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From the Brachystochrone to the Maximum Principle*

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§ 1. Introduction

Optimal control was born in 1696 —300 years ago this year— in the Netherlands, when Johann Bernoulli challenged his contemporaries with the "brachystochrone problem" (BP). The purpose of this paper is to give a brief outline of why this event truly deserves to be called the birth of optimal control, and how the research that began in 1696 has led to modern optimal control theory and, especially, to the maximum principle. In particular, we will argue that, as this path was followed, several opportunities were missed that would have led to much earlier discovery of the maximum principle. In at least one case —that of the formulation of Hamilton's equations— we will attempt to show that the discovery was missed for no reason other than the decision to rewrite an equation in terms of one formalism rather than another one that would have been equally suitable and was also available at the time. As a conclusion, we will show how a modern look at the BP, from the perspective of optimal control theory, can still yield new insights into this 300-year old problem.

Johann Bernoulli's challenge attracted a lot of attention, and some of the greatest mathematicians of the time submitted solutions. The May 1697 issue of *Acta Eruditorum* contains Johann's own solution, as well as a rather different one by his elder brother Jakob, and contributions by Newton, Tschirnhaus, l'Hôpital and Leibniz. So there is no doubt that something important happened in 1696-7. For example, D. J. Struik, referring to the articles published in the May 1697 *Acta Eruditorum*, writes ([3], p. 392) that "these papers opened the history of a new field, the calculus of variations."

We want to go a bit farther, however, and make a case for a 1696 birth of *optimal control theory.* This, naturally, requires some explanation.

The conventional wisdom holds that optimal control theory was born about 40 years ago with the work on the "Pontryagin maximum principle" by L. S. Pontryagin and his group, cf. [2], or perhaps a few years earlier with the work of McShane and Hestenes.

On the other hand, if we take a careful look at those features that make optimal control different from the calculus of variations, we can already find quite a few of them in the BP.

The calculus of variations deals mainly with optimization problems of the "standard" form

$$\begin{aligned}&\text{minimize } I=\int_a^b L(q(t),\dot q(t),t)dt\,,\\ &\text{subject to } q(a)=\bar q \text{ and } q(b)=\hat q\,,\end{aligned}\tag{1}$$

or, equivalently, of the form

$$\begin{aligned}&\text{minimize } I=\int_a^b L(q(t),u(t),t)dt\,,\\ &\text{subject to } q(a)=\bar q\,,\ q(b)=\hat q\,,\\ &\text{and } \dot q(t)=u(t) \text{ for } a\le t\le b\,.\end{aligned}\tag{2}$$

The distinctive feature of these problems is that the minimization of (1) takes place in the space of all curves, so *nothing interesting happens on the level of the set of curves under consideration, and all the nontrivial features of the problem arise because of the Lagrangian* L. (In 20th century mathematics, "all curves" means, of course, all curves in some appropriately chosen function space, such as that of all absolutely continuous curves, or that of all Lipschitz curves.)

Optimal control problems, by contrast, involve a minimization over *a set* $\mathcal{C}$ *of curves which is itself determined by some dynamical constraints.* For example, $\mathcal{C}$ might be the set of all curves $t\mapsto q(t)$ that satisfy a differential equation

$$\dot q(t)=f(q(t),u(t),t)\tag{3}$$

for some choice of the "control function" $t\mapsto u(t)$. (More precisely, since it may happen that a member of $\mathcal{C}$ does not uniquely determine the control u that generates it, we should really be talking *about trajectory-control pairs* $(q(\cdot),u(\cdot))$.)

So in an optimal control problem there are at least *two* objects that give the situation interesting structure, namely, the *dynamics* f and the *cost* functional I to be minimized. In particular, optimal control theory contains, at the opposite extreme from the calculus of variations, problems where the "Lagrangian" L is $\equiv 1$, i.e. completely trivial, and all the interesting action occurs because of the dynamics f. Such problems, in which it is desired to minimize time —i.e. the integral I of (2) with $L\equiv 1$— among all curves $t\mapsto q(t)$ that satisfy endpoint constraints as in (2) and are solutions of (3) for some control $t\mapsto u(t)$, are called *minimum time problems.* It is in these problems that the difference between optimal control and the calculus of variations is most clearly seen, and it is no accident that these were the problems that propelled the development of optimal control in the early 1960's, and that time-optimal control is prominently represented in today's research and in modern optimal control textbooks. (In addition, controlled dynamical systems of the form (3) are interesting objects to study even in the absence of a minimization problem, which is why *control theory* is a much richer subject than optimal control, which deals with only one of many kinds of important control problems.) More recently, it has become apparent that, as R. Hermann, R. Brockett, H. Hermes, A. Krener, C. Lobry, and others had argued since the 1960's and early

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70's, a controlled dynamics such as (3) is, really, a *family of vector fields*, and such an object has to be studied using the machinery of differential geometry, which allows us to associate an algebraic structure to a family of vector fields, by looking at their Lie brackets.

Within this framework, it is clear that *Johann Bernoulli's problem, as posed in the Acta Eruditorum, is a true minimum time problem of the kind that is studied today in optimal control theory.* This, incidentally, is why it is no coincidence that Johann Bernoulli called his fastest path the *brachystochrone* (from the Greek words *βράχιστος*: shortest, and *χρόνος*: time).

