq I$, and $\pi(K)$ is compact. Also, $a < b$, since if $a = b$ then L would be equal to $\{a\}$, which is compact. Since $L \subseteq [a, b]$ and L is not compact, there exists a sequence $\{t_j\}$ in L that converges to a limit $\bar{t} \notin L$. On the other hand, the open interval $]a, b[$ is contained in J. If $a < \bar{t} < b$, it would follow that $\xi(t_j) \to \xi(\bar{t})$, so $(\xi(\bar{t}), \bar{t})$ would be in K, and $\bar{t}$ would belong to L. Therefore either $\bar{t} = a$ or $\bar{t} = b$. Assume $\bar{t} = b$, the case when $\bar{t} = a$ being similar. Since $(\xi(t_j), t_j) \in K$, which is compact, we may assume, after passing to a subsequence, that $\bar{x} = \lim_{j\to\infty} \xi(t_j)$ exists. Clearly, $(\bar{x}, \bar{t}) \in K$. Let K_0', be a compact neighborhood of $\bar{x}$ in Ω, and let J_0' be a compact subinterval of I such that $\bar{t}$ belongs to the interior of J_0' relative to I. Let $K' = K_0' \times J_0'$. Let $\varepsilon > 0$ be such that $x \in \Omega$, $t \in I$, $||x - \bar{x}|| \leq \varepsilon$, $|\bar{t} - t| \leq \varepsilon$ implies $(x, t) \in K'$. Using the compactness property for $\mathcal{A}$, we can infer that $\mathcal{A}(K')$ is equicontinuous, so there exists a $\delta > 0$ such that $\eta \in \mathcal{A}$, $G(\eta) \subseteq K'$ implies $||\eta(t) - \eta(s)|| \leq \frac{\varepsilon}{3}$ whenever $|t - s| \leq \delta$. Pick j so large that $b - t_j < \delta$ and $||\xi(t_j) - \bar{x}|| < \frac{\varepsilon}{3}$. Then $\xi([t_j, b[) \subseteq K_0'$. (Otherwise, let $\tau = \inf\{t \in [t_j, b[: \xi(t) \notin K_0'\}$. Then $t_j < \tau < b$, $\xi(\tau) \in K_0'$, and $\xi\lceil[t_j, \tau] \in \mathcal{A}(K')$. Since $\tau - t_j < \delta$, we have $||\xi(\tau) - \xi(t_j)|| < \frac{\varepsilon}{3}$, so $||\xi(\tau) - \bar{x}|| < \frac{2\varepsilon}{3}$, and then $\xi(\tau)$ is an interior point of K_0'. But then $\xi(t) \in K_0'$ for $t > \tau$, $t - \tau$ small enough, contradicting the definition of τ.) Using equicontinuity again, for the set of arcs $\{\xi\lceil[t_j, t] : t_j \leq t < b\} \subseteq \mathcal{A}(K')$, we see that the limit of $\xi(t)$ as $t \uparrow b$ must exist. Therefore $\lim_{t\uparrow b} \xi(t) = \bar{x}$. Then the arcs $\xi\lceil[t_j, t]$ converge to η as $t \uparrow b$, where $\eta : [t_j, b] \mapsto \Omega$ is such that $\eta\lceil[t_j, b[= \xi\lceil[t_j, b[$ and $\eta(b) = \bar{x}$. By the compactness property, $\eta \in \mathcal{A}$. Therefore, if we extend ξ to $J \cup \{b\}$ by letting $\xi(b) = \bar{x}$, we see that the extension belongs to $\mathcal{A}^{\#}$. Since ξ is maximal, we conclude that $b \in J$ and $\xi(b) = \bar{x}$. Since $(\bar{x}, b) = \lim_{j\to\infty}(\xi(t_j), t_j)$, we see that $(\bar{x}, b) \in K$, so $b \in L$, contradicting the fact that $b = \bar{t} \notin L$. ∎

If X is a set, we use 2^X to denote the set of all subsets of X. If X is a topological space, we can topologize 2^X as follows. For each open subset U of X, let

$$V(U) = \{S \in 2^X \,:\, S \subseteq U\}\,. \tag{6}$$

We then let $\mathcal{T}_{USC}(X)$ denote the set of all subsets S of 2^X such that $S = \cup_\alpha V(U_\alpha)$ for some family $\{U_\alpha\}$ of open subsets of X. Then $\mathcal{T}_{USC}(X)$ is a topology on 2^X. We will call it the *USC topology* on 2^X, because of its relation to upper-semicontinuity of set valued maps. (If Y is a topological space, a set-valued map $F : Y \mapsto 2^X$ is upper semicontinuous if and only if it is continuous as a map from Y to $(2^X, \mathcal{T}_{USC}(X))$.) The name *upper semifinite topology* has also been used in the literature, cf. e.g. [17].

If Ω is a metric space and I is an interval, then for every subset K of $\Omega \times I$ we have a set-valued map $T_{\Omega,I,K} : AS(\Omega, I) \mapsto 2^{ARC(\Omega)}$ given by $T_{\Omega,I,K}(\mathcal{A}) = \mathcal{A}(K)$. As K ranges over all the compact subsets of $\Omega \times I$, we have a collection of maps $T_{\Omega,I,K}$ from $AS(\Omega, I)$ to $2^{ARC(\Omega)}$. Therefore there exists a weakest topology on $AS(\Omega, I)$ that renders all the maps $T_{\Omega,I,K}$ continuous, where $2^{ARC(\Omega)}$ is equipped with the USC topology. This topology will be called the *topology of trajectory convergence* (or simply *T-convergence*) and be denoted by $\mathcal{T}_T(\Omega, I)$. If $\{\mathcal{A}^\alpha\}_{\alpha\in A}$ is a net in $AS(\Omega, I)$ and $\mathcal{A} \in AS(\Omega, I)$, we say that $\{\mathcal{A}^\alpha\}_{\alpha\in A}$ *T-converges* to $\mathcal{A}$ (and write $\mathcal{A}^\alpha \xrightarrow{\mathrm{T}} \mathcal{A}$) if the $\mathcal{A}^\alpha$ converge to $\mathcal{A}$ in $\mathcal{T}_T(\Omega, I)$.

Let Ω be a metric space, let $\xi : I \mapsto \Omega$ be a curve, and assume that $\{\xi^\alpha\}_{\alpha\in A}$ is a net of curves in Ω, with $\mathrm{Dom}(\xi^\alpha) = I^\alpha$. We say that the ξ^α *converge to ξ on compact sets* if for every compact subset $J \subseteq I$, there exists an $\alpha^*(J)$ such that (i) $J \subseteq I^\alpha$ for $\alpha \succeq_A \alpha^*(J)$ and (ii) the $\xi^\alpha\lceil J$ converge uniformly to $\xi\lceil J$.

We say that a net $\{\mathcal{A}^\alpha\}_{\alpha\in A}$ in $AS(\Omega, I)$ *converges to $\mathcal{A} \in AS(\Omega, I)$ in the maximal curve convergence sense* if, whenever $\{(x^\alpha, t^\alpha)\}_{\alpha\in A}$ is a net of points in $\Omega \times I$ that converges to a limit $(\bar{x}, \bar{t}) \in \Omega \times I$, and $\{\xi^\alpha\}_{\alpha\in A}$ is a net such that for each $\alpha \in A$ $\xi^\alpha \in (\mathcal{A}^\alpha)^\#_{max}$ and $\xi^\alpha(t^\alpha) = x^\alpha$, it follows that there exist a $\xi^\infty \in \mathcal{A}^\#_{max}$, with domain $I^\infty \subseteq I$, such that $\xi^\infty(\bar{t}) = \bar{x}$, and a subnet $\{\xi^{\mu(\beta)}\}_{\beta\in B}$ of $\{\xi^\alpha\}_{\alpha\in A}$ that converges to ξ^∞ on compact sets.

Proposition 2 *Let Ω be a locally compact metric space, and let $I \subseteq \mathbb{R}$ be an interval. Suppose that $\{\mathcal{A}^\alpha\}_{\alpha\in A}$ is a net in $AS_{comp,le}(\Omega, I)$, and $\mathcal{A}$ belongs to $AS_{comp,le}(\Omega, I)$. Then the following conditions are equivalent:*

- *(I) the $\mathcal{A}^\alpha$ T-converge to $\mathcal{A}$;*
- *(II) every subnet of $\{\mathcal{A}^\alpha\}_{\alpha\in A}$ converges to $\mathcal{A}$ in the maximal curve convergence sense.*

PROOF. The implication (II)$\Rightarrow$(I) is trivial. Indeed, suppose (II) holds, and let K be a compact subset of $\Omega \times I$. Since $\mathcal{A}(K)$ is compact, a fundamental system of neighborhoods of $\mathcal{A}(K)$ in $2^{ARC(\Omega)}$ is given by the sets

$$V_\varepsilon = \{\xi \in ARC(\Omega) : d_{ARC}(\xi, \mathcal{A}(K)) < \varepsilon\}$$

for $\varepsilon > 0$. So it suffices to fix ε and prove that $\mathcal{A}^\alpha(K) \subseteq V_\varepsilon$ for sufficiently large α. If this was not so, there would exist a subnet $\{\mathcal{A}^{\mu(\beta)}\}_{\beta\in B}$ of $\{\mathcal{A}^\alpha\}_{\alpha\in A}$ such that there are $\xi^\beta \in \mathcal{A}^{\mu(\beta)}(K)$ for which $d_{ARC}(\xi^\beta, \mathcal{A}(K)) \geq \varepsilon$. Extend each ξ^β to a $\tilde{\xi}^\beta \in (\mathcal{A}^{\mu(\beta)})^{\#}_{max}$. Pick $t^\beta \in \mathrm{Dom}(\xi^\beta)$ in an arbitrary fashion, and let $x^\beta = \xi^\beta(t^\beta)$. Then $(x^\beta, t^\beta) \in K$, so by passing to a subnet if necessary we may assume that (x^β, t^β) converges to a limit $(\bar{x}, \bar{t})$. Using (II) we may assume, after passing to a subnet, that there exists a $\tilde{\xi} \in \mathcal{A}^{\#}_{max}$ such that $\tilde{\xi}(\bar{t}) = \bar{x}$ and $\tilde{\xi}^\beta \to \tilde{\xi}$ on compact sets. After passing once again to a subnet, we may assume that $a(\xi^\beta)$ and $b(\xi^\beta)$ have limits a, b, both belonging to I. Let $J = \mathrm{Dom}(\tilde{\xi})$. We will prove that $a \in J$ and $b \in J$. Since both proofs are similar, we will only show that $b \in J$. Suppose that $b \notin J$. Let $c = \sup J$, so $c \leq b$. Since $b \in I$, and J is relatively open in I, we have $c \notin J$. This implies in particular that $c > \bar{t}$, since $\bar{t} \in J$. Pick $c' \in [\bar{t}, c[$. Then $[\bar{t}, c'] \subseteq J$, so $[\bar{t}, c'] \subseteq \mathrm{Dom}(\tilde{\xi}^\beta)$ for large β, and $\tilde{\xi}^\beta\lceil[\bar{t}, c'] \to \tilde{\xi}\lceil[\bar{t}, c']$ uniformly. Since $b(\xi^\beta) \to b > c'$, we have $b(\xi^\beta) > c'$ for large β. Since $a(\xi^\beta) \leq t^\beta \to \bar{t} < c'$, we have $a(\xi^\beta) < c'$ for large β. Therefore $c' \in \mathrm{Dom}(\xi^\beta)$ for large β, so $\tilde{\xi}(c') = \lim \xi^\beta(c')$. Therefore $(\tilde{\xi}(c'), c') \in K$. Since this is true for $c' < c$ but arbitrarily close to c, and the set $L = \{t \in J : (\tilde{\xi}(t), t) \in K\}$ is compact by Proposition 1, we conclude that $c \in L$, so in particular $c \in J$, which is a contradiction.

So a and b belong to J. Therefore $[a, b] \subseteq \mathrm{Dom}(\tilde{\xi}^\beta)$ for large β, and $\tilde{\xi}^\beta\lceil[a, b]$ converges to $\tilde{\xi}\lceil[a, b]$. Since $a(\xi^\beta) \to a$, and $b(\xi^\beta) \to b$, given any $s \in [a, b]$ we can find $s^\beta \in \mathrm{Dom}(\xi^\beta)$ such that $a^\beta \leq s^\beta \leq b^\beta$ and $s^\beta \to s$. Therefore $\xi^\beta(s^\beta) \to \tilde{\xi}(s)$. So $(\tilde{\xi}(s), s) \in K$. This shows that the restriction ξ of $\tilde{\xi}$ to $[a, b]$ is in $\mathcal{A}(K)$. Therefore $d_{ARC}(\xi^\beta, \xi) \geq \varepsilon$. On the other hand, $\xi^\beta \to \xi$ in $ARC(\Omega)$, so $d_{ARC}(\xi^\beta, \xi) \to 0$. This contradiction proves that (II)$\Rightarrow$(I), as stated.

We now prove that (I)$\Rightarrow$(II). Assume that the net $\{\mathcal{A}^\alpha\}_{\alpha\in A}$ T-converges to $\mathcal{A}$. We want to show that every subnet of $\{\mathcal{A}^\alpha\}_{\alpha\in A}$ converges to $\mathcal{A}$ in the maximal curve convergence sense. Since every subnet of $\{\mathcal{A}^\alpha\}_{\alpha\in A}$ T-converges to $\mathcal{A}$, because the notion of T-convergence arises from a topology, it suffices to show that $\{\mathcal{A}^\alpha\}_{\alpha\in A}$ converges to $\mathcal{A}$ in the maximal curve convergence sense.

