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Continuous dependence of trajectories with respect to the input

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1. Introduction

The purpose of this paper is to outline the results of our journal article [2], where we characterize the weakest possible concept of convergence of a sequence of input functions such that the trajectories depend continuously on the input for *all* control-affine systems

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i f_i(x)\,, \;\; x \in \Omega\,, \tag{1}$$

with a right-hand side continuous with respect to the state variable x, and prove some results on the corresponding problem for more restricted classes of systems. These questions are in fact special cases of the following more general situation: we are given

(a) a set $\mathcal{U}$ of input functions defined on an interval I,
(b) a class $\mathcal{C}$ of control systems $\dot{x} = g(x, u)$, all of which admit the members of $\mathcal{U}$ as inputs.

We then seek to characterize the *weakest* topology $\mathcal{T}$ on $\mathcal{U}$ having the property that, if $\mathcal{U}$ is equipped with $\mathcal{T}$, then

(CD) the trajectories of Σ depend continuously on the input $u \in \mathcal{U}$ for all systems Σ in $\mathcal{C}$.

Let us write $\mathcal{T}(\mathcal{U}, \mathcal{C})$ to denote this topology, and assume temporarily that the definition of $\mathcal{T}(\mathcal{U}, \mathcal{C})$ has been justified, by

(J1) assigning a precise meaning to (CD)
(J2) proving the existence of a weakest topology on $\mathcal{U}$ for which (CD) holds.

Then the problems to be studied here are those of characterizing the topology $\mathcal{T}(\mathcal{U}, \mathcal{C})$ in two cases, namely, (i) when $\mathcal{C}$ is the class $C^0(m)$ of all systems of the form (1), with $f_0, \ldots, f_m$ continuous vector fields on some open subset Ω of some Euclidean space $\mathbb{R}^n$, and (ii) when $\mathcal{C}$ is an interesting subclass of $C^0(m)$, such as the set $C^1(m)$ of all systems (1) with C^1 vector fields. The input space $\mathcal{U}$ will be $L^1([0,T], \mathbb{R}^m)$, i. e., the set of all $\mathbb{R}^m$-valued Lebesgue-integrable functions $t \mapsto u(t) = (u_1(t), \ldots, u_m(t))$ on some fixed interval $[0,T]$. Convergence with respect to $\mathcal{T}(L^1([0,T], \mathbb{R}^m), C^0(m))$ will be called "$\mathcal{T}^0$-convergence." For case (i), we will provide a complete and rather simple characterization of sequential $\mathcal{T}^0$-convergence by showing that a sequence $\{u^j\}$ $\mathcal{T}^0$-converges to a limit u^∞ iff the indefinite integrals $t \mapsto \int_0^t u^j(s)ds$ converge uniformly on $[0,T]$ to $t \mapsto \int_0^t u^\infty(s)ds$ and in addition $\sup_j \|u^j\|_{L^1} < \infty$. (Naturally, we really should seek to characterize $\mathcal{T}^0$-convergence of *nets*, rather than just sequences. This, however, appears to be a much more difficult question, as explained below, on which we will only be able to present some partial results.) For case (ii), we provide an even simpler characterization of $\mathcal{T}(L^1([0,T], \mathbb{R}^1), C^1(1))$-convergence of general nets $\{u^\alpha\}_{\alpha \in A}$ to a limit u^∞, by showing that it is equivalent to uniform convergence on $[0,T]$ of the indefinite integrals $t \mapsto \int_0^t u^\alpha(s)ds$ to the limit $t \mapsto \int_0^t u^\infty(s)ds$. This result is, however, valid only for the single-input case. To explain why, we analyze the m-input situation and state two results (Theorems 4 and 5), according to which (a) a characterization similar to that for $m = 1$ is true for general m on the class $C^1_{comm}(m)$ of systems (1) with C^1 vector fields such that all the Lie brackets $[f_j, f_k]$ for $j, k \in \{1, \ldots, m\}$ vanish, and (b) the result becomes false for any class $\mathcal{C}$ that contains at least one system in $C^1(m) \backslash C^1_{comm}(m)$. This shows that Lie brackets are intimately related to input convergence, and ought to play a decisive role in any effort to achieve a better understanding of $\mathcal{T}(L^1([0,T], \mathbb{R}^m), C^1(m))$-convergence for $m > 1$.

So far, we have assumed that steps (J1) and (J2) had been carried out. We explain how this is done in Section 2, where we define, on the set $TVVF(\Omega, I)$ of all time-varying vector fields $f : \Omega \times I \mapsto \mathbb{R}^n$ (where Ω is an open subset of $\mathbb{R}^n$ and I is a subinterval of $\mathbb{R}$), a topology $\mathcal{T}_T(\Omega, I)$—called the *topology of trajectory convergence*, or *T-convergence*—and argue that this topology captures the concept of "continuous dependence with respect to f of the solutions of the ordinary differential equation $\dot{x} = f(x, t)$." This topology takes into account the possibility of explosions and nonuniqueness of solutions—a fact of paramount importance for us, since, typically, the time-varying vector fields that arise from the systems (1) are only continuous with respect to x—and we will point out in Section 2 that it has all the right properties, at least when restricted to the set $TVVF_{Car,LIB}(\Omega, I)$ of those f's that satisfy the Carathéodory condition (i.e., are such that f is measurable in t for each x and continuous in x for each t) and are "locally integrably bounded," i.e., that satisfy, for every compact subset K of Ω, a bound

$$||f(x,t)|| \le \varphi_K(t) \;\; \text{for} \;\; (x,t) \in K \times I\,,$$

with a locally integrable function $\varphi_K : I \mapsto \mathbb{R}$. Moreover, the topology induced by $\mathcal{T}_T(\Omega, I)$ on the subset $TVVF_{Car,LIB}(\Omega, I)$ of $TVVF(\Omega, I)$ is the weakest topology on $TVVF_{Car,LIB}(\Omega, I)$ that gives rise to joint continuous dependence of the solutions on f and the initial condition.

