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The Markov-Dubins problem with angular acceleration control*

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1. Introduction

The Markov-Dubins problem with angular acceleration control (MDPAAC) is of interest in optimal control theory because of the challenges it poses, due to the surprisingly couterintuitive properties of its solutions. In this note, we describe the problem and prove several results on the solutions, establishing in particular the existence of extremals involving infinite chattering.

The MDPAAC is the problem of describing the minimum-time trajectories for the four-dimensional control system

$$\begin{aligned} \dot{x} &= \cos z\,, \\ \dot{y} &= \sin z\,, \\ \dot{z} &= w\,, \\ \dot{w} &= u\,, \end{aligned} \tag{1}$$

where x, y, z and w are the state variables, and u is the control, subject to the inequality constraint

$$|u| \leq 1\,. \tag{2}$$

The variable z should be thought of as an angle, so that the true state space of the system is $\mathbb{R}^2 \times S^1 \times \mathbb{R}$.

System (1) is a dynamic extension of the widely studied three-dimensional *Dubins system*

$$\begin{aligned} \dot{x} &= \cos z\,, \\ \dot{y} &= \sin z\,, \\ \dot{z} &= u\,, \end{aligned} \tag{3}$$

with state variables x, y and z, state space $\mathbb{R}^2 \times S^1$, control variable u, and control constraint (2).

The minimum time problem for (3) was first posed by A. A. Markov in 1889 (cf. [6]), and then analyzed in detail by Dubins in 1957 (cf. [2], and also [8, 9, 10]), who studied it in the equivalent form of determining plane curves parametrized by arc-length, with a constraint $|u| \leq 1$ on the curvature, that minimize length given the initial and terminal points $(\bar{x}, \bar{y})$, $(\hat{x}, \hat{y})$ and the initial and terminal tangent vectors. Dubins showed that the optimal arcs are concatenations of at most three pieces, each of which is either "S" (a straight-line segment) or "C" (an arc of a circle of radius 1), and that optimality requires that further conditions be satisfied. (Precisely: only the forms S, C, S/C, C/S, C/C, $C/C/C$, $C/S/C$ are allowed —so that, for example, $S/C/S$ is excluded— and the middle C in a $C/C/C$ arc has to be of length $\geq \pi$.)

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The MDPAAC can be regarded as the problem of controlling in minimum time a vehicle that moves in the x, y plane with speed 1, and whose velocity vector points in the direction $z \in S^1$, which is changing at a rate w. The control u is thus the angular acceleration. It is intituively obvious (and easy to prove rigorously) that, if the initial and terminal positions of the vehicle lie in a straight line L forming an angle z_L with the x axis, the initial and terminal directions are both equal to z_L, and the initial and terminal angular velocities vanish, then the optimal control is given by $u \equiv 0$, which corresponds to motion along L, with zero angular velocity. It is then natural to expect that, if the endpoint conditions are $(\bar{x}, \bar{y}, \bar{z}, \bar{w})$ and $(\hat{x}, \hat{y}, \hat{z}, \hat{w})$, with $(\hat{x}, \hat{y})$ very far away from $(\bar{x}, \bar{y})$, $\bar{z}$ and $\hat{z}$ close to z_L (where L is the line joining $(\bar{x}, \bar{y})$ and $(\hat{x}, \hat{y})$), and $\bar{w}$, $\hat{w}$ close to zero, then the optimal motion will be along a straight line most of the time, with an initial maneuver to position the vehicle in the line, pointing in the right direction with zero angular velocity, and a terminal maneuver to adjust the final position, orientation and angular velocity.

A simple analysis, using the Pontryagin Maximum Principle (PMP, cf. [1, 7]), suggests that the "maneuvers" referred to above should be *bang-bang trajectories*, and it is natural to guess that these trajectories will in fact have a finite —and small— number of switchings. It turns out, however, that *this natural guess is wrong.* Indeed, we will prove rigorously, using the PMP, that *an optimal trajectory cannot contain points where the control jumps from* 1 *or* -1 *to* 0*, or from* 0 *to* 1 *or* -1. In view of this surprising fact, the simplest possible alternative to the "natural guess" is that *maneuvers involve infinite chattering* (IC), as in the celebrated Fuller problem (cf. [3, 4, 5, 11]). We will prove below the existence of *extremals* having the right kind of structure, namely, $IC/S/IC$. Further analysis, currently in progress, makes it possible to combine these extremals into *fields of extremals* and to apply Hamilton-Jacobi theory to prove their local optimality.