Let $\{\xi^\alpha\}_{\alpha\in A}$ be a net such that $\xi^\alpha \in (\mathcal{A}^\alpha)^{\#}_{max}$, and let I^α be the domain of ξ^α. Let $t^\alpha \in I^\alpha$ be such that $t^\alpha \to \bar{t} \in I$, and $x^\alpha \stackrel{\text{def}}{=} \xi^\alpha(t^\alpha) \to \bar{x} \in \Omega$. Pick a compact neighborhood K_0 of $\bar{x}$ in Ω and a compact interval J_0 which is a relative neighborhood of $\bar{t}$ in I. By passing to a subnet of $\{\xi^\alpha\}$, we may assume that

$x^\alpha \in K_0$ and $t^\alpha \in J_0$ for all α. Let $\mathbf{Q}$ be the set of all products $Q = K \times J$, where K is a compact subset of Ω such that $K_0 \subseteq K$, and J is a compact subinterval of I such that $J_0 \subseteq J$. For each $Q = K \times J \in \mathbf{Q}$, let $L^\alpha(Q) = \{t \in I^\alpha : (\xi^\alpha(t), t) \in Q\}$. Then $L^\alpha(Q)$ is compact. Let $S^\alpha(Q)$ be the connected component of $L^\alpha(Q)$ that contains t^α. Then $S^\alpha(Q)$ is a compact subinterval of I. Let $\xi^{\alpha,Q}$ be the restriction of ξ^α to $S^\alpha(Q)$. Then $\xi^{\alpha,Q} \in \mathcal{A}^\alpha(Q)$. Since $\mathcal{A}(Q)$ is compact, we can pick $\zeta^{\alpha,Q} \in \mathcal{A}(Q)$ such that $d_{ARC}(\xi^{\alpha,Q}, \zeta^{\alpha,Q}) = d_{ARC}(\xi^{\alpha,Q}, \mathcal{A}(Q))$. Since $\mathcal{A}^\alpha \xrightarrow{\mathrm{T}} \mathcal{A}$, the distance $d_{ARC}(\xi^{\alpha,Q}, \mathcal{A}(Q))$ goes to 0. So $d_{ARC}(\xi^{\alpha,Q}, \zeta^{\alpha,Q}) \to 0$. Let

$$U = \prod_{Q \in \mathbf{Q}} \mathcal{A}(Q) \,. \tag{7}$$

Then U is a product of compact topological spaces. By Tikhonov's theorem, U is compact with respect to the product topology. Let $Z^\alpha = \{\zeta^{\alpha,Q}\}_{Q \in \mathbf{Q}}$. Then each Z^α is a member of U. Therefore there is a subnet $\{Z^{\mu(\beta)}\}_{\beta \in B}$ of $\{Z^\alpha\}_{\alpha \in A}$ that converges to a limit $Z = \{\zeta^Q\}_{Q \in \mathbf{Q}} \in U$. Then $\zeta^{\mu(\beta),Q} \to \zeta^Q$ for every $Q \in \mathbf{Q}$. Since $d_{ARC}(\xi^{\mu(\beta),Q}, \zeta^{\mu(\beta),Q}) \to 0$, we have shown that $\xi^{\mu(\beta),Q} \to \zeta^Q$ for each Q.

If $Q, Q' \in \mathbf{Q}$ and $Q \subseteq Q'$, then it is easy to see that $L^\alpha(Q) \subseteq L^\alpha(Q')$ for all $\alpha \in A$. Therefore $S^\alpha(Q) \subseteq S^\alpha(Q')$, and then $G(\xi^{\alpha,Q}) \subseteq G(\xi^{\alpha,Q'})$. In particular, $G(\xi^{\mu(\beta),Q}) \subseteq G(\xi^{\mu(\beta),Q'})$ for all β and then, taking limits, it follows that $G(\zeta^Q) \subseteq G(\zeta^{Q'})$. Since for any two members Q, Q' of $\mathbf{Q}$ there is a $Q'' \in \mathbf{Q}$ such that $Q \cup Q' \subseteq Q''$, it follows that the arcs ζ^Q match, in the sense that the union of the graphs $G(\zeta^Q)$ is the graph of a map ζ. If $S(Q) = \mathrm{Dom}(\zeta^Q)$, then $\bar{t} \in S(Q)$, because $t^{\mu(\beta)} \in \mathrm{Dom}(\xi^{\mu(\beta),Q})$ for each β, and $t^{\mu(\beta)} \to \bar{t}$. Then $S = \cup_{Q \in \mathbf{Q}} S(Q)$ is an interval, and it is clear that $S = \mathrm{Dom}(\zeta)$. Obviously, $\zeta(\bar{t}) = \bar{x}$. We now have to show that $\zeta \in \mathcal{A}^{\#}_{max}$ and $\xi^{\mu(\beta)} \to \zeta$ on compact sets.

Write

$$S^\alpha(Q) = [a^\alpha(Q), b^\alpha(Q)] \,, \quad S(Q) = [a(Q), b(Q)] \,, \tag{8}$$

so $a(Q) = \lim_\beta a^{\mu(\beta)}(Q)$ and $b(Q) = \lim_\beta b^{\mu(\beta)}(Q)$.

Next, we make the following observation.

(*) Given any $\alpha \in A$, and any $Q = K \times J \in \mathbf{Q}$, the point $\xi^\alpha(b^\alpha(Q))$ belongs to $K \backslash \mathrm{Int}(K)$, unless $b^\alpha(Q) = \sup I$.

Indeed, if $b^\alpha(Q) < \sup I$ and $\xi^\alpha(b^\alpha(Q)) \in \mathrm{Int}(K)$, then $\xi^\alpha(t)$ would be defined for $t = b^\alpha(Q) + h$ if $h > 0$ is small enough, because the domain of ξ^α is relatively open in I. By continuity, $(\xi^\alpha(t), t)$ would be in Q for $t = b^\alpha(Q) + h$, $h > 0$, h small, contradicting the definition of $b^\alpha(Q)$. Thus (*) holds, as stated.

Let $\sigma = \sup S$. We distinguish three cases, namely,

(i) $\sigma \notin I$,

(ii) $\sigma \in I \backslash S$,

(iii) $\sigma \in S$.

In Cases (i) and (ii), if $b \in S$, then $b < \sigma$, so $b < b(Q)$ for some $Q \in \mathbf{Q}$, since $\sigma = \sup\{b(Q) : Q \in \mathbf{Q}\}$. Therefore $b < b^{\mu(\beta)}(Q)$ for large enough β. In Case (iii), we must have $\sigma = b(\bar{Q})$ for a $\bar{Q} \in \mathbf{Q}$. Let $Q' = K' \times J'$ be such that $\zeta(\sigma) \in \mathrm{Int}_\Omega(K')$ and pick a $Q = K \times J \in \mathbf{Q}$ such that $\bar{Q} \cup Q' \subseteq Q$. Then $b(Q) = \sigma$, and $\zeta^Q(b(Q)) = \zeta(b(Q)) \in \mathrm{Int}(K)$. So $\xi^{\mu(\beta),Q} \in \mathrm{Int}(K)$ for large β. Thus (*) implies that $b^{\mu(\beta)}(Q) = \sigma$ for large β.

Summarizing the above, we have shown the following.

(**) If $b \in S$, then there is a $Q \in \mathbf{Q}$ such that $b \leq b^{\mu(\beta)}(Q)$ for all sufficiently large β.

Naturally, a similar reasoning shows that if $a \in S$ then there is a Q such that $a^{\mu(\beta)}(Q) \leq a$ for all sufficiently large β. Therefore, if $S' = [a, b]$ is any compact subinterval of S, there exists a Q such that $[a, b] \subseteq S^{\mu(\beta)}(Q)$ for all large enough β. It follows that the restriction $\zeta\lceil[a, b]$ is the arc $\zeta^Q\lceil[a, b]$. Since $\zeta^Q \in \mathcal{A}$, we see that $\zeta\lceil[a, b] \in \mathcal{A}$. This is true for every S'. So we have shown that $\zeta \in \mathcal{A}^{\#}$. Moreover, it also follows that, for any S', there is a Q such that the inclusion $S' \subseteq S^{\mu(\beta)}(Q) \subseteq \mathrm{Dom}(\xi^{\mu(\beta)})$ holds for large enough β. This shows that $\xi^{\mu(\beta)} \to \zeta$ on compact sets.

To conclude, we have to show that $\zeta \in \mathcal{A}^{\#}_{max}$. Assume that this is not true, and let $\tilde{\zeta} : \tilde{S} \mapsto \Omega$ be a curve, defined on a subinterval $\tilde{S}$ of I such that $S \subseteq \tilde{S}$ but $S \neq \tilde{S}$, for which $\tilde{\zeta}\lceil S = \zeta$. Pick $s \in \tilde{S}\backslash S$. Then either $s \geq \sup S$ or $s \leq \inf S$. We assume that $s \geq \sup S$, the other case being similar. As before, we write $\sigma = \sup S$. Then $\sigma \in I$, because $\sigma \leq s \in I$ and $\sigma \geq t$ for some $t \in S$. Pick a $Q_1 = K_1 \times J_1 \in \mathbf{Q}$ such that $\tilde{\zeta}(\sigma) \in \mathrm{Int}(K_1)$. Pick a $\sigma' \in S$ such that $\sigma' \leq \sigma$, and $\zeta(t) \in \mathrm{Int}(K_1)$ whenever $\sigma' \leq t$ and $t \in S$. Using (**), pick a $Q_2 = K_2 \times J_2$ such that $\sigma' \leq b^{\mu(\beta)}(Q_2)$ for all sufficiently large β. Pick $Q = K \times J \in \mathbf{Q}$ such that $Q_1 \cup Q_2 \subseteq Q$. Then $\xi^{\mu(\beta)}(b^{\mu(\beta)}(Q)) \to \zeta(b(Q)) \in \mathrm{Int}(K)$. Thus $\xi^{\mu(\beta)}(b^{\mu(\beta)}(Q))$ belongs to $\mathrm{Int}(K)$ for large β, and then it follows from (*) that $b^{\mu(\beta)}(Q) = \sup I$ for large β. But then $b(Q) = \sup I$, and this implies that $\sigma = b(Q) = \sup I$, and $\sigma \in S$. Since $s \in I$, and $s \geq \sigma$, we must have $s = \sigma$, contradicting the fact that $s \notin S$. ∎

Remark 1 If the space Ω is σ-compact (i.e., a countable union of compact subsets), then instead of considering the product U defined in (7), we can use the product $U' = \prod_{j=1}^{\infty} \mathcal{A}(Q_j)$, where $\{Q_j\}$ is a sequence of compact subsets of $\Omega \times I$ whose relative interiors cover $\Omega \times I$. Then U' is metrizable. Therefore the statement and proof of Proposition 2 can be repeated using sequences instead of nets, and subsequences instead of subnets. ∎

We now specialize to the arc systems defined by vector fields. Let Ω be an open subset of $\mathbb{R}^n$, and let I be a subinterval of $\mathbb{R}$. A *time-varying vector field* on Ω with time domain I is a map $f : \Omega \times I \mapsto \mathbb{R}^n$. We use $TVVF(\Omega, I)$ to denote the set of all such maps. An $f \in TVVF(\Omega, I)$ satisfies the *Carathéodory condition* if it is continuous in x for each fixed $t \in I$ and Lebesgue measurable in t for each fixed $x \in \Omega$. We call f *locally integrably bounded* (LIB) if f is bounded in norm by a locally integrable function of t, as long as x stays in a compact subset of Ω. We use $TVVF_{Car,LIB}(\Omega, I)$ to denote the set of all $f \in TVVF(\Omega, I)$ that satisfy the Carathéodory condition, and are LIB.

If $f \in TVVF(\Omega, I)$, a *trajectory* (or *integral curve*) of f is an absolutely continuous function $\xi : J \mapsto \Omega$ defined on some nonempty subinterval $J \subseteq I$ such that $\dot{\xi}(t) = f(\xi(t), t)$ for almost all $t \in J$. A *maximal trajectory* of f is a trajectory $\xi : J \mapsto \Omega$ that cannot be extended to a trajectory of f defined on a strictly larger interval. We write $\mathrm{Traj}(f)$ to denote the set of all trajectories of f, and $\mathrm{Traj}_c(f)$ to denote the set of all $\xi \in \mathrm{Traj}(f)$ that are arcs, i.e., such that $\mathrm{Dom}(\xi)$ is a compact interval. Also, we write $\mathrm{Traj}(f)_{max}$ to denote the set of all maximal trajectories of f. Then it is clear that $\mathrm{Traj}_c(f)$ is an arc system, and $\mathrm{Traj}_c(f)^{\#}$ is precisely $\mathrm{Traj}(f)$. Therefore $\mathrm{Traj}(f)_{max}$ is exactly the set $\mathrm{Traj}_c(f)^{\#}_{max}$.

Thus $f \mapsto \mathrm{Traj}_c(f)$ is a map from $TVVF(\Omega, I)$ to $AS(\Omega, I)$. Using this map, we can pull back the topology of trajectory convergence on $AS(\Omega, I)$ and define a *topology of trajectory convergence*, or *T-convergence*, on $TVVF(\Omega, I)$. We can also pull back properties of arc systems and apply them to vector fields. (For example, we will say that f has the local existence property if the arc system $\mathrm{Traj}_c(f)$ does.) We call f *trajectory compact* if $\mathrm{Traj}_c(f)$ has the compactness property. We use $TVVF_{comp,le}(\Omega, I)$ to denote the set of those $f \in TVVF(\Omega, I)$ that are trajectory compact and have the local existence property. We say that f has the *uniqueness property* if for every initial condition $(\bar{x}, \bar{t})$ in $\Omega \times I$ the maximal trajectory of f whose graph contains $(\bar{x}, \bar{t})$ is unique.