Having defined $\mathcal{T}_T(\Omega, I)$, it is clear how to interpret (CD) and how to define $\mathcal{T}(\mathcal{U}, \mathcal{C})$: if $\Sigma \in \mathcal{C}$ is of the form $\dot{x} = g^\Sigma(x, u)$, $x \in \Omega^\Sigma$, then each input $t \mapsto u(t)$ belonging to $\mathcal{U}$ gives rise to a time-varying vector field

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$f^\Sigma(u) \in TVVF(\Omega^\Sigma, I)$, defined by

$$f^\Sigma(u) \stackrel{\text{def}}{=} \text{the map } (x,t) \mapsto g^\Sigma(x, u(t))\,. \qquad (2)$$

Then f^Σ is the *input-to-vector-field map* associated with Σ. If $\mathcal{T}$ is a topology on $\mathcal{U}$, then the statement that "the trajectories of Σ depend continuously on the input with respect to $\mathcal{T}$" can be translated as "f^Σ is continuous as a map from $\mathcal{U}$, equipped with $\mathcal{T}$, to $TVVF(\Omega^\Sigma, I)$, equipped with $\mathcal{T}_T(\Omega^\Sigma, I)$." Then the formal definition of $\mathcal{T}(\mathcal{U}, \mathcal{C})$ is simply the following:

> *$\mathcal{T}(\mathcal{U}, \mathcal{C})$ is the weakest topology on the input space $\mathcal{U}$ that renders all the input-to-vector-field maps $f^\Sigma : \mathcal{U} \mapsto TVVF(\Omega^\Sigma, I)$ continuous, for all $\Sigma \in \mathcal{C}$, when each $TVVF(\Omega^\Sigma, I)$ is given the topology $\mathcal{T}_T(\Omega^\Sigma, I)$ of trajectory convergence.*

To analyze in more detail the special case when $\mathcal{U}$ is $L^1([0,T], \mathbb{R}^m)$ and $\mathcal{C}$ is the class $C^0(m)$, let us agree to use f_i^Σ to denote the vector fields f_i corresponding to a given system $\Sigma \in C^0(m)$ of the form (1), and let us go on using Ω^Σ to denote the state space of Σ. Write $\mathcal{U}_T^m \stackrel{\text{def}}{=} L^1([0,T], \mathbb{R}^m)$. As before, use "$\mathcal{T}^0$-convergence" for "convergence with respect to $\mathcal{T}(\mathcal{U}_T^m, C^0(m))$." Then our problem is to determine necessary and sufficient conditions for a sequence $\{u^j\}_{j=1}^\infty$ of functions belonging to the input space $\mathcal{U}_T^m$ to $\mathcal{T}^0$-converge to an input $u^\infty \in \mathcal{U}_T^m$, i.e., for the following property to hold:

> (TC) For every possible system $\Sigma \in C^0(m)$, the time-varying vector fields $f^\Sigma(u^j)$ T-converge to $f^\Sigma(u^\infty)$.

As will be explained in Section 3, one may substitute for (TC) either one of the following conditions, both of which turn out in fact to be equivalent to (TC):

> (TC') For every system $\Sigma \in C^0(m)$, if (i) ξ^j are maximal trajectories of (1) corresponding to the u^j and satisfying the initial conditions $\xi^j(0) = \bar{x}^j \in \Omega^\Sigma$, (ii) $\bar{x}^j \to \bar{x} \in \Omega^\Sigma$, and (iii) for the limiting initial value problem
>
> $$\left\{ \begin{array}{lcl} \dot{x} & = & f^\Sigma(u^\infty)(x,t) \\ & \stackrel{\text{def}}{=} & f_0^\Sigma(x) + \sum_{i=1}^m u_i^\infty(t) f_i^\Sigma(x) \\ x(0) & = & \bar{x} \end{array} \right. \qquad (3)$$
>
> there is uniqueness of solutions as well as global existence on $[0,T]$, then the ξ^j are defined on $[0,T]$ for j large enough and converge uniformly on $[0,T]$ to the unique maximal solution ξ^∞ of (3).

> (TC") For every choice of the system $\Sigma \in C^0(m)$, if (i) $\bar{x} \in \Omega^\Sigma$, (ii) ξ^j are maximal trajectories of (1) corresponding to the u^j and satisfying $\xi^j(0) = \bar{x}$, and (iii) the initial value problem (3) has uniqueness of solutions as well as global existence on $[0,T]$, then the ξ^j are defined on $[0,T]$ for j large enough and converge uniformly on $[0,T]$ to the unique maximal solution ξ^∞ of (3).

Notice that (TC') asserts "joint continuous dependence on the input and the initial condition," whereas (TC") makes the weaker assertion of "continuous dependence on the input for each fixed initial condition," so the implication (TC')$\Rightarrow$(TC") is trivial, but the converse (TC")$\Rightarrow$(TC') is not at all obvious. (It can be shown that for a general sequence $\{f^j\}$ of time-varying vector fields it need not be true that convergence for each fixed initial condition of the trajectories of f^j to those of an $f \in TVVF(\Omega, I)$ implies convergence of trajectories for convergent initial conditions.)

Sequential $\mathcal{T}^0$-convergence will be characterized in terms of a type of convergence weaker than other weak convergence concepts considered in the literature. Precisely, let us say that a net $\{u^\alpha\}_{\alpha \in A} \subseteq \mathcal{U}_T^m$ based on a directed set $(A, \preceq_A)$ *converges in the integral sense* (or, for short, *I-converges*) to a $u^\infty \in \mathcal{U}_T^m$, if

> (IC) The indefinite integrals $U^\alpha(t) = \int_0^t u^\alpha(s)ds$ converge uniformly with respect to $t \in [0,T]$ to $U^\infty(t) = \int_0^t u^\infty(s)ds$,

and that it has the *uniform boundedness* property if

> (UB) There exist a constant $C > 0$ and an $\alpha_0 \in A$ such that $\|u^\alpha\|_{L^1} \le C$ whenever $\alpha_0 \preceq_A \alpha$.

Our main result (Theorem 1) then says that a sequence $\{u^j\}_{j=1}^\infty$ in $\mathcal{U}_T^m$ $\mathcal{T}^0$-converges to a function $u^\infty \in \mathcal{U}_T^m$ if and only if $\{u^j\}_{j=1}^\infty$ satisfies (UB) and I-converges to u^∞. An alternative formulation is in terms of *uniform weak convergence.* We say that a net $\{u^\alpha\}_{\alpha \in A}$ *UW-converges* to a limit u^∞ if the following holds.

> (UWC) For every continuous function $\varphi : [0,T] \mapsto \mathbb{R}$, if we define $\mathbb{R}^m$-valued functions U_φ^α on $[0,T]$ by $U_\varphi^\alpha(t) = \int_0^t u^\alpha(s)\varphi(s)ds$ for $\alpha \in A \cup \{\infty\}$, then the U_φ^α converge to U_φ^∞ uniformly on $[0,T]$.

For sequences, it is easy to see that the conjunction (IC)$\wedge$(UB) is equivalent to (UWC), so Theorem 1 says that a sequence $\{u^j\}_{j=1}^\infty$ $\mathcal{T}^0$-converges to a limit u^∞ if and only if it UW-converges to u^∞.

For more general nets, the condition (IC)$\wedge$(UB) still implies $\mathcal{T}^0$-convergence, which in turn implies UW-convergence, but the reverse implications no longer hold. Thus, although Theorem 1 provides a complete characterization of sequential $\mathcal{T}^0$-convergence in $\mathcal{U}_T^m$, it does not give a full characterization of the topology itself. To get such a characterization, one would have to understand $\mathcal{T}^0$-convergence of nets, and we have not yet been able to find a satisfactory characterization of $\mathcal{T}^0$-convergence of nets, comparable in simplicity to our results for sequences.