2. Bang and singular arcs

To apply the PMP, we introduce adjoint variables ξ, η, ζ, ω, dual to x, y, z and w, respectively, as well as the abnormal multiplier μ. The Hamiltonian H is then given by

$$H = \xi.\cos z + \eta.\sin z + \zeta.w + \omega.u + \mu\,, \tag{4}$$

and the PMP says that, if

$$[a, b] \ni t \mapsto \gamma(t) = (x(t), y(t), z(t), w(t))$$

is a minimizing arc, corresponding to a control $t \mapsto u(t)$, then there exist absolutely continuous functions $\xi(t)$, $\eta(t)$, $\zeta(t)$, $\omega(t)$, and a constant $\mu \geq 0$, such that

(i) H vanishes,

(ii) the function $\nu + \xi^2 + \eta^2 + \zeta^2 + \omega^2$ never vanishes,

(iii) the adjoint system of equations

$$\begin{aligned} \dot{\xi} &= 0\,, \\ \dot{\eta} &= 0\,, \\ \dot{\zeta} &= \xi.\sin z - \eta.\cos z\,, \\ \dot{\omega} &= -\zeta\,, \end{aligned} \tag{5}$$

holds,

and

(iv) $\omega(t)u(t) = -|\omega(t)|$ for a.e. $t \in [a, b]$.

The condition $\omega.u = -|\omega|$ will only be satisfied, on an interval I where ω never vanishes, if $u(\cdot)$ is a *bang control*, i.e. a constant control with value $+1$ or -1. On the other hand, if the switching function ω vanishes identically on I, then we are in the "singular case," in which it is often said that "the PMP does not give information about the control." It turns out, however, that the PMP gives quite a lot of information about the singular optimal controls. Indeed, if $\omega \equiv 0$ on I, then (5) implies that $\zeta \equiv 0$ on I as well. But then

$$\xi.\sin z - \eta.\cos z \equiv 0\,. \tag{6}$$

From this we can infer immediately that $\mu > 0$, for if $\mu = 0$ then (4) would imply that

$$\xi.\cos z + \eta.\sin z \equiv 0\,, \tag{7}$$

since $H \equiv \omega \equiv \zeta \equiv 0$, and this, together with (6), would imply $\xi = \eta = 0$, so $\xi \equiv \eta \equiv \zeta \equiv \omega \equiv 0 = \mu$, contradicting the nontriviality of the multiplier vector.

So $\mu > 0$, and then (4) yields

$$\xi.\cos z + \eta.\sin z = -\mu \neq 0\,. \tag{8}$$

This implies in particular that $(\xi, \eta) \neq (0, 0)$. Since the inner product of the vectors (ξ, η) and $(\cos z, \sin z)$ is equal to a fixed nonzero constant, and $z(t)$ is continuous, it follows that $z \equiv$ constant. Differentiation then yields $w \equiv 0$ and $u = 0$. So we have learned that

Proposition 1. *The only possible control which is singular on an interval I is $u \equiv 0$, and such a control can only occur in an extremal for which*

(i) $\mu > 0$,

(ii) $w \equiv \zeta \equiv \omega \equiv \xi.\sin z - \eta.\cos z \equiv 0$,

and

(iii) $(\xi, \eta) \neq (0, 0)$. □

Remark 1 Our system (1) is of the form

$$\dot{X} = f(X) + ug(X)\,, \tag{9}$$

with X belonging to a four-dimensional manifold M, and f, g smooth vector fields on M. For "most" systems of this type, one can find singular controls by differentiating the switching function φ twice. Precisely, one defines

$$\varphi(X, \lambda) \stackrel{\text{def}}{=} \lambda.g(X)\,, \tag{10}$$

if $X \in M$ and λ is a covector at X. Along a singular arc, the condition $\varphi \equiv 0$ must be satisfied, and differentiating this identity twice one gets

$$\varphi_1 = \varphi_2 = 0 \tag{11}$$

$$\varphi_3 + u\psi_1 = 0\,. \tag{12}$$

where $\varphi_1 \stackrel{\text{def}}{=} \varphi$, $\varphi_2 \stackrel{\text{def}}{=} \lambda.[f, g]$, $\varphi_3 \stackrel{\text{def}}{=} \lambda.[f, [f, g]]$, and $\psi_1 \stackrel{\text{def}}{=} \lambda.[g, [f, g]]$. In the "generic" case, when the four vector fields g, $[f, g]$, $[f, [f, g]]$ and $[g, [f, g]]$ are linearly independent on some open region $\Omega \subseteq M$, one can construct a singular control starting at a point $X \in \Omega$ by taking the initial value $\bar{\lambda}$ of λ to be such that