Moreover, *the BP is the first one ever to deal with a dynamical behavior and explicitly ask for the optimal selection of a path.*[†]

Finally, and most importantly, we will argue that *a large part of the subsequent history of the calculus of variations can be best understood as the search for the simplest and most general statement of the necessary conditions for optimality, which is provided by the maximum principle of optimal control theory.*

§ 2. Bernoulli, Euler, Lagrange, and Legendre

Johann's Bernoulli's BP is that of finding, for two given points A and B in a vertical plane, "the orbit AMB of the movable point M, along which it, starting from A, and under the influence of its own weight, arrives at B in the shortest possible time." In modern language, if we choose x and y axes in the plane with the y axis pointing downwards, use (a_1, a_2) and (b_1, b_2) to denote, respectively, the coordinates of the end points A and B, and fix a number $E \in \mathbb{R}$, we want to find a Lipschitz function $f : [0,T] \to \mathbb{R}^2$, with components $f_1(t), f_2(t)$, that satisfies the constraints $f(0) = A$, $f(T) = B$, and $\frac{1}{2}(|\dot{f}_1(t)|^2 + |\dot{f}_2(t)|^2) = E + g f_2(t)$ for almost all $t \in [0,T]$, and is such that T has the least possible value. Here g is the gravitational constant, and the number $E + ga_2$ is the initial kinetic energy of the body.

Obviously, such a path must be entirely contained in the closed half-plane $H_+(\eta) = \{(x,y) : y \geq -\eta\}$, where $\eta = \frac{E}{g}$.

It turns out that the brachystochrone —i.e. the optimal path— is a cycloid. When $E = 0$, it is the curve described by a point P in a circle that rolls without slipping on on the x axis, in such a way that P passes through A and then through B, without hitting the x axis in between. (It is easy to see that this defines the cycloid uniquely.)

Johann Bernoulli derived this fact using Fermat's minimum time principle, together with the fact that this principle implies Snell's law about the refraction of light. If we imagine for a moment that instead of dealing with the motion of a moving body we are dealing with a light ray, then the dynamical constraint gives us a formula for the "speed of light" c as a function of position: $c = \sqrt{2E + 2gy}$. From now on, let us change coordinates so that $E = 0$. (This amounts to shifting the x axis vertically.) Then $\eta = 0$, so our feasible paths live in the half plane $H_+ \stackrel{\text{def}}{=} H_+(0)$. Also, let us rescale —or "change our choice of physical units"— so that $2g = 1$. Then our problem is exactly equivalent to that of determining the light rays —i.e. the minimum-time paths— in a plane medium where the speed of light c varies continuously as a function of position according to the formula $c = \sqrt{y}$. Bernoulli solved this problem by dividing the half-plane H_+ into horizontal strips of height δ, treating c in each strip as a constant, finding the light rays for the discretized problem by means of Snell's law, and then taking the limit as $\delta \downarrow 0$. In the limit, he obtained the differential equation

$$y'(x) = \sqrt{\frac{C - y(x)}{y(x)}}\,, \tag{4}$$

for the y-coordinate of the brachystochrone as a function of its x-coordinate, where C is a constant. The curves parametrized by

$$x(\varphi) = x_0 + \frac{C}{2}(\varphi - \sin\varphi)\,,\ \ y(\varphi) = \frac{C}{2}(1 - \cos\varphi)\,, \tag{5}$$

with $0 \leq \varphi \leq 2\pi$, satisfy (4). It is easily seen that these equations specify the cycloid generated by a point P on a circle of diameter C that rolls without slipping on the horizontal axis, in such a way that P is at $(x_0, 0)$ when $\varphi = 0$. Moreover, it is also easy to check that

(*) given two points A and B in H_+ there is exactly one curve in this family passing through A and B.

The argument that we have presented is the one of Johann Bernoulli, and Equation (4) is essentially the one that he wrote in his paper, followed by the statement "from which I conclude that the *Brachystochrone* is the ordinary *Cycloid*." We should keep in mind, however, that he was *not* using the convention, customary in contemporary mathematics, according to which the symbol $\sqrt{r}$ stands for the *nonnegative* square root of r. What he meant was, clearly, what we would write today as $y'(x) = \pm\sqrt{\frac{C-y(x)}{y(x)}}$ or, even better, as

$$y(x)(1 + y'(x)^2) = \text{constant}\,. \tag{6}$$

Even with the more accurate rewriting (6), the differential equation derived by Johann Bernoulli also has spurious solutions, not given by (5). (For example, for any $\bar{y} > 0$, the constant function $y(x) = \bar{y}$ is a solution, corresponding to $C = \bar{y}$.)

These spurious solutions can be eliminated in a number of ways. For example, the calculus of variations approach, based on writing the Euler-Lagrange equation (8), gives the equation

$$1 + y'(x)^2 + 2y(x)y''(x) = 0\,, \tag{7}$$

which is *stronger* than (6), since (6) is equivalent to $y' + y'^3 + 2yy'y'' = 0$, i.e. to $y'(1 + y'^2 + 2yy'') = 0$, whose solutions are those of (7) plus the spurious solutions. It is easy to verify that the solutions of (7) are *exactly* the curves given by (5), without any extra spurious solutions, showing that *the Euler-Lagrange method gives better results than Johann Bernoulli's approach.*

[†] Other "calculus of variations" problems had been considered earlier. For example, the isoperimetric problem goes back to classical Greece, and Newton had solved a minimal drag problem in 1685. But in both cases the curves to be computed were not thought of as paths of a moving body or particle.