The Carathéodory existence theorem (cf. [6]) implies that every vector field $f \in TVVF_{Car,LIB}(\Omega, I)$ has the local existence property. Moreover, a simple application of the Ascoli–Arzelà theorem and the Lebesgue dominated convergence theorem shows that every time-varying vector field $f \in TVVF_{Car,LIB}(\Omega, I)$ is trajectory compact. So

$$TVVF_{Car,LIB}(\Omega,I) \subseteq TVVF_{comp,le}(\Omega,I)\,. \tag{9}$$

If S is a subset of $\Omega \times I$, we write $\mathrm{Traj}_c(f,S)$ to denote the set of all arcs $\xi \in \mathrm{Traj}_c(f)$ such that $G(\xi) \subseteq S$. We then have the following propositions.

Proposition 3 *Let f be in $TVVF(\Omega,I)$ and $\{f^\alpha\}_{\alpha\in A}$ be a net in $TVVF(\Omega,I)$. Then $\{f^\alpha\}_{\alpha\in A}$ T-converges to f if and only if for every compact subset K of $\Omega\times I$ and every open subset V of $ARC(\Omega)$ containing $\mathrm{Traj}_c(f,K)$ there exists an $\alpha^*(V)\in A$ such that $\mathrm{Traj}_c(f^\alpha,K)\subseteq V$ whenever $\alpha^*(V)\preceq\alpha$.*

Proposition 4 *If $f\in TVVF_{comp,le}(\Omega,I)$, then a net $\{f^\alpha\}_{\alpha\in A}$ in $TVVF(\Omega,I)$ T-converges to f if and only if*

$$\lim_\alpha \sup\{d_{ARC}(\xi,\mathrm{Traj}_c(f,K)) : \xi\in\mathrm{Traj}_c(f^\alpha,K)\} = 0 \tag{10}$$

for every compact subset K of $\Omega\times I$.

Proposition 5 *Let $\mathbf{f}=\{f^\alpha\}_{\alpha\in A}$ be a net in $TVVF_{comp,le}(\Omega,I)$, and suppose that $f\in TVVF_{comp,le}(\Omega,I)$. Then $\mathbf{f}$ T-converges to f if and only if, whenever $\{f^{\mu(\beta)}\}_{\beta\in B}$ is a subnet of $\mathbf{f}$, $(x^\beta,t^\beta)\in\Omega\times I$ converge to $(\bar{x},\bar{t})\in\Omega\times I$, and $\boldsymbol{\xi}=\{\xi^\beta\}_{\beta\in B}$ is a family such that each ξ^β is a maximal trajectory of $f^{\mu(\beta)}$ for which $\xi^{\mu(\beta)}(t^\beta)=x^\beta$, it follows that there exist a maximal trajectory ξ of f such that $\xi(\bar{t})=\bar{x}$ and a subnet $\{\xi^{\nu(\gamma)}\}_{\gamma\in C}$ of $\boldsymbol{\xi}$ that converges to ξ on compact sets.*

Proposition 6 *If $f\in TVVF_{comp,le}(\Omega,I)$, then a sequence $\mathbf{f}=\{f^j\}_{j=1}^\infty$ of members of $TVVF_{comp,le}(\Omega,I)$ T-converges to f if and only if, whenever J is an infinite set of positive integers, $(x^j,t^j)\in\Omega\times I$ converge to $(\bar{x},\bar{t})\in\Omega\times I$ as $j\to\infty$ via values in J, and $\boldsymbol{\xi}=\{\xi^j\}_{j\in J}$ is a family such that each ξ^j is a maximal trajectory of f^j for which $\xi^j(t^j)=x^j$, it follows that there exist a maximal trajectory ξ of f such that $\xi(\bar{t})=\bar{x}$ and an infinite subset J' of J such that $\{\xi^j\}_{j\in J'}$ converges to ξ on compact sets.*

Proposition 7 *Suppose that $f\in TVVF_{comp,le}(\Omega,I)$ and that a net $\mathbf{f}=\{f^\alpha\}_{\alpha\in A}$ of members of $TVVF_{comp,le}(\Omega,I)$ T-converges to f; assume that $(x^\alpha,t^\alpha)\in\Omega\times I$ converge to $(\bar{x},\bar{t})\in\Omega\times I$, $\boldsymbol{\xi}=\{\xi^\alpha\}_{\alpha\in A}$ is a family such that each ξ^α is a maximal trajectory of f^α for which $\xi^\alpha(t^\alpha)=x^\alpha$, and the maximal trajectory ξ of f for which $\xi(\bar{t})=\bar{x}$ is unique; then the net $\boldsymbol{\xi}$ converges to ξ on compact sets.*

Proposition 8 *Suppose that $f \in TVVF_{comp,le}(\Omega, I)$ has the uniqueness property, and let $\mathbf{f} = \{f^\alpha\}_{\alpha\in A}$ be a net of members of $TVVF_{comp,le}(\Omega, I)$; then $\mathbf{f}$ T-converges to f if and only if, whenever $(x^\alpha, t^\alpha) \in \Omega \times I$ converge to $(\bar{x}, \bar{t}) \in \Omega \times I$, $\boldsymbol{\xi} = \{\xi^\alpha\}_{\alpha\in A}$ is a family such that each ξ^α is a maximal trajectory of f^α for which $\xi^\alpha(t^\alpha) = x^\alpha$, and ξ is the maximal trajectory of f for which $\xi(\bar{t}) = \bar{x}$, it follows that the net $\boldsymbol{\xi}$ converges to ξ on compact sets.*

These properties are very easy to prove. Proposition 3 is just a restatement of the definition of T-convergence. Then Proposition 4 immediately follows from Proposition 3 by observing that if $\mathrm{Traj}_c(f, K)$ is compact, then the sets $\{\xi \in ARC(\Omega) : d_{ARC}(\xi, \mathrm{Traj}_c(f, K)) < \varepsilon\}$ form a fundamental system of neighborhoods of $\mathrm{Traj}_c(f)$. Proposition 5 follows from Proposition 2, and Proposition 6 follows from Remark 1. Finally, Propositions 7 and 8 follow trivially from Proposition 5. ∎

3 $\mathcal{T}^0$-convergence for control-affine systems

We are now ready to give the precise definition of the topology of $\mathcal{T}^0$-convergence on $L^1([0, T], \mathbb{R}^m)$.

Let m be a positive integer. A $C^0(m)$ *system* is an $(m + 2)$-tuple

$$\Sigma = (\Omega, f_0, \dots, f_m)\,, \tag{11}$$

where Ω is a nonempty open subset of $\mathbb{R}^n$ for some integer $n \geq 1$, and $f_0, \dots, f_m$ are continuous vector fields on Ω. The expressions f_i^Σ, Ω^Σ will be used to denote the vector fields f_i and the set Ω corresponding to a given system $\Sigma \in C^0(m)$.

If $T > 0$, and $u \in L^1([0, T], \mathbb{R}^m)$, then for each $\Sigma = (\Omega^\Sigma, f_0^\Sigma, \dots, f_m^\Sigma)$, we can consider the ordinary differential equation

$$\dot{x}(t) = f_0^\Sigma(x(t)) + \sum_{k=1}^{m} u_k(t) f_k^\Sigma(x(t))\,. \tag{12}$$

The right-hand side of (12) is a time-varying vector field on Ω^Σ with time domain $[0, T]$, which clearly belongs to $TVVF_{Car,LIB}(\Omega^\Sigma, [0, t])$. Thus we can associate with Σ the map

$$\Gamma_\Sigma \,:\, L^1([0, T], \mathbb{R}^m) \mapsto TVVF_{Car,LIB}(\Omega^\Sigma, [0, T]) \tag{13}$$

defined by

$$\Gamma_\Sigma(u)(x, t) = f^\Sigma(u)(x, t) \stackrel{\mathrm{def}}{=} f_0^\Sigma(x) + u_1(t) f_1^\Sigma(x) + \cdots + u_m(t) f_m^\Sigma(x)\,. \tag{14}$$

For each $\Sigma \in C^0(m)$ and $T > 0$, the set $TVVF_{Car,LIB}(\Omega^\Sigma, [0,T])$ is a topological space with the topology of trajectory convergence. Therefore, for any subclass $\mathcal{C}$ of $C^0(m)$ we can consider the topology $T(\mathcal{C})$ on $L^1([0,T],\mathbb{R}^m)$ induced by the family of mappings $\{\Gamma_\Sigma : \Sigma \in \mathcal{C}\}$, i.e. the weakest topology on $L^1([0,T],\mathbb{R}^m)$ that makes all the maps Γ_Σ continuous, for all $\Sigma \in \mathcal{C}$. In particular, the *topology of $\mathcal{T}^0$-convergence* on $L^1([0,T],\mathbb{R}^m)$ is, by definition, the $T(C^0(m))$-topology. We say that a net $\{u^\alpha\}_{\alpha\in A}$ *$\mathcal{T}^0$-converges* to a u^∞ in $L^1([0,T],\mathbb{R}^m)$ if it converges to u^∞ in this topology. Equivalently, a net $\{u^\alpha\}_{\alpha\in A}$ $\mathcal{T}^0$-converges to a limit u^∞ if and only if

(TCN) *For every possible choice of the system $\Sigma \in C^0(m)$, the time-varying vector fields $(x,t) \mapsto f^\Sigma(u^\alpha)(x,t)$ T-converge to $(x,t) \mapsto f^\Sigma(u^\infty)(x,t)$.*

(This is exactly the version for nets of Condition (TC) of the Introduction.)

If (TCN) holds, then Proposition 7 implies that in particular the following conditions hold as well:

(TCN') *For every possible choice of $\Sigma \in C^0(m)$, if ξ^α are maximal trajectories of (1) determined by the u^α, $x^\alpha = \xi^\alpha(0)$, and $x^\alpha \to \bar{x} \in \Omega^\Sigma$, then the ξ^α are defined on $[0,T]$ for α large enough and converge uniformly on $[0,T]$ to the maximal trajectory ξ^∞ for u^∞, provided that for the limiting initial value problem $\dot{x} = f_0^\Sigma(x) + \sum_{i=1}^m u_i^\infty(t) f_i^\Sigma(x)$, $x(0) = \bar{x}$ there is uniqueness of solutions as well as global existence on $[0,T]$.*

(TCN") *For every possible choice of the system $\Sigma \in C^0(m)$ and the initial condition $x(0) = \bar{x} \in \Omega^\Sigma$, if ξ^α are maximal trajectories of (1) determined by the u^α and $\bar{x}$, then the ξ^α are defined on $[0,T]$ for α large enough and converge uniformly on $[0,T]$ to the maximal trajectory ξ^∞ for u^∞, provided that for the limiting initial value problem $\dot{x} = f_0^\Sigma(x) + \sum_{i=1}^m u_i^\infty(t) f_i^\Sigma(x)$, $x(0) = \bar{x}$, there is uniqueness of solutions as well as global existence on $[0,T]$.*

4 A characterization of sequential $\mathcal{T}^0$-convergence

Theorem 1 *Let $\mathbf{u} = \{u^j\}_{j=1}^\infty$ be a sequence of controls in $L^1([0,T],\mathbb{R}^m)$, and let $u^\infty \in L^1([0,T],\mathbb{R}^m)$. Then the following conditions are equivalent*

(i) $\mathbf{u}$ *satisfies Condition (UB) and I-converges to u^∞,*

(ii) $\mathbf{u}$ *$\mathcal{T}^0$-converges to u^∞,*

(iii) *(TCN') holds,*

(iv) *(TCN") holds,*

(v) $\mathbf{u}$ *UW-converges to* u^∞*.*

If $\mathbf{u} = \{u^\alpha\}_{\alpha\in A}$ *is a net in* $L^1([0,T],\mathbb{R}^m)$*, and* u^∞ *belongs to* $L^1([0,T],\mathbb{R}^m)$*, then* (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii) $\Longrightarrow$(iv)$\Longrightarrow$(v)*, but in general* (v)$\not\Longrightarrow$(iv) *and* (ii)$\not\Longrightarrow$(i)*.*

Remark 2 At present, we do not know if the implications (iv)$\Longrightarrow$(iii)$\Longrightarrow$(ii) hold for general nets, i.e., whether (ii), (iii), and (iv) of Theorem 1 are equivalent for nets.

Theorem 2 *Let* $\mathbf{u} = \{u^\alpha\}_{\alpha\in A}$ *be a net of inputs belonging to* $L^1([0,T],\mathbb{R}^m)$*, and let* $u^\infty \in L^1([0,T],\mathbb{R}^m)$ *be such that* (i) *of Theorem 1 holds. Let* Ω *be open in* $\mathbb{R}^n$*, and let* $h^\alpha : \Omega\times[0,T] \mapsto \mathbb{R}^n$ *be time-varying vector fields given by*

$$h^\alpha(x,t) = \dot\theta^\alpha(t) + f_0(x) + \sum_{i=1}^m u_i^\alpha(t) f_i(x)\,,$$

where the $f_i : \Omega \mapsto \mathbb{R}^n$ *are continuous, the* $\theta^\alpha : [0,T] \mapsto \mathbb{R}^n$ *are absolutely continuous, and* $\theta^\alpha \to 0$ *uniformly. Then the net* $\{h^\alpha\}_{\alpha\in A}$ *T-converges to the vector field* h^∞ *given by* $h^\infty(x,t) = f_0(x) + \sum_{i=1}^m u_i^\infty(t) f_i(x)$*.*

PROOF OF THEOREMS 1 AND 2. We first prove that (i) implies (v). We have to show that (UB)$\wedge$(IC) implies (UWC). Pick a function $\varphi\in C^0([0,T],\mathbb{R})$ and a $\delta>0$, and find a piecewise constant function $\psi = \sum_{k=1}^N \psi_k\chi_{J_k}$ (where the J_k are subintervals of $[0,T]$, and χ_J denotes the indicator function of J) such that $|\varphi(t)-\psi(t)|<\delta$ for all $t\in[0,T]$. If (UB) holds, then there exists α^* and a $C>0$ such that $\|u_i^\alpha\|_{L^1}\le C$ for all $i=1,\ldots,m$, and $\alpha\succeq_A\alpha^*$ including $\alpha=\infty$. But then $|\int_0^t u_i^\alpha(s)\varphi(s)ds - \int_0^t u_i^\alpha(s)\psi(s)ds| \le C\delta$ for all i, t, and $\alpha\succeq_A\alpha^*$. In view of (IC), $\int_0^t u_i^\alpha(s)\psi(s)ds \to \int_0^t u_i^\infty(s)\psi(s)ds$ uniformly with respect to t. So $|\int_0^t u_i^\alpha(s)\psi(s)ds - \int_0^t u_i^\infty(s)\psi(s)ds| \le \delta$ for all t, if α is large enough. But then $|\int_0^t u_i^\alpha(s)\varphi(s)ds - \int_0^t u_i^\infty(s)\varphi(s)ds| \le (2C+1)\delta$ for all t if α is sufficiently large. Since δ is arbitrary, (UWC) follows.