$$\varphi_1(X, \bar{\lambda}) = \varphi_2(X, \bar{\lambda}) = 0\,, \tag{13}$$

while the numbers

$$\alpha = \varphi_3(X, \bar{\lambda})\,,\ \ \beta = \psi_1(X, \bar{\lambda}) \tag{14}$$

are completely arbitrary, subject only to the condition that $\beta \neq 0$. In a neighborhood of X, one then gets a "feedback control law"

$$u(X, \lambda) = -\frac{\lambda.[f, [f, g]](X)}{\lambda.[g, [f, g]](X)} = -\frac{\varphi_3(X, \lambda)}{\psi_1(X, \lambda)}\,, \tag{15}$$

which is defined as a function of X and the auxiliary variable λ, for (X, λ) in the subset Σ of the cotangent bundle $T^*\Omega$ of Ω, given —writing Y for (X, λ)— by

$$\Sigma = \{Y \in T^*\Omega : \varphi_1(Y) = \varphi_2(Y) = 0,\ \psi_1(Y) \neq 0\}\,. \tag{16}$$

Clearly, Σ is a 6-dimensional submanifold of $T^*\Omega$, and on this submanifold the "feedback" control law (15) gives rise to a smooth vector field V, whose integral curves project down to singular arcs. Since the adjoint vector only matters up to multiplication by a positive constant, it is easy to see that the flow of V is actually well defined on the "ray bundle" associated to Σ, i.e. the bundle Σ^r over Ω whose fiber $\Sigma^r(X)$ at each point $X \in \Omega$ is the set of rays in $T^*_X\Omega$ along which $\lambda.g(X) = \lambda.[f, g](X) = 0$. So, to construct a singular arc through X, we have one degree of freedom, namely, the choice of a point in $\Sigma^r(X)$. It follows that, generically, through every point X there passes a one-parameter family of singular arcs.

This is rather different from the situation arising from the MDPAAC, in which the singular arc passing through a point X only exists for X in a hypersurface of codimension 1, and when it exists it is actually unique. It follows that *the MDPAAC is highly nongeneric*. This can be confirmed by computing the relevant Lie brackets directly. It is easily verified that g and $[f, g]$ are linearly independent everywhere, but

$$[g, [f, g]] \equiv 0\,, \tag{17}$$

so $\psi_1 \equiv 0$. This means that it is never possible to choose $\bar\lambda$ so that $\beta \neq 0$. Moreover, (12) implies that $\varphi_3 \equiv 0$ along singular arcs. So, to obtain singular controls, we have to differentiate the equation $\varphi_3 = 0$ again, which leads to $\varphi_4 + u\psi_2 = 0$, where $\varphi_4 \stackrel{\text{def}}{=} \lambda.[f,[f,[f,g]]]$ and $\psi_2 \stackrel{\text{def}}{=} \lambda.[g,[f,[f,g]]]$. Since $[g,[f,[f,g]]] = [f,[g,[f,g]]]$, which vanishes thanks to (17), we have $\psi_2 \equiv 0$, so $\varphi_4 \equiv 0$. One more differntiation yields $\varphi_5 + u\psi_3 = 0$, where $\varphi_5 \stackrel{\text{def}}{=} \lambda.[f,[f,[f,[f,g]]]]$ and $\psi_3 \stackrel{\text{def}}{=} \lambda.[g,[f,[f,[f,g]]]]$. One can then compute $[g,[f,[f,[f,g]]]]$, and find that it has components $(-\cos z, -\sin z, 0, 0)$ (so that, in fact, $[g,[f,[f,[f,g]]]] \equiv -f - w[f,g]$). The "singular feedback control law" is then going to be defined on the set Σ_2 determined by the conditions

$$\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = 0,\ \psi_3 \neq 0\,. \tag{18}$$