To show how the calculus of variations leads to (7), it suffices to put the BP in the "standard" form (1). To do this, we consider curves $y = y(x)$ in the x, y plane, where $y : [a_1, b_1] \mapsto \mathbb{R}$ is a function. Then the dynamical constraint of Bernoulli's problem can be written (with $E = 0$ and $2g = 1$ as before) as $dx^2 + dy^2 = y\, dt^2$, which gives $dt = \frac{\sqrt{dx^2+dy^2}}{\sqrt{y}} = L(y, \dot{y})dx$, where $L(y, u) = \frac{\sqrt{1+u^2}}{\sqrt{y}}$, and we are using x rather than t for the time variable, and writing $\dot{y}$ for dy/dx. So Johann Bernoulli's problem becomes that of minimizing the integral $\int_{a_1}^{b_1} L(y(x), \dot{y}(x))dx$ subject to $y(a_1) = a_2$ and $y(b_1) = b_2$. The Euler-Lagrange equation for this problem turns out to be precisely (7).

Johann Bernoulli's BP was the first of a series of curve minimization problems studied by him, his brother Jakob, his student Euler, and Lagrange. Euler studied *general* problems and derived what is now known as the "Euler equation." Lagrange then found the necessary condition known today as the "Euler-Lagrange equation," or "Euler-Lagrange system," which says that, if $t \to q_*(t)$ is a solution of (1), then the equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q} \tag{8}$$

must hold along q_*. Equation (8) makes perfect sense and is a necessary condition for optimality for a vector-valued variable q as well as for a scalar one. It can be written as a system: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i}$, $i = 1, \ldots, n$. Alternatively, we can regard Equation (8) as a vector identity, in which $q = (q^1, \ldots, q^n)$ is an n-dimensional vector, and $\frac{\partial L}{\partial q}$, $\frac{\partial L}{\partial \dot{q}}$ stand for the n-tuples $\left(\frac{\partial L}{\partial q^1}, \ldots, \frac{\partial L}{\partial q^n}\right)$, $\left(\frac{\partial L}{\partial \dot{q}^1}, \ldots, \frac{\partial L}{\partial \dot{q}^n}\right)$.

A modern mathematician might be troubled by the use of $\dot{q}$ both as an "independent variable" and as a function of time evaluated along a trajectory, and might prefer to write (8) as

$$\boxed{\frac{d}{dt}\left[\frac{\partial L}{\partial u}\Big(\mathbf{q}_*(t)\Big)\right] = \frac{\partial L}{\partial q}\Big(\mathbf{q}_*(t)\Big)\,,\ \ i = 1, \ldots, n\,,} \tag{9}$$

where $\mathbf{q}_*(t) = (q_*(t), \dot{q}_*(t), t)$, and the "Lagrangian" $L(q, u, t)$ is a function of $q \in \mathbb{R}^n$, $u \in \mathbb{R}^n$, $t \in \mathbb{R}$. This makes it clear that the left-hand side of (8) is computed by first evaluating $\frac{\partial L}{\partial \dot{q}}$ "treating $\dot{q}$ as an independent variable," then plugging in $q_*(t)$ and $\dot{q}_*(t)$ for q, $\dot{q}$, and finally differentiating with respect to t.

The Euler-Lagrange system (8) —or (9)— only gave conditions for stationarity, i.e., for the first variation of I to be zero. The next natural step was to look at the second variation, and this was done by Legendre, who found an additional necessary condition for a minimum. His condition is

$$\boxed{\frac{\partial^2 L}{\partial \dot{q}^2}\Big(\mathbf{q}_*(t)\Big) \geq 0 \text{ (i.e. } \frac{\partial^2 L}{\partial u^2}\Big(\mathbf{q}_*(t)\Big) \geq 0)\,.} \tag{10}$$

With an appropriate reinterpretation, (10) is also a necessary condition in the vector case: we just have to read (10) as asserting that the Hessian matrix $\left\{\frac{\partial^2 L}{\partial u^i \partial u^j}\Big(\mathbf{q}_*(t)\Big)\right\}_{1 \leq i,j \leq n}$ has to be nonnegative definite.

§ 3. Two critical forks: Hamilton & Weierstrass

We are now close to the first and most critical fork in the road, involving the work of W. R. Hamilton. In a sense, the issue at stake will seem rather trivial, just a matter of rewriting the Euler-Lagrange system in a different formalism. But sometimes formalism can make a tremendous difference. We will argue that: (a) *more than one rewriting was possible,* (b) *it matters a lot which rewriting is chosen,* (c) *Hamilton's own choice may not have been the best one.*

To understand what happened and what could have happened but did not, let us try to make sense of the two necessary conditions for a minimum that have been presented so far. The Legendre condition is clearly the second-order necessary condition for a minimum of a function (namely, $L(q(t), u, t)$ as a function of u). But (8) does not look at all like the first-order condition for a minimum of that same function. It is natural to ask whether there might be a way to relate the two conditions. Is it possible that both can be expressed as necessary conditions for a minimum of one and the same function? The answer is "yes," and understanding how this can be done leads straight to optimal control theory, the maximum principle, and far-reaching generalizations of the classical theory.