For sequences, the condition (IC)$\wedge$(UB) or, equivalently, (UWC), is strictly weaker than weak convergence in $L^1([0,T], \mathbb{R}^m)$. In addition, the characterization of sequential $\mathcal{T}^0$-convergence in $\mathcal{U}_T^m$ depends crucially on the fact that we are working with the collection of *all* systems in $C^0(m)$. We could equally well have defined a concept of $\mathcal{T}^k$-convergence for any integer $k \ge 0$ by just taking $\mathcal{C}$ to be the class $C^k(m)$ of all systems of the form (1) with vector fields f_i of class C^k. But then it is no longer obvious that condition (UB) is still necessary for sequential $\mathcal{T}^k$-convergence for $k \ge 1$. (The proof that (UB) is necessary for sequential $\mathcal{T}^0$-convergence in $\mathcal{U}_T^m$ depends on using the uniform boundedness theorem and requires that we admit systems with an arbitrary continuous right-hand side. On the other hand, the necessity of (IC) depends only on the fact that the systems $\dot{x} = u_i$, $i = 1, \ldots, m$ are in our class, so (IC) is still necessary for trajectory convergence for practically any reasonable class of systems.) In fact, we show in [2] that a sequence $\{u^j\}_{j=1}^{\infty} \subseteq \mathcal{U}_T^1$ may $\mathcal{T}^k$-converge to a $u^\infty \in \mathcal{U}_T^1$ even though (UB) fails.

To our knowledge, the problem of characterizing sequential $\mathcal{T}^k$-convergence in $\mathcal{U}_T^m$ for $k \geq 1$ is still open for $m > 1$, and appears to be quite difficult. The case $m = 1$ is special, as explained in Section 5, where we give a complete characterization (in Theorem 3) of the topology $\mathcal{T}(\mathcal{U}_T^1, C^1(1))$ by showing that a net $\{u^\alpha\}_{\alpha \in A} \subseteq \mathcal{U}_T^1$ $\mathcal{T}(\mathcal{U}_T^1, C^1(1))$-converges to a limit u^∞ iff the u^α I-converge to u^∞. The profound reason why this characterization is possible for $m = 1$ but not for $m > 1$ is provided by Theorems 4 and 5.

2. The topology of trajectory convergence

We begin by giving a precise meaning to the "T-convergence" condition that appears in (TC).

In what follows, the word "interval" means "connected, nonempty subset of $\mathbb{R}$." An interval is *nontrivial* if it is not reduced to a single point.

If Ω is a metric space, we use d_Ω to denote the corresponding distance function. A *curve* in Ω is a continuous map ξ from an interval $I \subseteq \mathbb{R}$ to Ω. If I is compact, then ξ will be called an *arc* in Ω. We use $ARC(\Omega)$ to denote the set of all arcs in Ω. For every $-\infty < a \leq b < \infty$, we let $C^0([a,b],\Omega)$ denote the set of all arcs in Ω whose domain is $[a,b]$. For each $\xi \in ARC(\Omega)$, we let $a(\xi)$, $b(\xi)$ be the unique a, b such that ξ belongs to $C^0([a,b],\Omega)$. If $\xi \in ARC(\Omega)$, let $\tilde{\xi}$ be the continuous curve $\tilde{\xi} : \mathbb{R} \mapsto \Omega$ obtained by extending ξ to the whole real line by just letting $\tilde{\xi}$ be constant on $]-\infty, a(\xi)]$ and on $[b(\xi), +\infty[$. We topologize $ARC(\Omega)$ by means of the metric $d_{ARC} : ARC(\Omega) \times ARC(\Omega) \mapsto \mathbb{R}$ given by

$$
\begin{aligned}
d_{ARC}(\xi, \eta) \quad &= \quad |a(\xi) - a(\eta)| + |b(\xi) - b(\eta)| \\
&\quad + \sup\{d_\Omega(\tilde{\xi}(t), \tilde{\eta}(t)) \,:\, t \in \mathbb{R}\}\,.
\end{aligned}
$$

Then a sequence $\{\xi^j\}$ of arcs, possibly defined on different time intervals $[a^j, b^j]$, converges to an arc $\xi : [a,b] \mapsto \Omega$, if (i) $a^j \to a$, (ii) $b^j \to b$, and (iii) $\xi^j \to \xi$ uniformly, i.e., $\xi^j(t^j) \to \xi(t)$ whenever $t^j \in [a^j, b^j]$, $t^j \to t$.

If $\xi : J \mapsto \Omega$ is a curve, the *graph* of ξ is the set $G(\xi) = \{(\xi(t), t) : t \in J\}$.

If $\xi_i : [a_i, b_i] \mapsto \Omega$ for $i = 1, 2$ are arcs in Ω, then the *concatenation* $\xi_1 \# \xi_2$ is defined if and only if $b_1 = a_2$ and $\xi_1(b_1) = \xi_2(a_2)$, and is the arc whose graph is the set $G(\xi_1) \cup G(\xi_2)$. Concatenation is an associative partially defined binary operation on $ARC(\Omega)$.

A subset $\mathcal{A} \subseteq ARC(\Omega)$ is *bounded* if there is a compact subset K of $\Omega \times \mathbb{R}$ such that $G(\xi) \subseteq K$ for all $\xi \in \mathcal{A}$. We call $\mathcal{A}$ *equicontinuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_\Omega(\xi(t), \xi(s)) < \varepsilon$ whenever $\xi \in \mathcal{A}$, $t, s \in [a(\xi), b(\xi)]$, and $|t - s| \leq \delta$. It is then clear that a subset $\mathcal{A}$ of $ARC(\Omega)$ is relatively compact iff $\mathcal{A}$ is bounded and equicontinuous.

An *arc system in Ω with time domain I* is a subset $\mathcal{A}$ of $ARC(\Omega)$ such that

- (AS.1) $G(\xi) \subseteq \Omega \times I$ for every $\xi \in \mathcal{A}$,
- (AS.2) $\mathcal{A}$ contains every arc in Ω whose domain is of the form $\{t\}$ for $t \in I$,
- (AS.3) $\mathcal{A}$ is closed under concatenations and restrictions (that is: (i) if ξ_1, ξ_2 are in $\mathcal{A}$ and $\xi_1 \# \xi_2$ is defined, then $\xi_1 \# \xi_2 \in \mathcal{A}$, and (ii) if $\xi \in \mathcal{A}$ and $a(\xi) \leq \alpha \leq \beta \leq b(\xi)$, then the restriction $\xi\lceil[\alpha, \beta]$ of ξ to $[\alpha, \beta]$ is in $\mathcal{A}$).

We use $AS(\Omega, I)$ to denote the set of all arc systems in Ω with time domain I. If $\mathcal{A} \in AS(\Omega, I)$ and $K \subseteq \Omega \times I$, we write $\mathcal{A}(K) \stackrel{\text{def}}{=} \{\xi \in \mathcal{A} : G(\xi) \subseteq K\}$.