Now, one easily sees that f, g, $[f,g]$ and $[f,[f,g]]$ are linearly independent at every point, and the identity

$$[f,[f,[f,g]]] = -wf - w^2[f,g] \tag{19}$$

holds. So $g(X)$, $[f,g](X)$, $[f,[f,g]](X)$ and $[f,[f,[f,g]]](X)$ are linearly independent if and only if $w(X) \neq 0$. Therefore the projection of Σ_2 on X space is the hypersurface $\mathcal{H}$ defined by $w = 0$, and for a point $X \in \mathcal{H}$ the choice of $\bar\lambda$ is unique up to a multiplicative constant, since $\bar\lambda$ must be orthogonal to $g(X)$, $[f,g](X)$, and $[f,[f,g]](X)$. Moreover, the number $\delta = \bar\lambda.f(X)$ must be $\neq 0$, and $\bar\lambda$ is uniquely determined up to a *positive* multiplicative constant if we specify in addition the sign of δ. Of the two possible signs, only one leads to an extremal, since the vanishing of the Hamiltonian imposes the condition that $\delta < 0$. Therefore $\bar\lambda$ is completely determined by X, up to a positive multiplicative constant. It then follows that through every $X \in \mathcal{H}$ there passes a *unique* singular arc, in agreement with our earlier calculations. Finally, (19) implies that

$$[f,[f,[f,[f,g]]]] = -w^2[f,[f,g]]\,, \tag{20}$$

so that φ_5 vanishes identically on Σ_2. This implies that the singular feedback is given on Σ_2 by $u \equiv 0$, agreeing, once again, with our earlier calculations. □

3. Junctions

We now analyze the possible junctions between a bang control and a singular one. For simplicity, we only consider the case of a jump from $u = 1$ to $u = 0$. (The other three cases, namely, -1 to 0, 0 to 1, and 0 to -1, are similar.) Assume the jump occurs at time c, so $u(t) = 1$ for t in some interval $[a,c[$, and $u(t) = 0$ for t in some interval $I =]c,b]$, where $a < c < b$. Then we know that $\mu > 0$, and that ω, ζ and w vanish on I. We also know that the constant vector (ξ,η) is nonzero.

Successive differentiation of ω yields

$$\begin{aligned}
\omega_1 &= -\zeta\,,\\
\omega_2 &= -\dot\zeta\\
&= -\xi.\sin z + \eta.\cos z\,,\\
\omega_3 &= -\ddot\zeta\\
&= -(\xi.\cos z + \eta.\sin z)\dot z\\
&= -(\xi.\cos z + \eta.\sin z)w\,,
\end{aligned} \tag{21}$$

where ω_i is the i-th derivative of ω.

The right-hand sides of these three equations are continuous on $[a,b]$ and vanish identically on I. On the other hand $\xi.\cos z + \eta.\sin z = -\mu < 0$ on $[b,c]$, since $H \equiv 0$ on $[a,b]$ and the functions ζ and ω vanish on $[b,c]$. So, after making a larger, if necessary, we may assume that $\xi.\cos z + \eta.\sin z < 0$ on $[a,c]$. Since $\dot w = u = 1$ on $[a,c[$, and $w(c) = 0$, we conclude that $w < 0$ on $[a,c[$. But then $\omega_3 < 0$ on $[a,c[$. Since $\omega_2(c) = 0$, we see that $\omega_2 > 0$ on $[a,c[$. Since $\omega_1(c) = 0$, we can conclude that $\omega_1 < 0$ on $[a,c[$. But then, since $\omega(c) = 0$, we find that $\omega > 0$ on $[a,c[$, which implies $u \equiv -1$ on $[a,c[$.

We have thus reached a contradiction, and established that the jump from $u = 1$ to $u = 0$ is impossible:

Proposition 2. *An optimal control cannot have a jump discontinuity in which one of the values is* 0 *and the other one is* $+1$ *or* -1. □

4. Chattering

At this point, we need a finer analysis to investigate the transitions between bang-bang and singular arcs. Suppose that $a < c < b$, and we are given an optimal arc defined on $[a,b]$, which is singular on $[c,b]$ but such that the switching function ω does not vanish identically on any nontrivial subinterval of $[a,c]$. Let $Z(\omega,[a,c])$ denote the set of zeros of ω on $[a,c]$. Proposition 2 implies that c must be an accumulation point of $Z(\omega,[a,c])$. The simplest possibility would therefore be that "infinite chattering" occurs, i.e. that, if a is sufficiently close to c, then $Z(\omega,[a,c])$ consists of c together with an increasing sequence $\{t_j\}_{j=1}^{\infty}$ of points converging to c. In principle, it is conceivable that $Z(\omega,[a,c])$ could be much less simple (for example, uncountable, and even of positive measure, though of course it has to be closed and with empty interior). We would like to exclude these extremely pathological possibilities, and conclude that all that happens is the infinite chattering.