To see this, define a function $H(q, u, p, t)$ of three vector variables q, u, p in $\mathbb{R}^n$, and of $t \in \mathbb{R}$, by letting

$$\boxed{H(q, u, p, t) = \langle p, u \rangle - L(q, u, t)\,.} \tag{11}$$

Then define

$$p_*(t) = \frac{\partial L}{\partial u}\Big(\mathbf{q}_*(t)\Big)\,. \tag{12}$$

It is then clear that $\frac{\partial H}{\partial p} = u$, so along our optimal curve q_* —writing $\mathbf{qp}_*(t) = (q_*(t), \dot{q}_*(t), p_*(t), t)$— we have $\frac{dq_*}{dt}(t) = \frac{\partial H}{\partial p}\Big(\mathbf{qp}_*(t)\Big)$. Also, $\frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q}$, so (9), with $p_*(t)$ defined by (12), says that $\frac{dp}{dt}(t) = -\frac{\partial H}{\partial q}\Big(\mathbf{qp}_*(t)\Big)$. Finally, $\frac{\partial H}{\partial u} = p - \frac{\partial L}{\partial u}$, so (12) says: $\frac{\partial H}{\partial u}\Big(\mathbf{qp}_*(t)\Big) = 0$. The above system of three equations, which can be written more concisely as

$$\boxed{\frac{dq}{dt} = \frac{\partial H}{\partial p}\,,\ \ \frac{dp}{dt} = -\frac{\partial H}{\partial q}\,,\ \ \frac{\partial H}{\partial u} = 0\,,} \tag{13}$$

is exactly equivalent to (8), provided that H is defined as in (11).

In our view, Formula (11) is the definition that Hamilton should have given for the Hamiltonian, and Equations (13) are "Hamilton's equations as he should have written them."

However, what Hamilton actually wrote was (in our notation)

$$\frac{dq}{dt} = \frac{\partial \mathcal{H}}{\partial p}\,,\quad \frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial q}\,, \tag{14}$$

where $\mathcal{H}(q, p, t)$ is a function of p, q and t alone, defined by the formula $\mathcal{H}(q, p, t) = \langle p, \dot{q} \rangle - L(q, \dot{q}, t)$, which resembles (11) but is not at all the same. The difference is that in Hamilton's definition $\dot{q}$ is supposed to be treated *not* as an independent variable, but as a function of q, p, t, defined implicitly by the equation $p = \frac{\partial L}{\partial \dot{q}}(q, \dot{q}, t)$. It is easy to see that, *if the map* $(q, \dot{q}, t) \mapsto (q, p, t)$

defined by this equation can be inverted, then (14) is equivalent to (13).

It should be clear from the above discussion that the Hamiltonian reformulation of the Euler-Lagrange equations in terms of H (the "control Hamiltonian") is at least as natural as the one Hamilton used, and arguably even simpler. Moreover, it has at least one obvious advantage, namely,

(A1) *the control version of Hamilton's equations is equivalent to the Euler-Lagrange system under completely general conditions, whereas the classical version only makes sense in the special case when the transformation $(q, \dot{q}, t) \mapsto (q, p, t)$ can be inverted, at least locally, to solve for $\dot{q}$ as a function of q, p, t.*

We now show that (A1) is not the only advantage of the control view over the classical one. To see this, let us take another look at Legendre's condition (10). Clearly, (10) is equivalent to $\frac{\partial^2 H}{\partial u^2}\Big(\mathbf{qp}_* q(t)\Big) \leq 0$, i.e. to $\frac{\partial^2 H}{\partial \dot{q}^2}\Big(\mathbf{qp}_*(t)\Big) \leq 0$. If we write this side by side with the third equation of (13), we get:

$$\boxed{\frac{\partial H}{\partial u} = 0 \qquad \text{and} \qquad \frac{\partial^2 H}{\partial u^2} \leq 0\,.} \tag{15}$$

Staring at this for a few seconds, it becomes clear that *what has to be going on here is that H has a maximum as a function of u*. So we state this as a conjecture.

CONJECTURE M: besides (13) (or the equivalent form (8)), an additional necessary condition for optimality is that $H(q_*(t), u, p_*(t), t)$, as a function of u, have a maximum at $\dot{q}_*(t)$ for each t. □

The point of this is that Conjecture M is an extremely natural consequence of writing Hamilton's equations "as Hamilton should have done it." One can reasonably guess that, if Hamilton had actually done it, then he himself or some other 19th century mathematician would have written (15) and be led by it to the conjecture.

It turns out that Conjecture M is true, and knowing this leads to vast generalizations. But before we get there, we must move to the second critical fork in the road, and discuss the work of Weierstrass, who essentially discovered and proved Conjecture M, but did it using a language that obscured the simplicity of the result, and for that reason missed some profound implications of his discovery.

Weierstrass considered the problem of minimizing an integral I of the form $I = \int_a^b L(q(s), \dot{q}(s))\,ds$ for Lagrangians L such that

(W) *$L(q, \dot{q})$ is positively homogeneous with respect to the velocity $\dot{q}$ (that is, $L(q, \alpha\,\dot{q}) = \alpha L(q, \dot{q})$ for all $q, \dot{q}$ and all $\alpha \geq 0$) and does not depend on time.*

When Weierstrass imposed Condition (W) on his Lagrangians, he was doing so "without loss of generality," since it is not hard to see that every minimization problem of the form (1) can be transformed into one where (W) holds. However, "without loss of generality" is a dangerous phrase, and does not at all entail "without loss of insight." We will show that this restriction, together with the dominant view that Hamilton's equations had to be written in the form (14), may have hidden from Weierstrass the true meaning and the far-reaching implications of his new necessary condition.