If $\mathcal{A} \in AS(\Omega, I)$, then it follows easily that, if

- (i) the arcs $\xi_1, \ldots, \xi_m$ belong to $\mathcal{A}$,
- (ii) $[a(\xi_1), b(\xi_1)] \cup \cdots \cup [a(\xi_m), b(\xi_m)]$ is an interval,
- (iii) $\xi_i\lceil I_i \cap I_j = \xi_j\lceil I_i \cap I_j$ for all $i, j \in \{1, \ldots, m\}$, where we write $I_\ell = [a(\xi_\ell), b(\xi_\ell)]$ for $\ell \in \{1, \ldots m\}$,
- (iv) we let $\xi_1 \vee \cdots \vee \xi_m$ be the arc whose graph is the set $G(\xi_1) \cup \ldots \cup G(\xi_m)$,

then $\xi_1 \vee \ldots \vee \xi_m$ is also in $\mathcal{A}$.

If $\mathcal{A} \in AS(\Omega, I)$, we define $\mathcal{A}^\#$ to be the set of all curves $\xi : J \mapsto \Omega$ such that J is a subinterval of I and $\xi\lceil L \in \mathcal{A}$ for every compact subinterval L of J. The members of $\mathcal{A}^\#$ will be called *$\mathcal{A}$-curves*. An $\mathcal{A}$-curve $\xi \in \mathcal{A}^\#$ is *maximal* if there does not exist $\eta \in \mathcal{A}^\#$ such that $G(\xi)$ is a proper subset of $G(\eta)$. We use $\mathcal{A}^\#_{max}$ to denote the set of all maximal elements of $\mathcal{A}^\#$. A trivial application of Zorn's lemma shows that, if $\mathcal{A} \in AS(\Omega, I)$, then every $\xi \in \mathcal{A}^\#$ can be extended to an $\eta \in \mathcal{A}^\#_{max}$. In particular, through every point of $\Omega \times I$ there passes the graph of a $\xi \in \mathcal{A}^\#_{max}$.

If $\mathcal{A} \in AS(\Omega, I)$, we say that $\mathcal{A}$ has the *compactness property* if for every compact subset $K \subseteq \Omega \times I$ the set $\mathcal{A}(K)$ is compact in $ARC(\Omega)$. We say that $\mathcal{A}$ has the *local existence property* if for every $(x, t) \in \Omega \times I$ there exists $\xi \in \mathcal{A}$ such that t belongs to the interior of $[a(\xi), b(\xi)]$ relative to I and $\xi(t) = x$. We use $AS_{comp,le}(\Omega, I)$ to denote the set of all $\mathcal{A} \in AS(\Omega, I)$ that have the compactness property and the local existence property.

The following result is proved in [2]:

Proposition 1 *Let Ω be a locally compact metric space, and let $I \subseteq \mathbb{R}$ be an interval. Assume that $\mathcal{A} \in AS_{comp,le}(\Omega, I)$. Let $\xi : J \mapsto \Omega$ be a curve in $\mathcal{A}^\#$. Then $\xi \in \mathcal{A}^\#_{max}$ if and only if* (i) *J is relatively open in I and* (ii) *for every compact subset K of $\Omega \times I$ the set $\{t \in J : (\xi(t), t) \in K\}$ is compact.* $\diamondsuit$

If X is a set, we use 2^X to denote the set of all subsets of X. If X is a topological space, we can topologize 2^X as follows. For each open subset U of X, let $V(U)$ denote the set $\{S \in 2^X \,:\, S \subseteq U\}$. We then let $\mathcal{T}_{USC}(X)$ denote the set of all subsets S of 2^X such that $S = \cup_\alpha V(U_\alpha)$ for some family $\{U_\alpha\}$ of open subsets of X. Then $\mathcal{T}_{USC}(X)$ is a topology on 2^X. We will call it the *USC topology* on 2^X, because of its obvious relation to upper-semicontinuity of set valued maps (cf. [2]).

If Ω is a metric space and I is an interval, then for every subset K of $\Omega \times I$ we can define a set-valued map $T_{\Omega,I,K} : AS(\Omega, I) \mapsto 2^{ARC(\Omega)}$ by $T_{\Omega,I,K}(\mathcal{A}) = \mathcal{A}(K)$. As K ranges over all the compact subsets of $\Omega \times I$, we have a collection of maps $T_{\Omega,I,K}$ from $AS(\Omega, I)$ to

$2^{ARC(\Omega)}$. Therefore there exists a weakest topology on $AS(\Omega, I)$ that renders all the maps $T_{\Omega,I,K}$ continuous, where $2^{ARC(\Omega)}$ is equipped with the USC topology. This topology will be called the *topology of trajectory convergence* (or simply *T-convergence*) and be denoted by $\mathcal{T}_T(\Omega, I)$. If $\{\mathcal{A}^\alpha\}_{\alpha\in A}$ is a net in $AS(\Omega, I)$ and $\mathcal{A} \in AS(\Omega, I)$, we say that $\{\mathcal{A}^\alpha\}_{\alpha\in A}$ *T-converges* to $\mathcal{A}$ (and write $\mathcal{A}^\alpha \xrightarrow{\mathrm{T}} \mathcal{A}$) if the $\mathcal{A}^\alpha$ converge to $\mathcal{A}$ in $\mathcal{T}_T(\Omega, I)$.

Let Ω be a metric space, let $\xi : I \mapsto \Omega$ be a curve, and assume that $\{\xi^\alpha\}_{\alpha\in A}$ is a net of curves in Ω, with $\mathrm{Dom}(\xi^\alpha) = I^\alpha$. We say that the ξ^α *converge to* ξ *on compact sets* if for every compact subset $J \subseteq I$, there exists an $\alpha^*(J)$ such that (i) $J \subseteq I^\alpha$ for $\alpha \succeq_A \alpha^*(J)$ and (ii) the $\xi^\alpha\lceil J$ converge uniformly to $\xi\lceil J$.

A net $\{\mathcal{A}^\alpha\}_{\alpha\in A}$ in $AS(\Omega, I)$ *converges* to a member $\mathcal{A}$ of $AS(\Omega, I)$ *in the maximal curve convergence sense* if, whenever $\{(x^\alpha, t^\alpha)\}_{\alpha\in A}$ is a net of points in $\Omega \times I$ that converges to a limit $(\bar{x}, \bar{t}) \in \Omega \times I$, and $\{\xi^\alpha\}_{\alpha\in A}$ is a net such that for each $\alpha \in A$ $\xi^\alpha \in (\mathcal{A}^\alpha)^\#_{max}$ and $\xi^\alpha(t^\alpha) = x^\alpha$, it follows that there exist a $\xi^\infty \in \mathcal{A}^\#_{max}$, with domain $I^\infty \subseteq I$, such that $\xi^\infty(\bar{t}) = \bar{x}$, and a subnet $\{\xi^{\mu(\beta)}\}_{\beta\in B}$ of $\{\xi^\alpha\}_{\alpha\in A}$ that converges to ξ^∞ on compact sets.