To do this, we fist observe that it is possible to assume, after a linear change of coordinates in the x,y plane, that the singular arc is contained in the x axis. Then the function $t \mapsto z(t)$ actually vanishes on $[c,b]$. Also, we can assume that $\mu = 1$, and then the value of the Hamiltonian is equal to $\xi + 1$ on $[c,b]$, so $\xi = -1$. Moreover, since $\xi.\sin z - \eta.\cos z \equiv 0$ on $[c,b]$, we have $\eta = 0$. So the system of equations (21) has the simpler form

$$\begin{aligned}
\omega_1 &= -\zeta\,,\\
\omega_2 &= \sin z\,,\\
\omega_3 &= w.\cos z\,.
\end{aligned} \tag{22}$$

Now write $x_1 = \omega$, $x_2 = \omega_1$, $x_3 = z$, $x_4 = w$. Then

$$\begin{aligned}
\dot x_1 &= x_2\,,\\
\dot x_2 &= \sin x_3\,,\\
\dot x_3 &= x_4\,,\\
\dot x_4 &= u\,,\\
u &= -\mathrm{sgn}(x_1)\,,
\end{aligned} \tag{23}$$

where, "$u = -\mathrm{sgn}(x_1)$" is shorthand for "$u = -1$ if $x_1 > 0$, $u = +1$ if $x_1 < 0$, and $u \in [-1,1]$ if $x_1 = 0$." System (23) is a "small perturbation" of the system

$$\begin{aligned} \dot{x}_1 &= x_2\,, \\ \dot{x}_2 &= x_3\,, \\ \dot{x}_3 &= x_4\,, \\ \dot{x}_4 &= u\,, \\ u &= -\mathrm{sgn}(x_1)\,, \end{aligned} \tag{24}$$

which has been widely studied because it occurs in the theory of the solutions of Fuller's problem. The recent book by Zelikin and Borisov [11] is devoted to a detailed study of perturbations of (24). In particular, our system does satisfy the conditions of the fundamental perturbation theorem of Zelikin-Borisov, and one can the conclude that:

Proposition 3. *There exists a subset S_s of $\mathbb{R}^4$ which is forward-invariant under the flow of (23), is topologically a two-dimensional surface, and is the union of the loci of a one-parameter family $\{\gamma_\alpha\}$ of solutions of (23), consisting of trajectories that reach the origin in finite time and correspond to controls u_α that only take the values $+1$ and -1. Moreover, if we translate the origin of time so that each γ_α reaches the origin at time 0, then the set of discontinuities of u_α consists of 0 plus a sequence $\{t_j\}$ (that depends on α) such that $t_j < 0$ and $t_j \uparrow 0$, and the sequence $\{t_j\}$ is such that the lengths $L_j = t_{j+1} - t_j$ of the intervals between successive switchings are asymptotically in decreasing geometric progression, that is,*

$$\lim_{j\to\infty} \frac{L_{j+1}}{L_j} = \kappa \tag{25}$$

for some $\kappa \in]0,1[$. □

Remark 2 There is, of course, a similar result for the time-reversed system, yielding a two-dimensional surface S_u, which is the union of a one-parameter family of trajectories that approach the origin in finite time if they are run backwards.

One should think of S_s and S_u as analogues of the usual stable and unstable manifolds for smooth systems. If, in the equation $\dot{x}_4 = -\mathrm{sgn}(x_1)$, the right-hand side is replaced by a smooth approximation such as $-\frac{2}{\pi}\mathrm{arc}\tan(\gamma x_1)$, with $\gamma >> 0$, then the linearization of the resulting system at the origin is easily seen to have eigenvalues

$$\sigma_k = \left(\frac{\gamma}{2\pi}\right)^{1/4} j^k(1+j)\,,\ k = 1,2,3,4\,, \tag{26}$$

where $j = \sqrt{-1}$. So the linearized system has two stable and two unstable eigenvalues, and then the original, nonlinear system has a 2-dimensional stable manifold S_s^γ and a 2-dimensional unstable manifold S_u^γ. As $\gamma \to +\infty$, the absolute values of the eigenvalues go to ∞, so the rate of exponential decay along S_u^γ goes to infinity as well. So S_u is, roughly, "the stable manifold when $\gamma = +\infty$," and the corresponding "exponential decay rate" is now infinite, meaning that the origin is actually reached in finite time. □

Remark 3 It is desirable to give a direct proof of Proposition 3, simpler than that of Zelikin and Borisov. Such a proof is now in preparation, and will be included in the final version of the conference paper. □

Having established the existence of extremals with infinite chattering, one still has to show that they are optimal. Recently, we have succeded in proving local optimality by suitably embedding the chattering extremals in fields of extremals, so that Hamilton-Jacobi theory can be used. The proof will be included in the final version of the conference paper.

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