Next, we prove Theorem 2, which contains as a particular case the implication (i)$\Longrightarrow$(ii) of Theorem 1. Let $\mathbf{u} = \{u^\alpha\}_{\alpha\in A}$, and let $\Omega, f_0,\ldots,f_m$ and the θ^α be as in the statement of Theorem 2. Let $\Sigma = (\Omega, f_0,\ldots,f_m)$ be a $C^0(m)$ system. We have to show that the h^α T-converge to h^∞. Let K be a compact subset of $\Omega\times[0,T]$. We will show that the sets $\mathrm{Traj}_c(h^\alpha,K)$ converge to $\mathrm{Traj}_c(h^\infty,K)$. For this we first prove the following.

(CTR) If $\{\xi^\alpha\}_{\alpha\in A}$ is a net in $ARC(\Omega)$ such that ξ^α belongs to $\mathrm{Traj}_c(h^\alpha,K)$, then $\{\xi^\alpha\}_{\alpha\in A}$ has a subnet $\{\xi^{\nu(\beta)}\}_{\beta\in B}$ that converges to an $\eta\in\mathrm{Traj}_c(h^\infty,K)$.

Clearly, if we show that $\{\xi^\alpha\}_{\alpha\in A}$ has a subnet that converges in $ARC(\Omega)$, then (CTR) would follow. Indeed, let $\{\xi^{\nu(\beta)}\}_{\beta\in B}$ be a subnet of $\{\xi^\alpha\}_{\alpha\in A}$ that converges in $ARC(\Omega)$. Let η be the limit. Let $I^\alpha = \mathrm{Dom}(\xi^\alpha)$ and $I = \mathrm{Dom}(\eta)$. If I just contains one point, then η is in $\mathrm{Traj}_c(h^\infty, K)$ by definition. Otherwise, for β large enough, $I^{\nu(\beta)} \cap I \neq \emptyset$. Write $I^\alpha = [a^\alpha, b^\alpha]$, and let $I = [a,b]$. Then $a^{\nu(\beta)} \to a$ and $\xi^{\nu(\beta)}(a^{\nu(\beta)}) \to \eta(a)$.

Let

$$\sigma^\alpha(t) = \xi^\alpha(a^\alpha) + \theta^\alpha(t) - \theta^\alpha(a^\alpha)\,. \tag{15}$$

We then have, for all $t \in I^{\nu(\beta)} \cap I$:

$$\begin{aligned}\xi^{\nu(\beta)}(t) = \sigma^{\nu(\beta)}(t) \quad &+ \int_{a^{\nu(\beta)}}^t f_0\Big(\xi^{\nu(\beta)}(s)\Big)\,ds + \sum_{k=1}^m \int_{a^{\nu(\beta)}}^t u_k^{\nu(\beta)}(s) f_k\Big(\eta(s)\Big)\,ds \\ &+ \sum_{k=1}^m \int_{a^{\nu(\beta)}}^t u_k^{\nu(\beta)}(s)\Big\{f_k\Big(\xi^{\nu(\beta)}(s)\Big) - f_k\Big(\eta(s)\Big)\Big\}\,ds\,. \end{aligned}\tag{16}$$

Then (16) implies (since $f_k\Big(\xi^{\nu(\beta)}(s)\Big) - f_k\Big(\eta(s)\Big) \to 0$ uniformly, the L^1 norms of the u^α are bounded by a fixed constant for large α, (UWC) holds, and $\theta^{\nu(\beta)}(t) - \theta^{\nu(\beta)}(a^\beta) \to 0$ uniformly) that, for $t \in I$,

$$\eta(t) = \eta(a) + \int_a^t f_0(\eta(s))\,ds + \sum_{k=1}^m \int_a^t u_k^\infty(s) f_k(\eta(s))\,ds\,.$$

So η is a trajectory of h^∞. Clearly the graph $G(\eta)$ of η is contained in K. So $\eta \in \mathrm{Traj}_c(h^\infty, K)$ and then (CTR) follows.

We now show that the net $\{\xi^\alpha\}_{\alpha\in A}$ has a subnet that converges in $ARC(\Omega)$. Again let $I^\alpha = [a^\alpha, b^\alpha] = \mathrm{Dom}(\xi^\alpha)$. Then for $t \in I^\alpha$, we have

$$\begin{aligned}\xi^\alpha(t) &= \theta^\alpha(t) - \theta^\alpha(a^\alpha) + \xi^\alpha(a^\alpha) + \int_{a^\alpha}^t f_0\Big(\xi^\alpha(s)\Big)ds + \sum_{k=1}^m \int_{a^\alpha}^t u_k^\alpha(s) f_k\Big(\xi^\alpha(s)\Big)ds \\ &= \theta^\alpha(t) - \theta^\alpha(a^\alpha) + \xi^\alpha(a^\alpha) + \int_{a^\alpha}^t f_0\Big(\xi^\alpha(s)\Big)ds + \sum_{k=1}^m \int_{a^\alpha}^t u_k^\infty(s) f_k\Big(\xi^\alpha(s)\Big)ds \\ &\quad + \sum_{k=1}^m \int_{a^\alpha}^t \Big(u_k^\alpha(s) - u_k^\infty(s)\Big) f_k\Big(\xi^\alpha(s)\Big)ds\,.\end{aligned}$$

Let

$$\zeta^\alpha(t) = \xi^\alpha(a^\alpha) + \int_{a^\alpha}^t f_0\Big(\xi^\alpha(s)\Big)ds + \sum_{k=1}^m \int_{a^\alpha}^t u_k^\infty(s) f_k\Big(\xi^\alpha(s)\Big)ds\,.$$

It is obvious that the set $\{\zeta^\alpha : \alpha \in A\}$ is equicontinuous and uniformly bounded. The $\theta^\alpha(t) - \theta^\alpha(a^\alpha)$ converge to 0 uniformly by our assumption. So in order to show that $\{\xi^\alpha\}_{\alpha\in A}$ has a convergent subnet, we need only show that for each $k \in \{1,\ldots,m\}$, the integrals $\int_{a^\alpha}^t (u_k^\alpha(s) - u_k^\infty(s)) f_k(\xi^\alpha(s))ds$ converge to 0 uniformly with respect to t.

Fix an integer $k \in \{1,\ldots,m\}$. Write $\hat{u}_k^\alpha = u_k^\alpha - u_k^\infty$. Then (UB) implies that there exist an $\alpha^* \in A$ and a constant $C > 0$ such that $\|\hat{u}^\alpha\|_{L^1} \le C$ for $\alpha \succeq_A \alpha^*$. Let

$$\hat{\xi}^\alpha(t) = \xi^\alpha(t) - \theta^\alpha(t)\,. \tag{17}$$

Then $\hat{\xi}^\alpha(t) - \xi^\alpha(t)$ goes to 0 uniformly, so $f_k(\hat{\xi}^\alpha(t)) - f_k(\xi^\alpha(t))$ goes to 0 uniformly for each k. Since the L^1 norms of the functions $\hat{u}_k^\alpha$ are bounded, we can conclude that the integrals

$$\int_{a^\alpha}^t \Big(u_k^\alpha(s) - u_k^\infty(s)\Big)\Big(f_k\Big(\hat{\xi}^\alpha(s)\Big) - f_k\Big(\xi^\alpha(s)\Big)\Big)ds$$

converge to 0 uniformly with respect to t. So it suffices to show that

$$\int_{a^\alpha}^t \Big(u_k^\alpha(s) - u_k^\infty(s)\Big) f_k\Big(\hat{\xi}^\alpha(s)\Big)ds \to 0 \ \text{ uniformly}\,. \tag{18}$$

Let $\mathcal{K}$ be a compact subset of Ω such that $K \subseteq \mathcal{K} \times [0,T]$. For any given $\varepsilon > 0$, let $g_k : \Omega \mapsto \mathbb{R}^n$ be vector fields of class C^∞ with compact support such that

$$\sup_{x\in\mathcal{K}} \|f_k(x) - g_k(x)\| < \frac{\varepsilon}{C}\,. \tag{19}$$

Then

$$\begin{aligned}\int_{a^\alpha}^t \hat{u}_k^\alpha(s) f_k\Big(\hat{\xi}^\alpha(s)\Big)ds \ &= \ \int_{a^\alpha}^t \hat{u}_k^\alpha(s) g_k\Big(\hat{\xi}^\alpha(s)\Big)\,ds \\ &\quad + \int_{a^\alpha}^t \hat{u}_k^\alpha(s)\Big(f_k\Big(\hat{\xi}^\alpha(s)\Big) - g_k\Big(\hat{\xi}^\alpha(s)\Big)\Big)\,ds\,. \end{aligned} \tag{20}$$

The sup norm of the second term on the right-hand side of (20) is less than ε. Via integration by parts, we can rewrite the first term as

$$\hat{U}_k^\alpha(t)\cdot g_k\Big(\hat{\xi}^\alpha(t)\Big) - \int_{a^\alpha}^t \hat{U}_k^\alpha(s)\frac{d}{ds}\Big(g_k(\hat{\xi}^\alpha(s))\Big)\,ds\,, \tag{21}$$

where

$$\hat{U}_k^\alpha(t) = \int_{a^\alpha}^t \hat{u}_k^\alpha(s)\,ds\,. \tag{22}$$

Since $\hat{U}_k^\alpha(t) \to 0$ uniformly, the g_k are uniformly bounded, and the functions $t \mapsto \frac{d}{dt}(g_k(\hat{\xi}^\alpha(t)))$ are uniformly bounded in L^1 for large α, because

$$\frac{d}{dt}\Big(g_k(\hat{\xi}^\alpha(t))\Big) = \frac{\partial g_k}{\partial x}(\hat{\xi}^\alpha(t))\left(f_0(\xi^\alpha(t)) + \sum_{\ell=1}^m u_\ell^\alpha(t) f_\ell(\xi^\alpha(t))\right), \tag{23}$$

we can conclude that the integrals $\int_{a^\alpha}^t \hat{u}_k^\alpha(s) g_k(\hat{\xi}^\alpha(s))\, ds$ go to zero uniformly. This implies that

$$\limsup_\alpha \left\| \int_{a^\alpha}^t \hat{u}_k^\alpha(s) f_k\Big(\hat{\xi}^\alpha(s)\Big) ds \right\| \le \varepsilon\,. \tag{24}$$

Since ε is arbitrary, we have shown that $\int_{a^\alpha}^t \hat{u}_k^\alpha(s) f_k\Big(\hat{\xi}^\alpha(s)\Big)\, ds \to 0$ uniformly. As explained before, this shows that $\{\xi^\alpha\}_{\alpha\in A}$ has a convergent subnet in $ARC(\Omega)$.

The above argument clearly applies to any subnet of $\{h^\alpha\}$. Thus we have actually proved the following.

(CTR1) for every compact subset K of $\Omega \times I$, if $\{h^{\nu(\beta)}\}_{\beta\in B}$ is a subnet of $\{h^\alpha\}_{\alpha\in A}$, and $\{\xi^\beta\}_{\beta\in B}$ is a family in $ARC(\Omega)$ having the property that each ξ^β belongs to $\mathrm{Traj}_c(h^{\nu(\beta)}, K)$, then the net $\{\xi^\beta\}_{\beta\in B}$ has a subnet $\{\xi^{\mu(\gamma)}\}_{\gamma\in C}$ that converges to an $\eta \in \mathrm{Traj}_c(h^\infty, K)$.

It follows easily from (CTR1) that the h^α T-converge to h^∞. Indeed, (CTR1) clearly implies that

$$\sup\{d_{ARC}(\xi, \mathrm{Traj}_c(h^\infty, K)) : \xi \in \mathrm{Traj}_c(h^\alpha, K)\} \to 0$$

for every K, and then the T-convergence of h^α to h^∞ follows from Proposition 4. This completes the proof of Theorem 2. In particular, we have established that (i) of Theorem 1 implies (ii).

We now continue with the proof of Theorem 1. The implication (ii)$\Longrightarrow$(iii) follows from Proposition 7, and (iii) implies (iv) trivially. We now prove that (iv) implies (v). To see this, let $\varphi : [0, T] \mapsto \mathbb{R}$ be continuous. Consider the $C^0(m)$ system $\Sigma = (\mathbb{R}^{m+1}, f_0, \ldots, f_m)$ where—using $(x_0, x_1, \ldots, x_m)$ to denote the standard coordinates in $\mathbb{R}^{m+1}$, and $(e_0, e_1, \ldots, e_m)$ to denote the canonical basis—$f_0(x) \equiv e_0$, $f_i(x) \equiv \varphi(x_0)e_i$ for $i = 1, \ldots, m$. Then the differential equation associated to Σ and u^α is $\dot{x}_0 = 1$, $\dot{x}_i = \varphi(x_0)u_i^\alpha$ for $i = 1, \ldots, m$. Take the initial condition $\bar{x} = (0, \ldots, 0)$. Then the components ξ_i^α of the solution, for $i = 1, \ldots, m$, are precisely the components of the vector U_φ^α. Thus (UWC) holds.