The proof of the following proposition is given is [2].

Proposition 2 *Let Ω be a locally compact metric space, and let $I \subseteq \mathbb{R}$ be an interval. Suppose that $\{\mathcal{A}^\alpha\}_{\alpha\in A}$ is a net in $AS_{comp,le}(\Omega, I)$, and $\mathcal{A}$ belongs to $AS_{comp,le}(\Omega, I)$. Then the $\mathcal{A}^\alpha$ T-converge to $\mathcal{A}$ if and only if every subnet of $\{\mathcal{A}^\alpha\}_{\alpha\in A}$ converges to $\mathcal{A}$ in the maximal curve convergence sense.* $\diamondsuit$

We now specialize to the arc systems defined by vector fields. Let Ω be an open subset of $\mathbb{R}^n$, and let I be a subinterval of $\mathbb{R}$. A *time-varying vector field* on Ω with time domain I is a map $f : \Omega \times I \mapsto \mathbb{R}^n$. We use $TVVF(\Omega, I)$ to denote the set of all such maps. An $f \in TVVF(\Omega, I)$ satisfies the *Carathéodory condition* if it is continuous in x for each fixed $t \in I$ and Lebesgue measurable in t for each fixed $x \in \Omega$. We call f *locally integrably bounded* (LIB) if f is bounded in norm by a locally integrable function of t, as long as x stays in a compact subset of Ω. We use $TVVF_{Car,LIB}(\Omega, I)$ to denote the set of all $f \in TVVF(\Omega, I)$ that satisfy the Carathéodory condition, and are LIB.

If $f \in TVVF(\Omega, I)$, a *trajectory* (or *integral curve*) of f is an absolutely continuous function $\xi : J \mapsto \Omega$ defined on some nonempty subinterval $J \subseteq I$ such that $\dot{\xi}(t) = f(\xi(t), t)$ for almost all $t \in J$. A *maximal trajectory* of f is a trajectory $\xi : J \mapsto \Omega$ that cannot be extended to a trajectory of f defined on a strictly larger interval. We write $\mathrm{Traj}(f)$ to denote the set of all trajectories of f, and $\mathrm{Traj}_c(f)$ to denote the set of all $\xi \in \mathrm{Traj}(f)$ that are arcs, i.e., such that $\mathrm{Dom}(\xi)$ is a compact interval. Also, we write $\mathrm{Traj}(f)_{max}$ to denote the set of all maximal trajectories of f. Then it is clear that $\mathrm{Traj}_c(f)$ is an arc system, and $\mathrm{Traj}_c(f)^\#$ is precisely $\mathrm{Traj}(f)$. Therefore $\mathrm{Traj}(f)_{max}$ is exactly the set $\mathrm{Traj}_c(f)^\#_{max}$.

Thus $f \mapsto \mathrm{Traj}_c(f)$ is a map from $TVVF(\Omega, I)$ to $AS(\Omega, I)$. Using this map, we can pull back the topology of trajectory convergence on $AS(\Omega, I)$ and define a *topology of trajectory convergence*, or *T-convergence*, on $TVVF(\Omega, I)$. We can also pull back properties of arc systems and apply them to vector fields. (For example, we will say that f has the local existence property if the arc system $\mathrm{Traj}_c(f)$ does.) We call f *trajectory compact* if $\mathrm{Traj}_c(f)$ has the compactness property. We use $TVVF_{comp,le}(\Omega, I)$ to denote the set of those $f \in TVVF(\Omega, I)$ that are trajectory compact and have the local existence property. We say that f has the *uniqueness property* if for every initial condition $(\bar{x}, \bar{t})$ in $\Omega \times I$ the maximal trajectory of f whose graph contains $(\bar{x}, \bar{t})$ is unique.

The Carathéodory existence theorem implies that every vector field $f \in TVVF_{Car,LIB}(\Omega, I)$ has the local existence property. Moreover, a simple application of the Ascoli–Arzelà theorem and the Lebesgue dominated convergence theorem shows that every time-varying vector field $f \in TVVF_{Car,LIB}(\Omega, I)$ is trajectory compact. So

$$TVVF_{Car,LIB}(\Omega, I) \subseteq TVVF_{comp,le}(\Omega, I). \quad (4)$$

If S is a subset of $\Omega\times I$, we write $\mathrm{Traj}_c(f, S)$ to denote the set of all arcs $\xi \in \mathrm{Traj}_c(f)$ such that $G(\xi) \subseteq S$. We then have the following propositions, proved in [2].

Proposition 3 *Let f be in $TVVF(\Omega, I)$ and $\{f^\alpha\}_{\alpha\in A}$ be a net in $TVVF(\Omega, I)$. Then $\{f^\alpha\}_{\alpha\in A}$ T-converges to f if and only if for every compact subset K of $\Omega\times I$ and every open subset V of $ARC(\Omega)$ containing $\mathrm{Traj}_c(f, K)$ there exists an $\alpha^*(V) \in A$ such that $\mathrm{Traj}_c(f^\alpha, K) \subseteq V$ whenever $\alpha^*(V) \preceq \alpha$.* $\diamondsuit$

Proposition 4 *If $f \in TVVF_{comp,le}(\Omega, I)$, then a net $\{f^\alpha\}_{\alpha\in A}$ in $TVVF(\Omega, I)$ T-converges to f if and only if*

$$\lim_\alpha \sup\{d_{ARC}(\xi, \mathrm{Traj}_c(f, K)) : \xi \in \mathrm{Traj}_c(f^\alpha, K)\} = 0$$

for every compact subset K of $\Omega \times I$. $\diamondsuit$

Proposition 5 *Let $\mathbf{f} = \{f^\alpha\}_{\alpha\in A}$ be a net in $TVVF_{comp,le}(\Omega, I)$, and suppose that f belongs to $TVVF_{comp,le}(\Omega, I)$. Then the net $\mathbf{f}$ T-converges to f if and only if, whenever $\{f^{\mu(\beta)}\}_{\beta\in B}$ is a subnet of $\mathbf{f}$, $(x^\beta, t^\beta) \in \Omega \times I$ converge to $(\bar{x}, \bar{t}) \in \Omega \times I$, and $\boldsymbol{\xi} = \{\xi^\beta\}_{\beta\in B}$ is a family such that each ξ^β is a maximal trajectory of $f^{\mu(\beta)}$ for which $\xi^{\mu(\beta)}(t^\beta) = x^\beta$, it follows that there exist a maximal trajectory ξ of f such that $\xi(\bar{t}) = \bar{x}$ and a subnet $\{\xi^{\nu(\gamma)}\}_{\gamma\in C}$ of $\boldsymbol{\xi}$ that converges to ξ on compact sets.* $\diamondsuit$