Weierstrass introduced the "excess function"

$$\mathcal{E}(q, u, \bar{u}) = L(q, \bar{u}) - \frac{\partial L}{\partial u}(q, u) \cdot \bar{u}\,, \tag{16}$$

depending on three sets of independent variables q, u and $\bar{u}$. He then proved his *side condition*:

(SC) *For a curve $s \mapsto q_*(s)$ to be a solution of the minimization problem, the function $\mathcal{E}$ has to be ≥ 0 when evaluated for $q = q_*(s)$, $u = \dot{q}_*(s)$, and a completely arbitrary $\bar{u}$.*

Notice that, for Lagrangians with Property (W), $L(q, u) = \frac{\partial L}{\partial u}(q, u) \cdot u$, so Weierstrass could equally well have written his excess function as

$$\mathcal{E}(q, u, \bar{u}) = L(q, \bar{u}) - \frac{\partial L}{\partial u}(q, u) \cdot \bar{u} - \Big(L(q, u) - \frac{\partial L}{\partial u}(q, u) \cdot u\Big)\,. \tag{17}$$

Using $p = \frac{\partial L}{\partial u}(q, u)$ as in (12), we see that

$$\mathcal{E}(q, u, \bar{u}) = \Big(L(q, \bar{u}) - \langle p, \bar{u}\rangle\Big) - \Big(L(q, u) - \langle p, u\rangle\Big)\,, \tag{18}$$

which the reader will immediately recognize as

$$\mathcal{E}(q, u, \bar{u}) = H(q, u, p) - H(q, \bar{u}, p)\,, \tag{19}$$

where H is our "control Hamiltonian." So Weierstrass' condition, expressed in terms of the control Hamiltonian, simply says that

(MAX) *along an optimal curve $t \mapsto q_*(t)$, if we define $p_*(t)$ via (12), then, for every t, the value $u = \dot{q}_*(t)$ must maximize the (control) Hamiltonian $H(q_*(t), u, p_*(t), t)$ as a function of u.*

In Weierstrass' formulation, the condition was stated in terms of the excess function, for the special Lagrangians that satisfy (W). But, if one rewrites Weierstrass's condition as we have done, in terms of H, then one can take a general Lagrangian, not satisfying (W), transform the minimization problem into one in Weierstrass's form, write the Weierstrass condition in the form (MAX), and then undo the transformation and go back to the original problem. The result is (MAX), as written, with the control Hamiltonian of the original problem. So the Weierstrass condition, if reformulated as in (MAX), is valid for all problems, with exactly the same statement. For comparison, notice that if one applies the same procedure to the formulation in terms of $\mathcal{E}$, then one also ends up with a statement valid for all problems, but the expression for $\mathcal{E}$ is now much more complicated, since one can no longer use (16) instead of (17).

Moreover, (MAX) can be considerably simplified. Indeed, the requirement that $p(t)$ be defined via (12) is now redundant: if $H(q_*(t), u, p_*(t), t)$, regarded as a function of u, has a maximum at $u = \dot{q}_*(t)$, then $\frac{\partial H}{\partial u}(q_*(t), \dot{q}_*(t), p_*(t), t)$ has to vanish, so $p_*(t)$ has to be given by (12). Moreover, the vanishing of $\frac{\partial H}{\partial u}(q_*(t), \dot{q}_*(t), p_*(t), t)$ is also one of the conditions of (13). So we can state (13) and (MAX) together:

(NCO) *If a curve $t \mapsto q_*(t)$ is a solution of the minimization problem (1), then there has to exist a function $t \mapsto p_*(t)$ such that (p_* is absolutely continuous and) the following three conditions hold for all t:*

$$\dot{q}_*(t) = \frac{\partial H}{\partial p}(\mathbf{qp}_*(t)), \quad \dot{p}_*(t) = -\frac{\partial H}{\partial q}(\mathbf{qp}_*(t)), \quad (20)$$

$$H(\mathbf{qp}_*(t)) = \max_u H(q_*(t), u, p_*(t), t). \quad (21)$$

As a version of the necessary conditions for optimality, (NCO) encapsulates in one single statement the combined power of the Euler-Lagrange necessary conditions and the Weierstrass side condition as well, of course, as the Legendre condition, which obviously follows from (MAX). Notice the elegance and economy of language achieved by this unified statement: there is no need to bring in an extra entity called the "excess function." Nor does one need to include a formula specifying how $p_*(t)$ is defined, since (21) does this automatically.