Next we prove that (v) implies (i) for sequences. Let $\{u^j\}_{j=1}^\infty$ be a sequence in $L^1([0, T], \mathbb{R}^m)$, and let $u^\infty \in L^1([0, T], \mathbb{R}^m)$. Assume that (UWC) holds for $\{u^j\}$

and u^∞. Then, in particular, the integrals $\langle u_i^j, \varphi\rangle = \int_0^T u_i^j(t)\varphi(t)dt$ converge to $\langle u_i^\infty, \varphi\rangle = \int_0^T u_i^\infty(t)\varphi(t)dt$ for each i and for each continuous function $\varphi : [0,T] \mapsto \mathbb{R}$. So the u_i^j weak*-converge to u_i^∞ as linear functionals on the Banach space $C^0([0,T],\mathbb{R})$ of continuous real-valued functions on $[0,T]$. It is well known that, if ψ belongs to $L^1([0,T],\mathbb{R})$, then the norm of the linear functional $C^0([0,T],\mathbb{R}) \ni \rho \mapsto \int_0^T \rho(t)\psi(t)dt$ is precisely the L^1 norm of ψ. Therefore the L^1 norms of the u_i^j are bounded by a fixed constant, by the uniform boundedness theorem. Thus (UB) holds. Moreover, taking $\varphi \equiv 1$ we conclude that the u^j I-converge to u^∞.

The proof that in general (v) $\not\Longrightarrow$ (iv) and (ii)$\not\Longrightarrow$ (i) is given in the appendix, where we exhibit an example of a net that satisfies (v) but not (iv) and one of a net that satisfies (ii) but not (i). ∎

5 Other classes of systems

The equivalence for sequences of the five conditions of Theorem 1 depends very strongly on the fact that the class of systems under consideration is $C^0(m)$. One could equally well have considered $\mathrm{T}(\mathcal{C})$-convergence in $L^1([0,T],\mathbb{R}^m)$ with respect to other classes $\mathcal{C}$ of systems. For example, we could have taken $\mathcal{C}$ to be the class $C^k(m)$ of all the $\Sigma \in C^0(m)$ such that $f_0^\Sigma, \ldots, f_m^\Sigma$ are of class C^k on Ω^Σ, for an arbitrary integer $k > 0$. Let us call the $\mathrm{T}(C^k(m))$-convergence in $L^1([0,T],\mathbb{R}^m)$ simply $\mathcal{T}^k$*-convergence.*

It is then natural to ask whether explicit characterizations of $\mathcal{T}^k$-convergence of sequences exist. The necessary and sufficient conditions for $\mathcal{T}^0$-convergence are of course sufficient to ensure $\mathcal{T}^k$-convergence, but they are no longer necessary. For example, if a sequence $\{u^j\}$ $\mathcal{T}^k$-converges to u^∞ in $L^1([0,T],\mathbb{R}^m)$ for some $k \geq 1$, then we can no longer apply the uniform boundedness theorem to conclude that the L^1 norms of the u^j are bounded. The following theorem, which characterizes $\mathcal{T}^k$-convergence of nets in $L^1([0,T],\mathbb{R})$ for $k \geq 1$, implies in particular that it is indeed possible for a sequence $\{u^j\}_{j=1}^\infty \subseteq L^1([0,T],\mathbb{R})$ to $\mathcal{T}^1$-converge to a $u^\infty \in L^1([0,T],\mathbb{R})$ even if the $\|u^j\|_{L^1}$ are not bounded.

Theorem 3 *Let $T > 0$, and let $k > 0$ be an integer. A net $\{u^\alpha\}_{\alpha\in A}$ in $L^1([0,T],\mathbb{R})$ $\mathcal{T}^k$-converges to a $u^\infty \in L^1([0,T],\mathbb{R})$ if and only if $\{u^\alpha\}$ I-converges to u^∞, i.e. if and only if the integrals $\int_0^t u^\alpha(s)ds$ converge to $\int_0^t u^\infty(s)ds$ uniformly on $[0,T]$.*

Before proving Theorem 3, it is natural to ask whether a similar characterization might be valid for $m > 1$. The answer is negative, as will be shown by an example in the next section. The reason for this is by now well understood. The obstructions for the validity of Theorem 3 when $m > 1$ are the Lie brackets of the vector fields f_i for $i > 0$. Precisely, call a system $\Sigma \in C^1(m)$ *commutative* if the Lie brackets $[f_i^\Sigma, f_j^\Sigma]$ vanish identically for all $i, j \in \{1, \ldots, m\}$. (We do *not* require that the brackets involving f_0 vanish. Therefore for $m = 1$ *every* system is commutative.) For $k \geq 1$, Let $C^k_{comm}(m)$ be the class of all commutative systems in $C^k(m)$. A net $\mathbf{u}$ in $L^1([0,T], \mathbb{R}^m)$ $\mathcal{T}^k_{comm}$*-converges* to $u \in L^1([0,T], \mathbb{R}^m)$ if it converges to u in the topology $\mathcal{T}(C^k_{comm}(m))$, i.e., if the time-varying vector fields $f^\Sigma(u^\alpha)$ T-converge to $f^\Sigma(u)$ for all $\Sigma \in C^k_{comm}(m)$.

Theorem 4 *Let $T > 0$, and let m be a positive integer. For any integer $k \geq 1$, a net $\{u^\alpha\}_{\alpha \in A}$ in $L^1([0,T], \mathbb{R}^m)$ $\mathcal{T}^k_{comm}$-converges to a $u^\infty \in L^1([0,T], \mathbb{R}^m)$ iff $\{u^\alpha\}$ I-converges to u^∞, i.e. iff the integrals $\int_0^t u^\alpha(s)ds$ converge to $\int_0^t u^\infty(s)ds$ uniformly on $[0,T]$.*

Since for $m = 1$ every C^k system is commutative, Theorem 3 is a particular case of Theorem 4.

PROOF OF THEOREM 4. First of all, we observe that $\mathcal{T}^k_{comm}$-convergence obviously implies I-convergence. To show the converse, we assume that $\mathbf{u} = \{u^\alpha\}_{\alpha \in A}$ I-converges to u^∞, and seek to prove that the net $\{f^\Sigma(u^\alpha)\}_{\alpha \in A}$ T-converges to $f^\Sigma(u^\infty)$ for every $C^1_{comm}(m)$ system Σ. We consider first the special case of systems

$$\Sigma: \quad \dot{x} = f_0(x) + \sum_{i=1}^m u_i b_i = f_0(x) + u \cdot B\,,\ x \in \Omega\,, \tag{25}$$

where f_0 is continuous (but not necessarily of class C^1) and the vector fields $f_1, \ldots, f_m$ are constant. Let $U_i^\alpha(t) = \int_a^t u_i^\alpha(s)\,ds$ for $\alpha \in A \cup \{\infty\}$, so $U^\alpha \to U^\infty$ uniformly on $[0,T]$. Let $\xi^\alpha : [a^\alpha, b^\alpha] \mapsto \Omega$ be trajectories of Σ for the u^α, such that $G(\xi^\alpha) \subseteq K$ for some compact subset K of $\Omega \times [0,T]$. We want to find a subnet of $\{\xi^\alpha\}_{\alpha \in A}$ that converges to a $\xi^\infty \in \mathrm{Traj}_c(f^\Sigma(u^\infty), K)$. By passing to a subnet, we may assume that a^α, b^α have limits a, b, and $x^\alpha = \xi^\alpha(a^\alpha)$ converges to a limit $\bar{x}$. Then $(\bar{x}, a) \in K$. Since

$$\xi^\alpha(t) = x^\alpha + \int_{a^\alpha}^t f_0(\xi^\alpha(s))\,ds + (U^\alpha(t) - U^\alpha(a^\alpha)).B\,, \tag{26}$$

and the U^α converge uniformly to U^∞, it is clear that $\{\xi^\alpha\}$ is equicontinuous. Thus by passing to a subnet we may assume that ξ^α converges uniformly to a limit ξ^∞. Then a trivial argument allows us to pass to the limit in (26) and conclude that ξ^∞ is a trajectory for U^∞. The above argument clearly applies to any subnet of $\{\xi^\alpha\}_{\alpha \in A}$. Then Proposition 4 implies that the $f^\Sigma(u^\alpha)$ T-converge to $f^\Sigma(u^\infty)$ in this special case.

We now consider the case when Σ is such that the vector fields $f_1^\Sigma, \ldots, f_m^\Sigma$ are linearly independent at each point. In this case, each point $x \in \Omega^\Sigma$ has an open neighborhood U_x such that there is a C^1 diffeomorphism Φ_x from U_x to some other open subset V_x of $\mathbb{R}^n$ that transforms the f_i^Σ, for $i = 1, \ldots, m$, to constant vector fields $g_{i,x}$. Let $g_{0,x}$ be the vector field on V_x that corresponds to $f_0^\Sigma \lceil U_x$ under Φ_x, that is, $g_{0,x}(y) = D\Phi_x(z).f_0^\Sigma(z)$, where $z = \Phi_x^{-1}(y)$. (Notice that $g_{0,x}$ is continuous but not necessarily of class C^1.) Let Σ_x be the restriction of Σ to U_x and $\tilde{\Sigma}_x$ be the system $(V_x, g_{0,x}, g_{1,x}, \ldots, g_{m,x})$. Then $\tilde{\Sigma}_x$ is a $C^0(m)$ system. Since the vector fields $g_{1,x}, \ldots, g_{m,x}$ are constant, we know that that $f^{\tilde{\Sigma}_x}(u^\alpha) \xrightarrow{\mathrm{T}} f^{\tilde{\Sigma}_x}(u^\infty)$. Since T-convergence is preserved by C^1 diffeomorphisms, we conclude that $f^{\Sigma_x}(u^\alpha) \xrightarrow{\mathrm{T}} f^{\Sigma_x}(u^\infty)$. This says that our desired conclusion holds locally, in the sense that every $x \in \Omega$ has a neighborhood U_x such that the conclusion holds for the restriction of Σ to U_x. From this, we have to infer that the conclusion also holds globally. It suffices to show that, if (a) $\{\xi^\alpha\}_{\alpha \in A}$ is a net of maximal trajectories of the $f^\Sigma(u^\alpha)$, (b) $\{(x^\alpha, t^\alpha)\}_{\alpha \in A}$ is a net of points in $\Omega \times [0, T]$ such that $\xi^\alpha(t^\alpha) = x^\alpha$ for each α, and (c) $(x^\alpha, t^\alpha) \to (x^\infty, t^\infty) \in \Omega \times [0, T]$, then $\{\xi^\alpha\}_{\alpha \in A}$ converges on compact sets to the maximal trajectory ξ^∞ of $f^\Sigma(u^\infty)$ such that $\xi(t^\infty) = x^\infty$ (using the fact that Σ is a C^1 system and therefore has uniqueness of trajectories). For $\alpha \in A \cup \{\infty\}$, let $I^\alpha = \mathrm{Dom}(\xi^\alpha)$, so I^α is relatively open in $[0, T]$ and $t^\alpha \in I^\alpha$. Call a compact subinterval J of I^∞ "good" if there exists an $\alpha^*(J) \in A$ such that $J \subseteq I^\alpha$ whenever $\alpha \succeq_A \alpha^*(J)$, and the net $\{\xi^\alpha\}_{\alpha \in A; \alpha \succeq_A \alpha^*(J)}$ converges to ξ^∞ uniformly on J. We have to prove that every compact subinterval of I^∞ is good. It clearly suffices to consider intervals J of the form $[a, t^\infty]$ or $[t^\infty, b]$. The proof is similar in both cases, so we will only consider the second case, and assume that $J = [t^\infty, b]$ for some $b \in [0, T]$. Pick an integer $N > 0$ such that $\frac{1}{N}$ is a Lebesgue number of the covering $\{(\xi^\infty)^{-1}(U_x) : x \in \Omega\}$ of J. Let $b_k = t^\infty + \frac{k}{N}(b - t^\infty)$ for $k = 0, 1, \ldots, N$, so $b_0 = t^\infty$ and $b_N = b$. Let $J_k = [b_0, b_k]$, so $J = J_N$. Let $L_k = [b_{k-1}, b_k]$ for $k = 1, \ldots, N$. Pick $x_k \in \Omega$ such that $\xi^\infty(L^k) \subseteq U_{x_k}$. Write $\Omega_k = U_{x_k}$, $\Sigma_k = \Sigma_{x_k}$. We prove by induction on k that J_k is good. First, we show that J_1 is good. Fix an $\varepsilon > 0$. For $\alpha \in A \cup \{\infty\}$, let η^α be the maximal trajectory of $f^{\Sigma_1}(u^\alpha)$ that goes through x^α at time t^α. Since $f^{\Sigma_1}(u^\alpha) \xrightarrow{\mathrm{T}} f^{\Sigma_1}(u^\infty)$, Σ_1 has uniqueness of trajectories, and $J_1 \subseteq \mathrm{Dom}(\eta^\infty)$, there exists an α^* such that $J_1 \subseteq \mathrm{Dom}(\eta^\alpha)$ and $\|\eta^\alpha(t) - \eta^\infty(t)\| < \varepsilon$ on J_1 whenever $\alpha \succeq_A \alpha^*$. On the other hand, η^α is a trajectory of $f^\Sigma(u^\alpha)$, so $\eta^\alpha = \xi^\alpha \lceil \mathrm{Dom}(\eta^\alpha)$. Therefore $J_1 \subseteq \mathrm{Dom}(\xi^\alpha)$ and $\|\xi^\alpha(t) - \xi^\infty(t)\| < \varepsilon$ for all $t \in J_1$ whenever $\alpha \succeq_A \alpha^*$. This establishes that J_1 is good. Now suppose that $0 < k \leq N$ and J_{k-1} is good. Pick α_1^* such that $J_{k-1} \subseteq I^\alpha$ whenever $\alpha \succeq_A \alpha_1^*$ and $\{\xi^\alpha\}_{\alpha \in A; \alpha \succeq_A \alpha_1^*}$ converges to ξ^∞ uniformly on J_{k-1}. Then $\{\xi^\alpha(b_{k-1})\}_{\alpha \in A; \alpha \succeq_A \alpha_1^*}$ converges to $\xi^\infty(b_{k-1})$. Thus there exists $\alpha_2^* \in A$ such that $\alpha_2^* \succeq_A \alpha_1^*$ and $\xi^\alpha(b_{k-1}) \in \Omega_k$ for $\alpha \succeq_A \alpha_2^*$. For $\alpha \succeq_A \alpha_2^*$, or $\alpha = \infty$, let η^α be the maximal trajectory of $f^{\Sigma_k}(u^\alpha)$ that goes through $\xi^\alpha(b_{k-1})$ at time b_{k-1}. Since $\xi^\infty \lceil L_k$ is a trajectory of $f^{\Sigma_k}(u^\infty)$, we have $L_k \subseteq \mathrm{Dom}(\eta^\infty)$. Since $f^{\Sigma_k}(u^\alpha) \xrightarrow{\mathrm{T}} f^{\Sigma_k}(u)$, Σ_k has uniqueness of trajectories, and $L_k \subseteq \mathrm{Dom}(\eta^\infty)$, there exists an α^* such that $\alpha^* \succeq_A \alpha_2^*$, $L_k \subseteq \mathrm{Dom}(\eta^\alpha)$ whenever $\alpha \succeq_A \alpha^*$, and $\eta^\alpha \to \eta^\infty$ uniformly on L_k. On the other hand, η^α is a trajectory of $f^\Sigma(u^\alpha)$ and $\eta^\alpha(b_{k-1}) = \xi^\alpha(b_{k-1})$, so $\eta^\alpha = \xi^\alpha \lceil \mathrm{Dom}(\eta^\alpha)$. Therefore $L_k \subseteq \mathrm{Dom}(\xi^\alpha)$ whenever $\alpha \succeq_A \alpha^*$, and $\xi^\alpha \to \xi^\infty$ uniformly on L_k. Since $\alpha^* \succeq_A \alpha_1^*$, we have in fact shown that $J_k \subseteq \mathrm{Dom}(\xi^\alpha)$ whenever $\alpha \succeq_A \alpha^*$, and $\xi^\alpha \to \xi^\infty$ uniformly on J_k. So J_k is good. This completes the inductive proof that all the J_k are good, and in particular $J = J_N$ is good. So $\xi^\alpha \to \xi^\infty$ on compact sets. Therefore $f^\Sigma(u^\alpha) \xrightarrow{\mathrm{T}} f^\Sigma(u^\infty)$.