Proposition 6 *If $f \in TVVF_{comp,le}(\Omega, I)$, then a sequence $\mathbf{f} = \{f^j\}_{j=1}^\infty$ of members of $TVVF_{comp,le}(\Omega, I)$ T-converges to f if and only if, whenever J is an infinite set of positive integers, $(x^j, t^j) \in \Omega \times I$ converge to $(\bar{x}, \bar{t}) \in \Omega \times I$ as $j \to \infty$ via values in J, and $\boldsymbol{\xi} = \{\xi^j\}_{j\in J}$ is a family such that each ξ^j is a maximal trajectory of f^j for which $\xi^j(t^j) = x^j$, it follows that there exist a maximal trajectory ξ of f such that $\xi(\bar{t}) = \bar{x}$ and an infinite subset J' of J such that $\{\xi^j\}_{j\in J'}$ converges to ξ on compact sets.* $\diamondsuit$

Proposition 7 *Suppose that $f \in TVVF_{comp,le}(\Omega, I)$ and that a net $\mathbf{f} = \{f^\alpha\}_{\alpha\in A}$ of members of $TVVF_{comp,le}(\Omega, I)$ T-converges to f; assume that $(x^\alpha, t^\alpha) \in \Omega\times I$ converge to $(\bar{x}, \bar{t}) \in \Omega\times I$, $\boldsymbol{\xi} = \{\xi^\alpha\}_{\alpha\in A}$ is a family such that each ξ^α is a maximal trajectory of f^α for which $\xi^\alpha(t^\alpha) = x^\alpha$, and the maximal trajectory ξ of f for which $\xi(\bar{t}) = \bar{x}$ is unique; then the net $\boldsymbol{\xi}$ converges to ξ on compact sets.* $\diamondsuit$

Proposition 8 *Suppose that $f \in TVVF_{comp,le}(\Omega, I)$ has the uniqueness property, and let $\mathbf{f} = \{f^\alpha\}_{\alpha\in A}$ be a net of members of $TVVF_{comp,le}(\Omega, I)$; then $\mathbf{f}$ T-converges to f if and only if, whenever $(x^\alpha, t^\alpha) \in \Omega\times I$ converge to $(\bar{x}, \bar{t}) \in \Omega \times I$, $\boldsymbol{\xi} = \{\xi^\alpha\}_{\alpha\in A}$ is a family such that each ξ^α is a maximal trajectory of f^α for which $\xi^\alpha(t^\alpha) = x^\alpha$, and ξ is the maximal trajectory of f for which $\xi(\bar{t}) = \bar{x}$, it follows that the net $\boldsymbol{\xi}$ converges to ξ on compact sets.* $\diamondsuit$

3. $\mathcal{T}^0$-convergence for control-affine systems

We are now ready to give the precise definition of the topology of $\mathcal{T}^0$-convergence on $L^1([0,T],\mathbb{R}^m)$.

Let $m \in \mathbb{N}$, $m > 0$. A $C^0(m)$ *system* is an $(m+2)$-tuple $\Sigma = (\Omega, f_0, \ldots, f_m)$, where Ω is a nonempty open subset of $\mathbb{R}^n$ for some integer $n \geq 1$, and $f_0, \ldots, f_m$ are continuous vector fields on Ω. The expressions f_i^Σ, Ω^Σ will be used to denote the vector fields f_i and the set Ω corresponding to a given system $\Sigma \in C^0(m)$.

If $u \in L^1([0,T],\mathbb{R}^m)$, each $\Sigma \in C^0(m)$ gives rise to an O.D.E. $\dot{x}(t) = f_0^\Sigma(x(t)) + \sum_{k=1}^m u_k(t) f_k^\Sigma(x(t))$, whose right-hand side is in $TVVF_{Car,LIB}(\Omega^\Sigma,[0,t])$. So we can define a map Γ_Σ from the input space $L^1([0,T],\mathbb{R}^m)$ to $TVVF_{Car,LIB}(\Omega^\Sigma,[0,T])$ by letting $f^\Sigma(u)(x,t) \stackrel{\text{def}}{=} f_0^\Sigma(x) + u_1(t)f_1^\Sigma(x) + \cdots + u_m(t)f_m^\Sigma(x)$ and $\Gamma_\Sigma(u) = f^\Sigma(u)$. For each $\Sigma \in C^0(m)$ and $T > 0$, the set $TVVF_{Car,LIB}(\Omega^\Sigma,[0,T])$ is a topological space with the topology of trajectory convergence. Therefore, for any subclass $\mathcal{C}$ of $C^0(m)$ we can consider the topology $T(\mathcal{C})$ on $L^1([0,T],\mathbb{R}^m)$ induced by the family of mappings $\{\Gamma_\Sigma : \Sigma \in \mathcal{C}\}$, i.e. the weakest topology on $L^1([0,T],\mathbb{R}^m)$ that makes all the maps Γ_Σ continuous, for all $\Sigma \in \mathcal{C}$. In particular, the *topology of $\mathcal{T}^0$-convergence* on $L^1([0,T],\mathbb{R}^m)$ is, by definition, the $T(C^0(m))$-topology. We say that a net $\{u^\alpha\}_{\alpha\in A}$ *$\mathcal{T}^0$-converges* to a u^∞ in $L^1([0,T],\mathbb{R}^m)$ if it converges to u^∞ in this topology. Equivalently, a net $\{u^\alpha\}_{\alpha\in A}$ $\mathcal{T}^0$-converges to a limit u^∞ if and only if

> (TCN) *For every system $\Sigma \in C^0(m)$, the time-varying vector fields $(x,t) \mapsto f^\Sigma(u^\alpha)(x,t)$ T-converge to $(x,t) \mapsto f^\Sigma(u^\infty)(x,t)$.*

(This is exactly the version for nets of Condition (TC) of the Introduction.)

If (TCN) holds, then Proposition 7 implies that in particular the following conditions hold as well:

> (TCN') *For every $\Sigma \in C^0(m)$, if ξ^α are maximal trajectories of (1) determined by the u^α, $x^\alpha = \xi^\alpha(0)$, and $x^\alpha \to \bar{x} \in \Omega^\Sigma$, then the ξ^α are defined on $[0,T]$ for α large enough and converge uniformly on $[0,T]$ to the maximal trajectory ξ^∞ for u^∞, provided that uniqueness of solutions, as well as global existence on $[0,T]$, hold for the limiting initial value problem $\dot{x} = f_0^\Sigma(x) + \sum_{i=1}^m u_i^\infty(t) f_i^\Sigma(x)$, $x(0) = \bar{x}$.*

> (TCN") *For every $\Sigma \in C^0(m)$ and $\bar{x} \in \Omega^\Sigma$, if ξ^α are maximal trajectories of (1) determined by the u^α and $\bar{x}$, then the ξ^α are defined on $[0,T]$ for α large enough and converge uniformly on $[0,T]$ to the maximal trajectory ξ^∞ for u^∞, provided that for the limiting initial value problem $\dot{x} = f_0^\Sigma(x) + \sum_{i=1}^m u_i^\infty(t) f_i^\Sigma(x)$, $x(0) = \bar{x}$, there is uniqueness of solutions as well as global existence on $[0,T]$.*

4. Sequential $\mathcal{T}^0$-convergence

The following result, proved in [2], characterizes $\mathcal{T}^0$-convergence of sequences.