It is clear that (MAX) —or, more precisely, the Weierstrass side condition part of (MAX)— is exactly Conjecture M. So we can now add two new items to our list of advantages of the "control formulation" of Hamilton's equations over the classical one:

(A2) *using the control Hamiltonian, it would have been an obvious next step to write Legendre's condition in "Hamiltonian form," as in (15), and this would have led immediately to the formulation of Conjecture M, a proof of which would then have been found soon after.*

(A3) *using the control Hamiltonian, the Weierstrass side condition has a much simpler statement, not requiring the introduction of an "excess function," and can be combined with the Hamilton equations into an elegant unified formulation (NCO) of the necessary conditions for optimality, in which there is no need to write an equation defining p_*.*

But this is by no means the end of our story. There is much more to the new formulation (NCO) than just elegance and simplicity. Quite remarkably, **in (NCO) the derivatives with respect to the u variable have completely disappeared.** This makes it reasonable to state a new conjecture:

CONJECTURE M2: (NCO) should still be a necessary condition for optimality for problems with a constraint $\dot{q} \in U \subseteq \mathbb{R}^n$, and with L not required to be differentiable with respect to u. □

Conjecture M2 can be easily tested by looking at some simple toy problems where the answer is reasonably easy to guess directly. For example, we can test the differentiability part of Conjecture M2 by looking at the following:

PROBLEM 1: Given $L > 0$, and $\alpha > 0$ find a real-valued (Lipschitz) function $t \mapsto x(t)$ on the interval $[0, L]$ that satisfies $x(0) = 1$, and $x(L) = 1$, and minimizes the integral $\int_0^L |\dot{x}(t)|^\alpha \, dt$ among all such functions. This looks exactly like a calculus of variations problem of the classical sort, with a Lagrangian $L(x, \dot{x}, t) = |\dot{x}|^\alpha$ (i.e. $L(x, u, t) = |u|^\alpha$), except that L is no longer everywhere differentiable[‡] with respect to u. For this problem the classical Euler-Lagrange equation cannot even be written, let alone solved. But *if we apply (NCO) as stated we arrive at the correct solution*, namely, the curve $x_*(t) \equiv 1$. □

We can also test the other part of Conjecture M2, by looking at a problem where the range of u is no longer the whole space.

PROBLEM 2: Given $L > 0$, find a real-valued (Lipschitz) function $t \mapsto x(t)$ on $[0, L]$ that satisfies $x(0) = 1$, $x(L) = 1$, and $|\dot{x}(t)| \leq 1$ for almost all $t \in [0, L]$, and minimizes the integral $\int_0^L x(t)^2 dt$ among all such functions. Once again, this looks exactly like a calculus of variations problem of the classical type —with a Lagrangian $L(x, \dot{x}, t) = x^2$— except that the derivative $\dot{x}$ is required to satisfy an "inequality constraint" $|\dot{x}| \leq 1$. It is easy to see that there is no way at all to satisfy the Euler-Lagrange equation together with the boundary conditions. On the other hand, if we apply (NCO) formally, making the sensible guess that in this case the maximization with respect to u should be made over the set $U = [-1, 1]$ of permissible values of u, then we get the right answer. (The details are simple and we omit them.) □

These two elementary examples show that (NCO), which is none other than the combination of the classical Euler-Lagrange condition and the Weierstrass side condition, is a surprisingly powerful tool, *provided only that it is properly reinterpreted as we have done.* The formal application of version of (NCO) solves at least some problems that do not fit within the framework of the classical calculus of variations, either because of nondifferentiability of L with respect to u, or because of the presence of an inequality constraint $\dot{q} \in U$ on the velocity.

Finally, now that we have liberated ourselves from the constraint that L be differentiable with respect to u, it ought to be possible for u —i.e. $\dot{q}$— to be anything, and (NCO) will still work. Once this is understood, the next natural step is to allow $\dot{q}$ to be even "more arbitrary," for example a general function of some other variable u, and of q and t. So instead of letting $\dot{q}$ be u, we can write $\dot{q} = f(q, u, t)$ for a general function $f(q, u, t)$. Naturally, the expression $\langle p, u \rangle$ that occurs in (11) should now be replaced by $\langle p, f(q, u, t) \rangle$. We then get to

CONJECTURE M3: (NCO) should still be a necessary condition for optimality even for problems where q is restricted to satisfy a differential equation $\dot{q} = f(q, u, t)$, with the "control function" $t \mapsto u(t)$ taking values in some set U and allowed to be a "completely arbitrary" U-valued function of t, provided that the formula

$$H(q, u, p, t) = \langle p, f(q, u, t) \rangle - L(q, u, t) \quad (22)$$

is used to define the Hamiltonian. □

As before, we test this with a simple problem, namely, a "soft landing in minimum time problem":

[‡] And not Lipschitz either, if $\alpha < 1$, so it would not do to try to write the Euler-Lagrange system using a Clarke generalized gradient of L with respect to u.

PROBLEM 3: Suppose we have a point in $\mathbb{R}^2$ with coordinates x, y, moving according to $\dot{x} = y$, $\dot{y} = u$, $|u| \leq 1$. Starting at $x = \bar{x}$, $y = \bar{y}$, we want to get to $x = 0$, $y = 0$ in minimum time. It is not hard to guess directly what the solution is, and to prove rigorously that the guess is right. Using (NCO) with the Hamiltonian defined according to Subconjecture M3.a, we also find the correct solution, provided that we first transform our problem, which is intrinsically a "variable time interval problem," into one with a fixed time interval. This can be done by introducing a 'pseudotime' parameter s, not to be thought of as true physical time, just making sure that $dt/ds > 0$. We write $dt/ds = v$, $dx/ds = vy$, $dy/ds = vu$, where the controls u, v satisfy $v > 0$ and $|u| \leq 1$. With such a reparametrization, we can always work on a fixed pseudotime interval, for example $[0, 1]$. The cost functional is $\int_0^1 v(s)ds$, so our new Lagrangian is v. Conjecture M3 gives the correct solution. □