We now remove the restriction that $f_1^\Sigma, \ldots, f_m^\Sigma$ be linearly independent at each point. Let $\Sigma = (\Omega, f_0, f_1, \ldots, f_m)$ be an arbitrary system in $C^1_{comm}(m)$. Let $\hat{\Omega} = \Omega \times \mathbb{R}^m$, and let $\hat{\Sigma}$ be the system on $\hat{\Omega}$ given by

$$\dot{x} = f_0(x) + u_1 f_1(x) + \cdots + u_m f_m(x)\,, \quad \dot{y}_1 = u_1\,, \ldots, \dot{y}_m = u_m\,. \tag{27}$$

Then $\hat{\Sigma} \in C^1_{comm}(m)$, and it is clear that

$$\left(f^{\hat{\Sigma}}(u^\alpha) \xrightarrow{\mathrm{T}} f^{\hat{\Sigma}}(u^\infty)\right) \Rightarrow \left(f^\Sigma(u^\alpha) \xrightarrow{\mathrm{T}} f^\Sigma(u^\infty)\right). \tag{28}$$

Moreover, the vector fields $f_1^{\hat{\Sigma}}, \ldots, f_m^{\hat{\Sigma}}$ are linearly dependent at each point. So we conclude that $f^{\hat{\Sigma}}(u^\alpha) \xrightarrow{\mathrm{T}} f^{\hat{\Sigma}}(u^\infty)$, and therefore $f^\Sigma(u^\alpha) \xrightarrow{\mathrm{T}} f^\Sigma(u^\infty)$. ∎

6 Beyond I-convergence: The role of Lie brackets

Theorem 4 characterizes the topology of $\mathcal{T}^1_{comm}$-convergence on the input space $L^1([0,T],\mathbb{R}^m)$, by showing that $\mathcal{T}^1_{comm}$-convergence is exactly equivalent to I-convergence. It is natural to ask whether I-convergence actually implies T-convergence for at least some noncommutative systems. The answer to this question turns out to be negative:

Theorem 5 *Let $\mathcal{C}$ be any subclass of $C^1(m)$ that contains at least one noncommutative system. Then there exists a sequence $\{u^j\}_{j=1}^\infty$ in $L^1([0,T],\mathbb{R}^m)$ such that* (i) *$\{u^j\}$ I-converges to a limit $u^\infty \in L^1([0,T],\mathbb{R}^m)$,* (ii) *for every $\Sigma \in \mathcal{C}$ the time-varying vector fields $f^\Sigma(u^j)$ T-converge to a limit h^Σ, but* (iii) *it is not true that $h^\Sigma = f^\Sigma(u^\infty)$ for all $\Sigma \in \mathcal{C}$.*

This result—which will follow from Proposition 9—says that noncommutativity is the fundamental obstruction for I-convergence of the inputs to imply convergence of trajectories. The best way to understand why this is so is to think of the space $L^1([0,T],\mathbb{R}^m)$ of ordinary inputs as embedded in a larger space $\mathcal{GI}(m)$ of *generalized* m-dimensional inputs, whose elements have components not only for the m input channels corresponding to the vector fields $f_1,\ldots,f_m$ but also in "Lie bracket channels" such as $[f_1,f_2]$. An "ordinary input" $u\in L^1([0,T],\mathbb{R}^m)$ gives rise, for each system Σ, to a time-varying vector field $f^\Sigma(u)$. Thus an ordinary input can be thought of as a "functional" that to every system Σ in a given class $\mathcal{C}$ assigns a time-varying vector field. It is then natural to extend the class of ordinary inputs by allowing more general functionals. By analogy with other constructions of spaces of "generalized functions" (e.g., Schwartz distributions), we should also require that such a functional be approximable by ordinary inputs. So we first define, for a class $\mathcal{C}\subseteq C^0(m)$ and an interval $[a,b]$, the space $\mathcal{PGI}(\mathcal{C},a,b)$ of all maps v that assign to every $\Sigma=(\Omega,f_0,\ldots,f_m)\in\mathcal{C}$ a time-varying vector field $v_\Sigma\in TVVF(\Omega,[a,b])$, and call the elements of $\mathcal{PGI}(\mathcal{C},a,b)$ "pseudogeneralized inputs." Then $\mathcal{PGI}(\mathcal{C},a,b)$ is just the product $\Pi_{(\Omega,f_0,\ldots,f_m)\in\mathcal{C}}TVVF(\Omega,[a,b])$, and we can topologize it by means of the product topology. An ordinary input $u\in L^1([a,b],\mathbb{R}^m)$ can then be regarded as a member of $\mathcal{PGI}(\mathcal{C},a,b)$ by associating to it the map —also labeled u— $\Sigma\mapsto f^\Sigma(u)$. A "generalized input" on $[a,b]$ for the class $\mathcal{C}$ is then a member of $\mathcal{PGI}(\mathcal{C},a,b)$ which is a limit of ordinary inputs. We use $\mathcal{GI}(\mathcal{C},a,b)$ to denote the set of all generalized inputs, so by construction $\mathcal{GI}(\mathcal{C},a,b)$ is a topological space in which the space $L^1([a,b],\mathbb{R}^m)$ is densely embedded.

Particular examples of members of $\mathcal{PGI}(\mathcal{C},a,b)$ can be constructed by means of *formal Lie bracket expressions* such as

$$X_0+v_1(t)X_1+v_2(t)X_2+v_3(t)[X_1,X_2]+v_4(t)[X_2,[X_1,X_2]]\,.$$

Here the indeterminates X_i should be thought of as "slots" where, for each particular system Σ with vector fields $f_0,\ldots,f_m$, the f_i are to be plugged in, thus giving rise to a time-varying vector field. Precisely, we can define a *formal pseudogeneralized input on* $[a,b]$ to be an integrable function with values in the free Lie algebra $L(m+1)$ in $m+1$ indeterminates $X_0,\ldots,X_m$. (The field of scalars is of course $\mathbb{R}$.) Use $\mathcal{FPGI}(m,a,b)$ to denote the set of all such objects. Then a $v\in\mathcal{FPGI}(m,a,b)$ gives rise to a true pseudogeneralized input—also labeled v— for any class $\mathcal{C}\subseteq C^0(m)$ of control-affine systems that has enough differentiability for all the Lie brackets occurring in the formal expression to make sense. For example, to an ordinary input $u=(u_1,\ldots,u_m)$ we associate the $L(m+1)$-valued function $t\mapsto X_0+u_1(t)X_1+\ldots+u_m(t)X_m$, and in this case it is possible to plug in $f_0,\ldots,f_m$ for $X_0,\ldots,X_m$ for every system $\mathcal{C}\in C^0(m)$. On the other hand, if

$m \geq 2$, then an expression v such as $X_0 + v_1(t)X_1 + v_2(t)X_2 + v_3(t)[X_1, X_2]$ will give rise to a member of $\mathcal{PGI}(\mathcal{C}, a, b)$ only if $\mathcal{C}$ consists of systems $(\Omega, f_0, \ldots, f_m)$ for which the Lie bracket $[f_1, f_2]$ is well defined. This will be the case, in particular, if $\mathcal{C} \subseteq C^1(m)$.

Thus, for example, the formal pseudogeneralized input $w = X_0 + \frac{1}{2}[X_1, X_2]$ gives rise to a pseudo-generalized input w belonging to $\mathcal{PGI}(C^1(2), a, b)$. With this terminology, the following result exhibits an example of a sequence $\{u^j\} \subseteq L^1([0,T], \mathbb{R}^2)$ that I-converges to zero but converges in $\mathcal{PGI}(C^1(2), 0, T)$ to w. This shows that $w \in \mathcal{GI}(C^1(2), a, b)$, i.e., that w is a true generalized input. It is now clear why, when we look only at classes $\mathcal{C}$ of systems for which $[f_1, f_2] = 0$, as we did in Theorem 4, the sequence $\{u^j\}$ $\mathcal{T}(\mathcal{C})$-converges to 0, whereas, as soon as $\mathcal{C}$ contains one system for which $[f_1, f_2] \neq 0$, the $\mathcal{T}(\mathcal{C})$-limit of $\{u^j\}$ is no longer 0. The reason is that the "true" limit is w, and w contains "Lie bracket components" that are only seen when w is tested against a noncommutative system.

Proposition 9 *Let $T > 0$. For $j = 1, 2, \ldots$, let $u^j \in L^1([0,T], \mathbb{R}^2)$ be the input given by*

$$u^j(t) = (j^{\frac{1}{2}} \cos jt, j^{\frac{1}{2}} \sin jt)\,. \tag{29}$$

Let $\Sigma = (\Omega, f_0, f_1, f_2)$ be a system of class C^1. Let

$$f^j(x,t) = f^{\Sigma}(u^j)(x,t) = f_0(x) + j^{\frac{1}{2}} \cos jt\, f_1(x) + j^{\frac{1}{2}} \sin jt\, f_2(x) \tag{30}$$

and

$$f^{\infty}(x,t) = f_0(x) + \frac{1}{2}[f_1, f_2](x) \tag{31}$$

for $(x,t) \in \Omega \times [0,T]$. Then the f^j T-converge to f^{∞} in $TVVF_{LIB}(\Omega, [0,T])$.

PROOF. Suppose K is a compact subset of the product $\Omega \times [0,T]$. We have to show that $\mathrm{Traj}_c(f^j, K)$ converges to $\mathrm{Traj}_c(f^{\infty}, K)$. Let $\{\xi^j\} \subseteq ARC(\Omega)$ be a sequence of functions such that $\xi^j \in \mathrm{Traj}_c(f^j, K)$ for each j. We will first show that there exists a subsequence of $\{\xi^j\}$ that converges to a $\xi \in \mathrm{Traj}_c(f^{\infty}, K)$ in $ARC(\Omega)$. Let $I^j = [a^j, b^j] = \mathrm{Dom}(\xi^j)$. On I^j we have

$$\dot{\xi}^j(t) = f_0(\xi^j(t)) + j^{\frac{1}{2}} \cos jt\, f_1(\xi^j(t)) + j^{\frac{1}{2}} \sin jt\, f_2(\xi^j(t))\,. \tag{32}$$

But $j^{\frac{1}{2}} \cos jt\, f_1(\xi^j(t))$ is equal to

$$\dot{\theta}_1^j(t) - j^{-\frac{1}{2}} \sin jt\, g_1(\xi^j(t)) - \frac{1}{2} \sin 2jt\, g_2(\xi^j(t)) - \sin^2 jt\, g_3(\xi^j(t))\,,$$

where $g_1 = Df_1 \cdot f_0$, $g_2 = Df_1 \cdot f_1$, $g_3 = Df_1 \cdot f_2$, Df_1 denotes the Jacobian matrix of f_1, and $\theta_1^j = j^{-\frac{1}{2}} \sin jt\, f_1(\xi^j(t))$. Similarly, $j^{\frac{1}{2}} \sin jt\, f_2(\xi^j(t))$ is equal to

$$\dot\theta_2^j(t) + j^{-\frac{1}{2}} \cos jt\, g_4(\xi^j(t)) + \cos^2 jt\, g_5(\xi^j(t)) + \frac{1}{2} \sin 2jt\, g_6(\xi^j(t))\,,$$

where $g_4 = Df_2 \cdot f_0$, $g_5 = Df_2 \cdot f_1$, $g_6 = Df_2 \cdot f_2$, and $\theta_2^j = -j^{-\frac{1}{2}} \cos jt\, f_2(\xi^j(t))$. Thus, if we let

$$\begin{aligned} h^j(x,t) &= f_0(x) + \dot\theta_1^j(t) + \dot\theta_2^j(t) \\ &\quad -j^{-\frac{1}{2}} \sin jt\, g_1(x) - \frac{1}{2} \sin 2jt\, g_2(x) - \sin^2 jt\, g_3(x) \\ &\quad +j^{-\frac{1}{2}} \cos jt\, g_4(x) + \cos^2 jt\, g_5(x) + \frac{1}{2} \sin 2jt\, g_6(x)\,, \end{aligned}$$

then $\xi^j \in \mathrm{Traj}_c(h^j, K)$.