Theorem 1 *Let $\mathbf{u} = \{u^j\}_{j=1}^\infty$ be a sequence of controls in $L^1([0,T],\mathbb{R}^m)$, and let $u^\infty \in L^1([0,T],\mathbb{R}^m)$. Then the following conditions are equivalent*

(i) $\mathbf{u}$ *satisfies Condition (UB) and I-converges to u^∞,*
(ii) $\mathbf{u}$ *$\mathcal{T}^0$-converges to u^∞,*
(iii) *(TCN') holds,*
(iv) *(TCN") holds,*
(v) $\mathbf{u}$ *UW-converges to u^∞.*

If $\mathbf{u} = \{u^\alpha\}_{\alpha\in A}$ is a net in $L^1([0,T],\mathbb{R}^m)$, and u^∞ belongs to $L^1([0,T],\mathbb{R}^m)$, then (i)$\Rightarrow$(ii)$\Rightarrow$(iii) $\Rightarrow$(iv)$\Rightarrow$(v), *but in general* (v)$\not\Rightarrow$(iv) *and* (ii)$\not\Rightarrow$(i). $\diamondsuit$

Remark 1 At present, we do not know if the implications (iv)$\Longrightarrow$(iii)$\Longrightarrow$(ii) hold for general nets. $\diamondsuit$

Theorem 2 *Let $\mathbf{u} = \{u^\alpha\}_{\alpha\in A}$ be a net of inputs belonging to $L^1([0,T],\mathbb{R}^m)$, and let $u^\infty \in L^1([0,T],\mathbb{R}^m)$ be such that* (i) *of Theorem 1 holds. Let Ω be open in $\mathbb{R}^n$, and let h^α be time-varying vector fields given by $h^\alpha(x,t) = \dot{\theta}^\alpha(t) + f_0(x) + \sum_{i=1}^m u_i^\alpha(t) f_i(x)$, where the $f_i : \Omega \mapsto \mathbb{R}^n$ are continuous, the $\theta^\alpha : [0,T] \mapsto \mathbb{R}^n$ are absolutely continuous, and $\theta^\alpha \to 0$ uniformly. Then the net $\{h^\alpha\}_{\alpha\in A}$ T-converges to the vector field h^∞ given by $h^\infty(x,t) = f_0(x) + \sum_{i=1}^m u_i^\infty(t) f_i(x)$.* $\diamondsuit$

5. Other classes of systems

The equivalence for sequences of the five conditions of Theorem 1 depends very strongly on the fact that the class of systems under consideration is $C^0(m)$. One could equally well have considered T($\mathcal{C}$)-convergence in $L^1([0,T],\mathbb{R}^m)$ with respect to other classes $\mathcal{C}$ of systems. For example, we could have picked any positive integer k, and chosen $\mathcal{C}$ to be the class $C^k(m)$ of all the $\Sigma \in C^0(m)$ such that $f_0^\Sigma, \ldots, f_m^\Sigma$ are of class C^k on Ω^Σ. Let us call the T($C^k(m)$)-convergence in $L^1([0,T],\mathbb{R}^m)$ simply *$\mathcal{T}^k$-convergence.*

It is then natural to ask whether explicit characterizations of $\mathcal{T}^k$-convergence of sequences exist. The necessary and sufficient conditions for $\mathcal{T}^0$-convergence are of course sufficient to ensure $\mathcal{T}^k$-convergence, but they are no longer necessary. The following theorem, which characterizes $\mathcal{T}^k$-convergence of nets in $L^1([0,T],\mathbb{R})$ for $k \geq 1$, implies in particular that it is indeed possible for a sequence $\{u^j\}_{j=1}^\infty \subseteq L^1([0,T],\mathbb{R})$ to $\mathcal{T}^1$-converge to a $u^\infty \in L^1([0,T],\mathbb{R})$ even if the $\|u^j\|_{L^1}$ are not bounded.

Theorem 3 *Let $T > 0$, and let $k > 0$ be an integer. A net $\{u^\alpha\}_{\alpha\in A}$ in $L^1([0,T],\mathbb{R})$ $\mathcal{T}^k$-converges to a $u^\infty \in L^1([0,T],\mathbb{R})$ if and only if $\{u^\alpha\}$ I-converges to u^∞, i.e. if and only if the integrals $\int_0^t u^\alpha(s)ds$ converge to $\int_0^\tau u^\infty(s)ds$ uniformly on $[0,T]$.* $\diamondsuit$

A similar characterization is not valid for $m > 1$, because for $m > 1$ the vector fields f_i, $i = 1, \ldots, m$, need not commute. Let us call a system $\Sigma \in C^1(m)$ *commutative* if the Lie brackets $[f_i^\Sigma, f_j^\Sigma]$ vanish identically for all $i, j \in \{1, \ldots, m\}$. (Notice that for $m = 1$ every system is commutative.) For $k \geq 1$, let $C_{comm}^k(m)$ be the class of all commutative systems in $C^k(m)$. A net $\mathbf{u}$ in $L^1([0,T],\mathbb{R}^m)$ $\mathcal{T}^k_{comm}$*-converges* to $u \in L^1([0,T],\mathbb{R}^m)$ if it converges to u in the topology $\mathcal{T}(C^k_{comm}(m))$.

The following result, proved in [2], holds for $m > 1$.

Theorem 4 *Let $T > 0$, and let m be a positive integer. For any integer $k \geq 1$, a net $\{u^\alpha\}_{\alpha\in A}$ in $L^1([0,T],\mathbb{R}^m)$ $\mathcal{T}^k_{comm}$-converges to a $u^\infty \in L^1([0,T],\mathbb{R}^m)$ iff $\{u^\alpha\}$ I-converges to u^∞, i.e. iff the integrals $\int_0^t u^\alpha(s)ds$ converge to $\int_0^t u^\infty(s)ds$ uniformly on $[0,T]$.* $\diamondsuit$

Clearly, Theorem 3 is a particular case of Theorem 4.