§ 4. The maximum principle

Conjecture M3 is essentially the maximum principle, except for a minor adjustment. To see why some adjustment is needed, consider a fourth toy problem:

PROBLEM 4: An $L > 0$ is given, and we want to maximize the integral $\int_0^L \dot{x}(t)^2 dt$ among all Lipschitz functions $x(\cdot) : [0, L] \mapsto \mathbb{R}$ that satisfy the velocity constraint $\dot{x}(t) \geq 0$ for almost all t. as well as the endpoint conditions $x(0) = x(L) = 1$. In this case the Lagrangian is $L(x, u, t) = -u^2$, so the Hamiltonian is $H(x, u, p, t) = pu + u^2$. It is easy to see that no trajectories at all satisfy the conditions of (NCO). However, the curve $t \mapsto x(t) = 1$ is the solution of our optimization problem. □

So Conjecture M3 is not true. It turns out, however, that only a minor change suffices to make it true. All we have to do is introduce a new p-variable p_0 —known as the "abnormal multiplier.[§]"— and write the Hamiltonian as

$$H(q, u, p, p_0, t) = \langle p, f(q, u, t)\rangle - p_0 L(q, u, t) . \quad (23)$$

Everything we had done until now corresponded to taking $p_0 = 1$. We now impose, instead, the weaker requirements that $\dot{p}_0 = 0$ (i.e. p_0 is a constant), $p_0 \geq 0$, and $(p, p_0) \neq (0, 0)$. With Conjecture M3 adjusted in this way, we have, finally, reached the *Maximum Principle*:

(MP) *For a problem of minimizing a cost functional $I = \int_a^b L(q(t), u(t), t)\, dt$, subject to a dynamical constraint (3), and constraints $q(a) = \bar{q}$, $q(b) = \hat{q}$, $u(t) \in U$, where q takes values in $\mathbb{R}^n$ —or in a open subset Q of $\mathbb{R}^n$— and the time interval $[a, b]$ is fixed, a necessary condition for a function $t \mapsto u_*(t)$ on $[a, b]$ and a corresponding solution $t \mapsto q_*(t)$ of (3) to be a minimizer is that there exist a function $t \mapsto p_*(t) \in \mathbb{R}^n$ and a constant $p_0 \geq 0$ such that, for $t \in [a, b]$,*

(1) $(p_*(t), p_0) \neq (0, 0)$;

(2) $(-\dot{p}_*(t), q_*(t)) = \nabla H(\Xi_*(t))$;

(3) $H(\Xi_*(t)) = \max_{u \in U} H(q_*(t), u, p_*(t), p_0, t)$.

where we have written $\Xi_*(t) = (q_*(t), u_*(t), p_*(t), p_0, t)$, and the Hamiltonian $H(q, u, p, p_0, t)$ is given by (23). □

And we hope to have convinced *all* readers, even those who are not control theorists, that (MP) is a very natural conclusion. It should be clear from our discussion that (MP) could essentially have been guessed almost immediately from "Hamilton's equations as Hamilton should have written them," together with the Legendre condition, and would have been an almost obvious conjecture to make once the Weierstrass side condition is known, *if only the "correct" Hamiltonian formalism had been used.*

§ 5. The brachystochrone 300 years later

We conclude by returning to the BP, this time from the perspective of modern optimal control theory.

Due to lack of space, we will limit ourselves to four remarks.

First of all, it is clear that Johann Bernoulli's problem can be formulated in optimal control terms: the motion takes place in the x, y plane, the dynamics is given by[¶] $\dot{x} = u\sqrt{|y|}$, $\dot{y} = v\sqrt{|y|}$, and the control is a 2-dimensional vector (u, v) subject to the constraint $(u, v) \in U = \{(u, v) : u^2 + v^2 = 1\}$.

A simple computation shows that the maximum principle, applied to this problem, gives the correct answer, namely the cycloids, without any "spurious solutions."

Second, we point out that the question of the *existence* of optimal trajectories[‖], which is delicate when one uses the calculus of variations approach[**], becomes trivial in the control setting: to prove that any two points can be joined by a minimizer, it suffices to show that any two points can be joined by *some* feasible path. (Here the fact that $\sqrt{|y|}$ is not Lipschitz is essential. If, for example, the "speed of light" was $|y|$ rather than $\sqrt{|y|}$, then no point in the x axis can be joined by a feasible path to a point not in the x axis.) Once this is established, a trivial application of Ascoli's theorem gives the desired conclusion.

Our *third* remark concerns the *reflected BP*. Suppose we want to find the light rays for a medium which is the whole plane, with "speed of light" equal to $\sqrt{|y|}$. Can the necessary conditions help us find the solution? The problem here is that the dynamical law of the BP has a right-hand side which is not Lipschitz with respect to (x, y). So none of the versions of (MP) that require a Lipschitz reference vector field apply.

§The need for the abnormal multiplier had already been noticed by Bolza in 1913, cf. [1].

¶ It is better to shift gears slightly and regard the BP as defined on the *whole plane* rather than a half-plane. This is why we are now using the absolute value of y.