Theorem 2 then implies that the sequence of time-varying vector fields h^j T-converges to the vector field h^∞ given by

$$h^\infty(x,t) = f_0(x) + \frac{1}{2}(g_5(x) - g_3(x)) = f_0(x) + \frac{1}{2}[f_1, f_2](x)\,.$$

(We apply the theorem to the control system $\dot x = f_0(x) + \sum_{i=1}^6 v_i g_i(x)$, and the controls v^j given by

$$v^j(t) = \Big(- j^{-\frac{1}{2}} \sin jt, -\frac{1}{2} \sin 2jt, -\sin^2 jt, j^{-\frac{1}{2}} \cos jt, \cos^2 jt, \frac{1}{2} \sin 2jt \Big)\,.$$

It is clear that the controls v^j $\mathcal{T}^0$-converge to $(0, 0, -\frac{1}{2}, 0, \frac{1}{2})$. Since $\theta^j \to 0$ uniformly, Theorem 2 applies.)

It is clear that $h^\infty = f^\infty$. So $\{\xi^j\}$ has a subsequence that converges to a $\xi^\infty \in \mathrm{Traj}_c(f^\infty, K)$.

The above arguments clearly apply to any subsequence of $\{u^j\}$. Then from Proposition 4 we conclude that the f^j T-converge to f^∞. ∎

PROOF OF THEOREM 5. Let $\Sigma \in \mathcal{C}$ have vector fields $f_0, f_1, \ldots, f_m$ such that $[f_i, f_\ell] \neq 0$ for some $i, \ell \in \{1, \ldots, m\}$. Define inputs u^j by letting $u_k^j(t) = 0$ if $k \notin \{i, \ell\}$, $u_i^j(t) = j^{\frac{1}{2}} \cos jt$, $u_\ell^j(t) = j^{\frac{1}{2}} \sin jt$. By Proposition 9, the time-varying vector fields $f^\Sigma(u^j)$ T-converge to $f_0 + \frac{1}{2}[f_i, f_\ell]$. ∎

To understand the relationship between ordinary and generalized inputs, it is useful to keep in mind the following simple analogy. Integrable functions on, say, the interval $[0, 1]$, are members of the more general class of Borel measures on $[0, 1]$, modulo an obvious identification. A measure μ has a decomposition

$\mu = \mu_{ac} + \mu_{sing}$ into an absolutely continuous part and a singular part. (We include in μ_{sing} all the atoms of μ.) If μ is absolutely continuous, i.e., if $d\mu = f\,dx$, where $f \in L^1$, then of course μ has no singular part. A sequence $\{\mu^j\}$ of absolutely continuous measures will converge to an absolutely continuous measure μ provided that the μ^j weak*-converge to μ and the functions $f^j = \frac{d\mu^j}{dx}$ are "kept under control," e.g., if the sequence $\{f^j\}$ is equi-integrable. If, on the other hand, the μ^j weak*-converge to μ but $\{f^j\}$ is not equi-integrable, then the limit μ can have a singular part even though the μ^j do not. Similarly, the general picture for ordinary differential equations $\dot{x} = f_0(x) + \sum_i u_i(t) f_i(x)$ is that each right-hand side is in fact a member of a larger class of "generalized" objects, which contain Lie bracket terms as well as "ordinary" terms. An "ordinary" right-hand side has no Lie bracket terms, but a sequence $\{f^j\}$ of ordinary right-hand sides can converge to a "generalized" one, that contains Lie bracket terms in addition to the averaged limit, unless $\{f^j\}$ is "kept under control," e.g., by means of the boundedness conditions of our Theorem 4.1.

There is a more general theory that shows in a systematic way how iterated Lie brackets can occur as limits of the right-hand sides of *highly oscillatory* differential equations, such as the ones considered in Proposition 9. The results described in Theorem 1 can be thought of as the zeroth level of this more general theory, in which Lie bracket terms do not occur in the limit. High-level results, in which the structure of general limits involving Lie bracket terms is analyzed in detail, are described in [13], [14], [15]. (Some special cases of these limiting processes have also been studied by Kurzweil and Jarnik in [10] and [11].)

7 Comparison with other conditions

We recall that a sequence $\{u^j\}_{j=1}^{\infty}$ of functions in $L^1([0,T],\mathbb{R}^m)$ *converges weakly* to a limit $u^\infty \in L^1([0,T],\mathbb{R}^m)$ if $\lim_{j\to\infty} \int_0^T \varphi(t)u^j(t)dt = \int_0^T \varphi(t)u^\infty(t)dt$ for all functions $\varphi \in L^\infty([0,T],\mathbb{R})$. It is well known (cf. Theorem 3.1 on page 53 of [5]) that $\{u^j\}_{j=1}^{\infty}$ converges weakly to u^∞ if and only if (i) the integrals $U^j(t) = \int_0^t u^j(s)ds$ converge to $U^\infty(t) = \int_0^t u^\infty(s)ds$ for each t, and (ii) the sequence $\{u^j\}$ is *equiintegrable* in the sense that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\int_E \|u^j(t)\|dt \le \varepsilon$ for every j and every measurable subset E of $[0,T]$ whose measure is $\le \delta$.

It is clear that equi-integrability implies that the sequence $\{U^j\}$ is equicontinuous. Thus (i) and (ii) actually imply (IC). Moreover, the equi-integrability condition also implies (UB). Thus one particular consequence of our main result is that *weak convergence implies $\mathcal{T}^0$-convergence.* On the other hand, the following example shows that a sequence can satisfy the conditions of Theorem 1 without being weakly convergent, so our necessary and sufficient condition for sequences is strictly weaker than weak convergence.

Example 7.1. Let $m = 1$, $u^j(t) = j^{\frac{1}{2}} \cos jt$ for $0 \le t \le j^{-\frac{1}{2}}$ and $u^j(t) = 0$ for $j^{-\frac{1}{2}} < t \le T$. Then $\{u^j\}_{j=1}^{\infty}$ satisfies (IC) and (UB), with $u^{\infty} \equiv 0$. It follows from Theorem 1 that $\{u^j\}_{j=1}^{\infty}$ $\mathcal{T}^0$-converges to u^{∞}.

On the other hand, it is not hard to see that *the sequence* $\{u^j\}_{j=1}^{\infty}$ *is not weakly convergent in* $L^1[0,T]$. Indeed, the integrals $\int_0^{j^{-1/2}} |u^j(t)|dt$ converge to $\frac{2}{\pi}$ as $j \to \infty$, and then $\{u^j\}$ is not equi-integrable. ∎

The preceding example also makes it possible to show that our convergence condition is weaker than those of Buttazzo–Conti, Kurzweil–Vorel, and Neustadt, cf. [4], [9], [19]. Indeed, the Kurzweil–Vorel conditions amount to saying that

$$\int_0^t u^j(s)ds \to \int_0^t u^{\infty}(s)ds \quad \text{and} \quad \sup_j |u^j| \in L^1[0,T]\,.$$

(Our example violates the latter condition because if we let $v(t) = \sup_j |u^j(t)|$, then $v(t) \ge j^{1/2}|\cos jt|$ on the interval $[0, j^{-1/2}]$. So

$$\int_0^{j^{-\frac{1}{2}}} v(t)dt \ge j^{1/2} \int_0^{j^{-1/2}} |\cos jt|dt$$

Since the latter integral goes to $\frac{2}{\pi}$ as $j \to \infty$, we see that $\int_0^{j^{-1/2}} v(t)dt$ is bounded away from zero as $j \to \infty$. Therefore v is not integrable.) The Neustadt theorem requires that $\int_0^t u^j(s)ds \to \int_0^t u^{\infty}(s)ds$ and $\sup_j \int_t^{\tau} |u^j(s)|ds \overset{\tau \to t+}{\longrightarrow} 0$ uniformly in t. Our example violates this if $t = 0$. The Buttazzo–Conti theorem, when specialized to control-affine systems (1), requires that the vector fields $f_0, \ldots, f_m$ be Lipschitz, although they assume only that the u^j and u^{∞} satisfy (IC) and (UB).

Appendix A

We give three examples. The first two pertain, respectively, to the assertions made in Theorem 1 that in general (v) $\not\Longrightarrow$(iv) and (ii)$\not\Longrightarrow$ (i). The third one illustrates the difference, for general vector fields, between "continuous dependence of the trajectories for every fixed initial condition," and "joint continuous dependence of the trajectories on the right-hand side and the initial condition."

As shown in Theorem 1, uniform weak convergence is necessary for a net $\{u^{\alpha}\}_{\alpha \in A}$ to $\mathcal{T}^0$-converge to u^{∞} in $L^1([0,T], \mathbb{R}^m)$ and is also sufficient if $\{u^{\alpha}\}_{\alpha \in A}$

is a sequence. The following example shows that **(UWC)** is no longer sufficient in general for a net $\{u^\alpha\}$ to $\mathcal{T}^0$-converge to u^∞, and therefore proves that condition (v) of Theorem 1 does not in general imply (iv).

Example A.1. Let $T = 2\pi$. For every integer $n \geq 1$, every choice $\varphi_1, \ldots, \varphi_n$ of n real-valued continuous functions on $[0, T]$ and every $\varepsilon > 0$, set

$$V(\varphi_1, \ldots, \varphi_n, \varepsilon) = \left\{ \psi \in L^1([0,T], \mathbb{R}) \colon \sup_{0 \leq t \leq T} \left| \int_0^t \varphi_i(s)\psi(s)ds \right| < \varepsilon, i = 1, \ldots, n \right\}.$$

Let $\mathcal{V}$ be the set of all sets $V(\varphi_1, \ldots, \varphi_n, \varepsilon)$, for all possible integers $n \geq 1$, all n-tuples $(\varphi_1, \ldots, \varphi_n)$ of continuous functions on $[0, T]$, and all positive numbers ε. We define a directed set $(I, \preceq_I)$ by letting

$$I = \{(V, N) \ : \ V \in \mathcal{V}, N \text{ is an integer} > 0\},$$

with the partial ordering $\preceq_I$ such that

$$(V_1, N_1) \preceq_I (V_2, N_2) \Longleftrightarrow (V_2 \subseteq V_1 \text{ and } N_2 \geq N_1).$$

Let $m = 2$. We define a net $\{u^\alpha\}_{\alpha \in I}$ in $L^1([0,T], \mathbb{R}^2)$ as follows. For each index $\alpha = (V, N) \in I$, we let $u_1^\alpha(t) = N\cos(j_\alpha t)$, $u_2^\alpha(t) = N\sin(j_\alpha t)$, where j_α is the smallest integer $j > 0$ such that the functions $N\cos(jt), N\sin(jt)$ belong to V. (It follows from the Riemann-Lebesgue Lemma that, for each fixed V and N, the functions $N\cos(jt)$ and $N\sin(jt)$ belong to V when j is large enough. Thus j_α exists.) Now it is easy to see that the net $\{u^\alpha\}_{\alpha \in I}$ thus defined satisfies (UWC) with $u^\infty = (0,0)$. (To see this, let $\varepsilon > 0$, and let φ be a continuous function on $[0,T]$. Let $\alpha_0 = (V(\varphi, \varepsilon), 1)$. Then, if $\alpha = (V, N) \succeq_I \alpha_0$, it follows that $V \subset V_0(\varphi, \varepsilon)$. By definition, $(u_1^\alpha, u_2^\alpha) \in V \subset V_0(\varphi, \varepsilon)$, which implies $\sup_{0 \leq t \leq T} |\int_0^t \varphi(s) u_k^\alpha(s)ds| < \varepsilon, k = 1, 2$.)

We will now show that the above net does not $\mathcal{T}^0$-converge to the limit u^∞=$(0,0)$ in $L^1([0,T], \mathbb{R}^2)$. Let Σ=$(\mathbb{R}^2, f_0, f_1, f_2)$ with $f_0 = (0,0)$, $f_1 = (1,0)$, $f_3 = (0, x^{\frac{1}{5}})$. It is clear that this system is in $C^0(2)$. Consider, for each input $u \in L^1([0,T], \mathbb{R}^2)$, the system of differential equations $\dot{x}_1 = u_1$, $\dot{x}_2 = x_1^{1/5} u_2$. This system has the global existence and uniqueness property for every initial condition and every $u \in L^1([0,T], \mathbb{R}^2)$. Fix the initial condition $x(0) = (0,0)$. We prove that $\{u^\alpha\}$ does not $\mathcal{T}^0$-converge to u^∞ by showing that the trajectories x^α of the above differential equations that correspond to the u^α do not converge to $x^\infty \equiv (0,0)^T$. First, we prove a lemma.

Lemma A.1. Fix a $V = (\varphi_1, \ldots, \varphi_n, \varepsilon) \in \mathcal{V}$. *Then there exists an* $N_V > 0$ *such that if* $N > N_V$, *and* $\alpha = (V, N)$, *then* $j_\alpha \leq N^5$.