6. Beyond I-convergence: Lie brackets

Theorem 4 characterizes the topology of $\mathcal{T}^1_{comm}$-convergence on $L^1([0,T],\mathbb{R}^m)$, by showing that $\mathcal{T}^1_{comm}$-convergence is exactly equivalent to I-convergence. It is natural to ask whether I-convergence actually implies T-convergence for at least some noncommutative systems. The answer is negative:

Theorem 5 *Let $\mathcal{C}$ be any subclass of $C^1(m)$ that contains at least one noncommutative system. Then there exists a sequence $\{u^j\}_{j=1}^\infty$ in $L^1([0,T],\mathbb{R}^m)$ such that* (i) *$\{u^j\}$ I-converges to a limit $u^\infty \in L^1([0,T],\mathbb{R}^m)$,* (ii) *for every $\Sigma \in \mathcal{C}$ the time-varying vector fields $f^\Sigma(u^j)$ T-converge to a limit h^Σ, but* (iii) *it is not true that $h^\Sigma = f^\Sigma(u^\infty)$ for all $\Sigma \in \mathcal{C}$.* $\diamondsuit$

This result—which follows from Proposition 9 below—says that noncommutativity is the fundamental obstruction for I-convergence of the inputs to imply convergence of trajectories. The best way to understand why this is so is to think of the space $L^1([0,T],\mathbb{R}^m)$ of ordinary inputs as embedded in a larger space $\mathcal{GI}(m)$ of *generalized* m-dimensional inputs, whose elements have components not only for the m input channels corresponding to the vector fields $f_1,\ldots,f_m$ but also in "Lie bracket channels" such as $[f_1,f_2]$. An "ordinary input" $u \in L^1([0,T],\mathbb{R}^m)$ gives rise, for each system Σ, to a time-varying vector field $f^\Sigma(u)$. Thus an ordinary input can be thought of as a "functional" that to every system Σ in a given class $\mathcal{C}$ assigns a time-varying vector field. It is then natural to extend the class of ordinary inputs by allowing more general functionals. By analogy with other constructions of spaces of "generalized functions" (e.g., Schwartz distributions), we should also require that such a functional be approximable by ordinary inputs. So we first define, for a class $\mathcal{C} \subseteq C^0(m)$ and an interval $[a,b]$, the space $\mathcal{PGI}(\mathcal{C},a,b)$ of all maps v that assign to every system $\Sigma \in \mathcal{C}$ a time-varying vector field $v_\Sigma \in TVVF(\Omega,[a,b])$, and call the elements of $\mathcal{PGI}(\mathcal{C},a,b)$ "pseudogeneralized inputs." Then $\mathcal{PGI}(\mathcal{C},a,b)$ is just the product $\Pi_{\Sigma\in\mathcal{C}}TVVF(\Omega,[a,b])$, and we can topologize it by means of the product topology. An ordinary input $u \in L^1([a,b],\mathbb{R}^m)$ can then be regarded as a member of $\mathcal{PGI}(\mathcal{C},a,b)$ by associating to it the map —also labeled u— $\Sigma \mapsto f^\Sigma(u)$. A "generalized input" on $[a,b]$ for the class $\mathcal{C}$ is then a member of $\mathcal{PGI}(\mathcal{C},a,b)$ which is a limit of ordinary inputs. We use $\mathcal{GI}(\mathcal{C},a,b)$ to denote the set of all generalized inputs, so by construction $\mathcal{GI}(\mathcal{C},a,b)$ is a topological space in which the space $L^1([a,b],\mathbb{R}^m)$ is densely embedded.

Particular examples of members of $\mathcal{PGI}(\mathcal{C},a,b)$ can be constructed by means of *formal Lie bracket expressions* such as

$$\begin{aligned} v &= X_0 + v_1(t)X_1 + v_2(t)X_2 + v_3(t)[X_1,X_2] \\ &\quad + v_4(t)[X_2,[X_1,X_2]] \,. \end{aligned} \tag{5}$$

Precisely, if we let $L(m+1)$ denote the free Lie algebra over $\mathbb{R}$ in $m+1$ indeterminates $X_0,\ldots,X_m$, then we can define an $\mathbb{R}^m$*-valued formal pseudogeneralized input on* $[a,b]$ to be an integrable function with values in $L(m+1)$. Use $\mathcal{FPGI}(m,a,b)$ to denote the set of all such objects. Then every $v \in \mathcal{FPGI}(m,a,b)$ gives rise to a true pseudogeneralized input—also labeled v—for any class $\mathcal{C}$ of control-affine systems such that $\mathcal{C} \subseteq C^0(m)$, provided that $\mathcal{C}$ has enough differentiability for all the Lie brackets occurring in the formal expression to make sense. For example, to an ordinary input $u = (u_1,\ldots,u_m)$ we associate the $L(m+1)$-valued function

$$t \mapsto X_0 + u_1(t)X_1 + \ldots + u_m(t)X_m \,,$$

and in this case it is possible to plug in $f_0,\ldots,f_m$ for $X_0,\ldots,X_m$ for every system $\mathcal{C} \in C^0(m)$. On the other hand, if $m \geq 2$, then an expression v such as (5) will give rise to a member of $\mathcal{PGI}(\mathcal{C},a,b)$ only if $\mathcal{C}$ consists of systems $(\Omega,f_0,\ldots,f_m)$ for which the Lie bracket $[f_1,f_2]$ is well defined. This will be the case, in particular, if $\mathcal{C} \subseteq C^1(m)$. Thus, for example, the formal pseudogeneralized input $w = X_0 + \frac{1}{2}[X_1,X_2]$ gives rise to a pseudo-generalized input w belonging to $\mathcal{PGI}(C^1(2),a,b)$. With this terminology, Proposition 9—proved in [2]—exhibits an example of a sequence $\{u^j\} \subseteq L^1([0,T],\mathbb{R}^2)$ that I-converges to zero but converges in $\mathcal{PGI}(C^1(2),0,T)$ to w. This shows that $w \in \mathcal{GI}(C^1(2),a,b)$, i.e., that w is a true generalized input. It is now clear why, when we look only at classes $\mathcal{C}$ of systems for which $[f_1,f_2]=0$, as we did in Theorem 4, the sequence $\{u^j\}$ $\mathcal{T}(\mathcal{C})$-converges to 0, whereas, as soon as $\mathcal{C}$ contains one system for which $[f_1,f_2] \neq 0$, the $\mathcal{T}(\mathcal{C})$-limit of $\{u^j\}$ is no longer 0. The reason is that the "true" limit is w, and w contains "Lie bracket components" that are only seen when w is tested against a noncommutative system.

Proposition 9 *Let $T > 0$. For $j \in \mathbb{N}$, let u^j be given by $u^j(t) = (u_1^j(t), u_2^j(t))$, where $u_1^j(t) = j^{\frac{1}{2}}\cos jt$ and $u_2^j = j^{\frac{1}{2}}\sin jt$. Let $\Sigma = (\Omega,f_0,f_1,f_2)$ be a system of class C^1. For $j \in \mathbb{N}\cup\{+\infty\}$, let $f^j(x,t) = f^\Sigma(u^j)(x,t)$, where $f^\Sigma(u^j)(x,t) = f_0(x) + u_1^j(t)f_1(x) + u_2^j(t)_2(x)$ if $j < +\infty$ and $f^\infty(x,t) = f_0(x) + \frac{1}{2}[f_1,f_2](x)$. Then the f^j T-converge to f^∞ in $TVVF_{LIB}(\Omega,[0,T])$.* $\diamondsuit$

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