‖Proving existence is a crucial step for a rigorous proof that Bernoulli's cycloids are optimal: the necessary conditions for optimality imply that they are the only possible minimizers. So, if one knows that a minimizer exists, it follows that it is a cycloid.

**We thank F. H. Clarke for bringing this point to our attention.

It turns out, however, that a recent version of the maximum principle (cf. [4]) applies, since this result does not require Lipschitz continuity —or even continuity— of the right-hand side, and works as long as the reference trajectory arises from a semidifferentiable flow. We refer the reader to [4] for the details.

Our *fourth* and last observation is much more "differential geometric." We first point out that the mathematical formulation of the BP, as presented so far, is not completely natural, because it takes it for granted that we know that energy is conserved. A much better way to pose the problem would be to write down the equations of motion that correspond verbatim to Johann Bernoulli's problem as he stated it in June 1696. One should then let the mathematical analysis lead us to the discovery of energy conservation and the simplification resulting from it.

The true equations of motion are

$$\dot{x} = v, \quad \dot{y} = w\,, \quad \dot{v} = uw\,, \quad \dot{w} = -uv - g\,. \tag{24}$$

Here x and y are the coordinates of our moving point, and v, w are the components of the velocity vector. The requirement that the point is freely falling "under the influence of its own weight" means that the force effectively acting on it is equal to a vector proportional to $[0, -g]$ plus a "virtual force" that does no work, i.e. is perpendicular to the velocity. Equation (24) captures these requirements, by introducing a virtual force vector of the form $[uw, -uv]$, where u is an arbitrary "control," taking values in $\mathbb{R}$.

Using $q = [x, y, v, w]^\dagger$, we get the equation

$$\dot{q} = F(q) + uG(q)\,, \tag{25}$$

where $F = [v, w, 0, -g]^\dagger$, and $G = [0, 0, w, -v]^\dagger$.

This is a *4-dimensional system*. From now on, we will work on the set $\tilde{Q} = \{(x, y, v, w) : v^2 + w^2 \neq 0\}$. If we apply (MP), we get *nothing*. This is because, if one computes Lie brackets, one finds that

$$[F, [F, G]] = \psi_1 F + \psi_2 G + \psi_3 [F, G]\,, \ [G, [F, G]] = F, \tag{26}$$

where the ψ_i are smooth functions of q. From this one can easily show that every iterated Lie bracket of F and G is a linear combination of F, G, and $[F, G]$ with smooth coefficients. Since $F(q)$, $G(q)$, and $[F, G](q)$ are linearly independent at each point $q \in \tilde{Q}$, we can conclude that the Lie algebra of vector fields generated by F and G is 3-*dimensional* at each point. This means that, at least locally, there is a nontrivial integral of motion, i.e. a function with nonzero gradient which is constant along all integral curves of F and G, and then also along all solutions of (25). This integral of motion can be computed and turns out to be the energy E. It is not hard to show that, whenever a control system satisfies a nontrivial "holonomic constraint" such as $E =$ constant, then the maximum principle is uninformative, because *every trajectory is an extremal.*

On the other hand, using the integral of motion E, we can regard the 4-dimensional q-space as foliated by the 3-dimensional level hypersurfaces of E, and conclude that every trajectory is contained in one of the leaves. If L is a leaf, then the maximum principle on manifolds can be applied to the problem restricted to L, and the conclusion turns out to be exactly the same as that for the unrestricted problem, except that now the nontriviality condition says that $p_*(t)$ cannot be orthogonal to L at $q_*(t)$. (In differential-geometric terms, $p_*(t)$ is really a *covector* on L at $q_*(t)$, and has to be nontrivial as such. Equivalently, if one insists on regarding $p_*(t)$ as a vector in $\mathbb{R}^4$, then it should not be orthogonal to the tangent space to L at $q_*(t)$.)

Applying the maximum principle with this stronger nontriviality constraint yields a formula for the optimal control in *feedback form*, namely, $u = -\psi_1(q)$. (The reader is urged to carry out all these computations explicitly as an exercise, and verify that the final result is once again Johann Bernoulli's family of cycloids.)

The specification of a value of E lowers the dimensionality of the problem from 4 to 3. Since u is completely unrestricted, there is no obvious argument showing that any two points in the same leaf L can be joined by an optimal path. Our calculation of an optimal feedback control law shows that there is a smooth vector field V on L such that every optimal trajectory in L is in fact an integral curve of V. How can these facts be reconciled with our earlier discussion, in which the problem —for a fixed value of E— was *two-dimensional*, and for every pair A, B of points in $H_+(\eta)$ there was a solution? (This last fact says, in particular, that through a point A there pass not just one optimal curve, but a one-parameter family of such paths, whose union covers the whole 2-dimensional region $H_+(\eta)$.)

The answer is that, once we have specified E, and consequently singled out a leaf L, the kinetic energy at $A = (a_1, a_2)$ is determined, and equals $E + ga_2$. This determines the length of the velocity vector at A, but its direction is still arbitrary. That means that A is in fact the projection of a circle in L, namely, the set $\{(x, y, v, w) : x = a_1, y = a_2, v^2 + w^2 = 2(E + ga_2)\}$. (In modern terminology, L is, up to normalization, the unit tangent bundle of $H_+(\eta)$.) Through each point of this circle there passes an optimal path, namely, an integral curve of V. The projections of these curves form a one-parameter family of curves passing through A, which is, of course, the family of cycloids given by (5).

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