PROOF: Let $A = \max_{k=1,\dots,n}\{\|\varphi_k\|\}$, where $\|\cdot\|$ denotes the sup norm on $[0,T]$. Let $N_V = \frac{4A\pi}{\varepsilon}(\frac{nAT}{\varepsilon}+n)^{\frac{1}{2}}$. We show that $j_\alpha \le N^5$ if $\alpha = (V,N)$, $N > N_V$. Suppose there is an $N > N_V$ such that $j_\alpha > N^5$. Then by the definition of the j_α, for every $1 \le j \le N^5$, we have

$$\sup_{0\le t\le T, k=1,\dots,n} \max\left(\left|\int_0^t \cos(js)\varphi_k(s)ds\right|, \left|\int_0^t \sin(js)\varphi_k(s)ds\right|\right) \ge \frac{\varepsilon}{N}. \tag{33}$$

Let $K = [\frac{2ANT}{\varepsilon}]+1$, where $[\cdot]$ denotes the integer part function (i.e. $[r]$ is the largest integer ν such that $\nu \le r$). Divide the interval $[0,T]$ into K equal parts. Let $t_\ell = \frac{T\ell}{K}, \ell = 0,\dots,K$, be the partition points, and define $\varphi_{k,\ell}(t) = \varphi_k(t)\chi_{[0,t_\ell]}$, where $\chi_{[0,t_\ell]}$ is the indicator function of $[0,t_\ell]$. For any two functions φ,ψ in $L^2[0,T]$, let us use $\langle\varphi,\psi\rangle = \int_0^T \varphi(s)\psi(s)ds$ to denote the inner product of φ,ψ. Then $\{\frac{\cos jt}{\sqrt{\pi}} : j = 1,2,\dots\}\cup\{\frac{\sin jt}{\sqrt{\pi}} : j = 1,2,\dots\}$ is an orthonormal set in $L^2[0,T]$. From Bessel's inequality we know that

$$\begin{aligned}&\sum_{\ell=1}^{K}\sum_{k=1}^{n}\sum_{j=1}^{N^5}\left\{\left|\left\langle\frac{\cos(jt)}{\sqrt{\pi}},\varphi_{k,\ell}\right\rangle\right|^2+\left|\left\langle\frac{\sin(jt)}{\sqrt{\pi}},\varphi_{k,\ell}\right\rangle\right|^2\right\}\\ \le\ &\sum_{\ell=1}^{K}\sum_{k=1}^{n}\|\varphi_{k,\ell}\|^2_{L^2[0,T]} \le \sum_{\ell=1}^{K}\sum_{k=1}^{n}\|\varphi_k\|^2_{L^2[0,T]} \le 2\pi nKA^2\,.\end{aligned} \tag{34}$$

On the other hand it follows from (33) that for each $j\in\{1,\dots,N^5\}$ there exist a number $k(j)\in\{1,\dots,n\}$ and a $t(j)\in[0,T]$ such that at least one of the numbers

$$\left|\int_0^{t(j)}\cos(js)\varphi_{k(j)}(s)ds\right|,\ \left|\int_0^{t(j)}\sin(js)\varphi_{k(j)}(s)ds\right|$$

is $\ge \frac{\varepsilon}{N}$. Then $t(j)\in[t_{\ell-1},t_\ell]$ for some $\ell = \ell(j)$. It is clear that the integrals of $\cos(js)\varphi_{k(j)}(s)ds$ and $\sin(js)\varphi_{k(j)}(s)ds$ from $t(j)$ to $t_{\ell(j)}$ are bounded by $\frac{AT}{K}$. Since $K > \frac{2ANT}{\varepsilon}$, we have $\frac{AT}{K} < \frac{\varepsilon}{2N}$. Thus at least one of the numbers

$$\left|\int_0^{t_{\ell(j)}}\cos(js)\varphi_{k(j)}(s)ds\right|,\quad \left|\int_0^{t_{\ell(j)}}\sin(js)\varphi_{k(j)}(s)ds\right|,$$

is $\ge \frac{\varepsilon}{2N}$. Therefore

$$\left|\left\langle\frac{\cos(jt)}{\sqrt{\pi}},\varphi_{k(j),\ell(j)}\right\rangle\right|^2+\left|\left\langle\frac{\sin(jt)}{\sqrt{\pi}},\varphi_{k(j),\ell(j)}\right\rangle\right|^2 \ge \frac{\varepsilon^2}{4\pi N^2}\ .$$

Thus we have

$$\sum_{\ell=1}^{K}\sum_{k=1}^{n}\sum_{j=1}^{N^5}\left\{\left|\left\langle\frac{\cos(jt)}{\sqrt{\pi}},\varphi_{k,\ell}\right\rangle\right|^2+\left|\left\langle\frac{\sin(jt)}{\sqrt{\pi}},\varphi_{k,\ell}\right\rangle\right|^2\right\}\geq N^5\frac{\varepsilon^2}{4\pi N^2}=\frac{\varepsilon^2N^3}{4\pi}\,. \quad (35)$$

If we compare this with (34), we see that

$$N^3\leq\frac{8\pi^2nKA^2}{\varepsilon^2}\ .$$

Since $K<\frac{2ANT}{\varepsilon}+2$, and $N\geq 1$, we easily get the inequality

$$N^2\leq\frac{16\pi^2A^2}{\varepsilon^2}\left(\frac{nAT}{\varepsilon}+n\right),$$

so that $N\leq N_V$. This contradiction completes our proof. ▮

Using the lemma, it is easy to see that the x^α do not converge to x^∞. Indeed, for any $\alpha=(V,N)\in I$, the solution x^α is given by

$$x_1^\alpha(t)=\frac{N}{j_\alpha}\sin(j_\alpha t)\,,\qquad x_2^\alpha(t)=\frac{N^{\frac{6}{5}}}{j_\alpha^{\frac{1}{5}}}\int_0^t(\sin j_\alpha s)^{\frac{6}{5}}ds\,. \quad (36)$$

Take $\varepsilon_0=1$. If the net $\{x^\alpha\}$ converged to x^∞, then there would exist an $\alpha_0=(V,N_0)\in I$ such that $\|x^\alpha(t)\|\leq 1$ for all $\alpha\succeq_I\alpha_0$ and all $t\in[0,T]$. Let $\alpha=(V,N)$ with $N>\max\{N_0,N_V\}$. Then $\alpha\succeq_I\alpha_0$ and

$$x_2^\alpha(T)=\frac{N^{\frac{6}{5}}}{j_\alpha^{\frac{1}{5}}}\int_0^T\sin^{\frac{6}{5}}j_\alpha s\,ds\ .$$

Clearly, $\int_0^T\sin^{\frac{6}{5}}j_\alpha s\,ds$ is bounded away from 0. Since $j_\alpha\leq N^5$, we conclude that $x_2^\alpha(T)\to\infty$ as $N\to\infty$. This contradiction proves that $\{x^\alpha\}$ does not converge. ▮

Our second example shows that a net $\{u^\alpha\}_{\alpha\in A}$ in $L^1([0,T],\mathbb{R}^m)$ may $\mathcal{T}^0$-converge to a $u\in L^1([0,T],\mathbb{R}^m)$ while failing to satisfy (UB). This will prove the assertion that in general (ii)$\not\Longrightarrow$(i) in Theorem 1.

Example A.2. Take $m=1,T=1$. Let $\mathcal{F}_1$ be the collection of all pairs (Σ,K), where Σ is an element of $C^0(1)$ and K is a compact subset of $\Omega^\Sigma\times[0,1]$. Let $\mathcal{F}_2$ be the set of all finite subsets of $\mathcal{F}_1$. Let $\mathcal{F}=\mathcal{F}_2\times\mathbb{N}$, where $\mathbb{N}$ is the set of positive integers. Define a partial ordering $\preceq_{\mathcal{F}}$ on $\mathcal{F}$ by letting $(F_1,n_1)\preceq_{\mathcal{F}}(F_2,n_2)$ if $F_1\subseteq F_2$ and $n_1\leq n_2$. Then $(\mathcal{F},\preceq_{\mathcal{F}})$ is a directed set.

We define a net $\{u^{(F,n)}\}_{(F,n)\in\mathcal{F}}$ in $L^1([0,1],\mathbb{R})$ as follows. For each $n,k\in\mathbb{N}$, let $u_n^k(t)=n\sin kt$. Then from Theorem 1 we know that, for each fixed n, the sequence $\{u_n^k\}_{k=1}^{\infty}$ $\mathcal{T}^0$-converges to $u=0$ in $L^1([0,1],\mathbb{R})$. Therefore, for each $(\Sigma,K)\in\mathcal{F}_1$ the set $\mathrm{Traj}_c(f^\Sigma(u_n^k),K)$ converges, as $k\to\infty$, to $\mathrm{Traj}_c(f^\Sigma(u),K)$ in $2^{ARC(\Omega^\Sigma)}$ under the topology $\mathcal{T}_{USC}(ARC(\Omega^\Sigma))$. Let $k_{\Sigma,K,n}$ be an integer such that $\mathrm{Traj}_c(f^\Sigma(u_n^k),K)\subseteq U_{\frac{1}{n}}(f^\Sigma(u),K)$ if $k\geq k_{\Sigma,K,n}$, where $U_\varepsilon(f,K)$ is the ε-neighborhood of $\mathrm{Traj}_c(f,K)$, i.e.,

$$U_\varepsilon(f,K)=\{\xi\in ARC(\Omega):d_{ARC}(\xi,\zeta)<\varepsilon \text{ for some } \zeta\in\mathrm{Traj}_c(f,K)\}\,.$$

For any $F\in\mathcal{F}$, we let $k_{F,n}=\max\{k_{\Sigma,K,n}\ :\ (\Sigma,K)\in F\}$, and then define $u^{(F,n)}(t)=u_n^{k_{F,n}}(t)$. Clearly $\{u^{(F,n)}\}_{(F,n)\in\mathcal{F}}$ does not satisfy (UB). However, it is easy to show that $\{u^{(F,n)}\}_{(F,n)\in\mathcal{F}}$ $\mathcal{T}^0$-converges to u in $L^1([0,1],\mathbb{R})$. To see this, let $\Sigma\in C^0(1)$ be a system and K be a compact subset of $\Omega^\Sigma\times[0,1]$. We need to show that $\mathrm{Traj}_c(f^\Sigma(u^{(F,n)}),K)$ converges to $\mathrm{Traj}_c(f^\Sigma(u),K)$. For any given $\varepsilon>0$, let $n(\varepsilon)$ be an integer such that $\frac{1}{n(\varepsilon)}<\varepsilon$. Let $F_0=\{(\Sigma,K)\}$. Then if $(F_0,n(\varepsilon))\preceq_{\mathcal{F}}(F,n)$, by definition, $n\geq n(\varepsilon)$ and $(\Sigma,K)\in F$. Therefore,

$$\mathrm{Traj}_c(f^\Sigma(u^{(F,n)}),K)\subseteq U_{\frac{1}{n}}(f^\Sigma(u),K)\subseteq U_\varepsilon(f^\Sigma(u),K)\,. \qquad \blacksquare$$

Our third example shows that, for general vector fields, the fact that for every fixed initial condition $x(\bar t)=\bar x$ the trajectory of f^j converges to the trajectory of f does not imply that f^j T-converges to f.

Example A.3. Let $\varphi:\mathbb{R}\to\mathbb{R}$ be a smooth function such that $\varphi(s)=0$ if $s\leq 0$ or $s\geq 2$, and $\varphi(s)>0$ if $0<s<2$. Define $f^j:\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}^2$ by letting $f^j(x,y,t)=(0,j\varphi(jx))$, and let $f(x,y,t)\equiv 0$. Then f and the f^j are time-varying vector fields on $\mathbb{R}^2$. Given any initial condition $(\bar x,\bar y,\bar t)$, the solution $\xi_{j,\bar x,\bar y,\bar t}:\mathbb{R}\to\mathbb{R}^2$ of the initial value problem $\dot\xi(t)=f^j(\xi(t),t)$, $\xi(\bar t)=(\bar x,\bar y)$, is given by $\xi_{j,\bar x,\bar y,\bar t}(t)=(\bar x,\bar y+j(t-\bar t)\varphi(j\bar x))$. For each $\bar x$, the number $\varphi(j\bar x)$ is equal to zero for sufficiently large j. So for each initial condition $(\bar x,\bar y,\bar t)$ the curve $t\to\xi_{j,\bar x,\bar y,\bar t}(t)$ converges uniformly, as $j\to\infty$, to the constant curve $\xi_{\infty,\bar x,\bar y,\bar t}\equiv(\bar x,\bar y)$, and $\xi_{\infty,\bar x,\bar y,\bar t}$ is precisely the solution of the limiting initial value problem $\dot\xi=f(\xi,t)$, $\xi(\bar t)=(\bar x,\bar y)$. On the other hand, if we consider a j-dependent initial condition $(\bar x^j,\bar y^j,\bar t^j)$ given by $\bar x^j=\frac{1}{j}$, $\bar y^j=0$, $\bar t^j=0$, and let $t^j=\frac{1}{j}$, then $\xi_{j,\bar x^j,\bar y^j,\bar t^j}(t^j)=(\frac{1}{j},\varphi(1))$, which does not converge to $\xi_{\infty,0,0,0}(0)$, even though $(\bar x^j,\bar y^j,\bar t^j)\to(0,0,0)$ and $t^j\to 0$. Thus the curves $\xi_{j,\bar x^j,\bar y^j,\bar t^j}$ do not converge uniformly to $\xi_{\infty,0,0,0}(0)$ on the compact interval $[0,1]$. (Alternatively, we could have taken $\bar t^j=-\frac{1}{j}$, and then $\xi_{j,\bar x^j,\bar y^j,\bar t^j}(0)=(\frac{1}{j},\varphi(1))$, so $\xi_{j,\bar x^j,\bar y^j,\bar t^j}$ does not even converge pointwise to $\xi_{\infty,0,0,0}$.) $\blacksquare$

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