A General Result on the Stabilization of Linear Systems Using Bounded Controls*

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Abstract

We present two constructions of controllers that globally stabilize linear systems subject to control saturation. The only conditions imposed are the obvious necessary ones, namely that no eigenvalues of the uncontrolled system have positive real part and that the standard stabilizability rank condition holds. We use essentially arbitrary saturations σ, subject only to the requirement that (i) σ is locally Lipschitz, (ii) $s\sigma(s) > 0$ whenever $s \neq 0$, (iii) σ is differentiable at 0 and $\sigma'(0) > 0$, and (iv) $\liminf_{|s|\to\infty} |\sigma(s)| > 0$.

1. Introduction

We consider linear time-invariant continuous-time systems

$$\Sigma: \quad \dot{x} = Ax + Bu\,, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ for some integers n and m, for which the control values $u(t)$ are restricted to lie in a bounded set $U \subseteq \mathbb{R}^m$. We assume that U contains zero in its interior.

The study of such systems is motivated by the possibility of actuator saturation or constraints on actuators, reflected sometimes also in bounds on available power supply or rate limits. These systems cannot be naturally dealt with within the context of standard (algebraic) linear control theory, but are ubiquitous in control applications. To quote the recent textbook [7] (page 171): "saturation is probably the most commonly encountered nonlinearity in control engineering."

We will present results on *global stabilization* of systems of this form, concentrating on several explicit architectures for controllers. Of course, there are general limits as to what can be achieved, no matter what type of control law is allowed. Asymptotic null-controllability of the system (that is, the existence of open-loop controls that steer each state to the origin, in the limit as $t \to \infty$) is equivalent to the following pair of algebraic conditions:

(a) all eigenvalues of A have nonpositive real part, and

(b) all eigenvalues of the uncontrollable part of Σ have strictly negative real parts (that is, the pair (A, B) is stabilizable in the ordinary sense).

(The theory of controllability of linear systems with bounded controls is a well-studied topic; see e.g. the fundamental paper [5], as well as the different, more algebraic approach discussed in [8].) Note that under Conditions (a) and (b) there may very well be nontrivial Jordan blocks corresponding to critical eigenvalues, so that the system $\dot{x} = Ax$ need not be asymptotically stable, or even Lyapunov-stable. This is what makes the problem interesting, and allows inclusion of examples of practical importance such as systems involving integrators.

In very special cases, including all one- and two-dimensional systems, stabilization is possible by simply using a saturated linear feedback law of the type:

$$u = \overline{\sigma}(Fx)\,, \tag{2}$$

where F is an $m \times n$ matrix and $\overline{\sigma}$ is a function that computes a saturation in each coordinate of the vector Fx, for instance, $u_i = \tanh((Fx)_i)$, or $u_i = \mathrm{sat}((\mathrm{Fx})_\mathrm{i})$ where,

$$\mathrm{sat(s)} = \mathrm{sign(s)}\min\{|\mathrm{s}|, 1\}\,. \tag{3}$$

A similar solution is possible in the case of systems that are neutrally stable (i.e. such that the Jordan form of A has no off-diagonal ones corresponding to imaginary eigenvalues), using the "Jurdjevic-Quinn" approach. (See [2] and [6].) Thus it is natural to ask if simple control laws as in Equation (2) can also be used for more general systems. This was negatively answered in a paper by A.T. Fuller as far back as the late 1960s. He showed in [1] that already for triple integrators such saturated linear feedback is not sufficient, at least under certain assumptions on the saturation σ. (A stronger negative result, which applies to basically arbitrary σ's, was more recently given, independently, in [12].)

The fact that linear feedback laws when saturated can lead to instability has motivated a large amount of research. (See for instance [3] and [4], and references therein, for estimates of the size of the regions of attraction that result when using linear saturated controllers.) Here we take a different approach. Rather than working with linear saturated control laws $u = \sigma(Fx)$, and trying to show that they are globally stabilizing, or to estimate their domains of attraction, we allow more general bounded (and hence necessarily nonlinear) laws. This is not a new idea since, for example, bounded stabilizing control laws have been derived using optimal control techniques. Optimal control laws, however, may be highly discontinuous. *If, on the other hand, we ignore optimality questions, then it reasonable to expect to be able to find more regular controllers with a simpler structure.* Indeed, taking this point of view, we were able to obtain, in the previous work [11], a general result on bounded stabilization by means of infinitely dif-

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ferentiable feedback laws. The result in [11] holds under the weakest possible conditions, namely those reviewed above for open-loop asymptotic null-controllability.

Unfortunately, the construction in [11] relied on a complicated and far from explicit inductive procedure. On the other hand, since saturated linear feedbacks suffice for up to two dimensions, it is natural to look, in higher dimensions, for other simple control structures consisting, for example, of *linear combinations* and *compositions* of saturation nonlinearities. (In the language of neural networks, one wants control laws that are implementable by feedforward nets using "hidden layers", rather than the "perceptrons" represented by (2)). Recently, and motivated in part by [11] and [12], Andrew Teel showed in [13] how, in the particular case of single-input multiple integrators, such combinations of saturations are indeed sufficient to obtain stabilizing feedback controllers. In this paper we obtain a general solution of the same type, for the full case treated in [11]. The approach is explicit and constructive. Our solution is inspired by the techniques introduced in [13] for the particular case treated there, but the details are far more complicated, due to the possibilities of having both multiple inputs and (perhaps multiple) purely imaginary eigenvalues, and to the need to deal with arbitrary saturations.

In addition to the design that employs linear combinations and compositions of saturation nonlinearities, we also exhibit a different design that uses linear combinations of saturated linear functions. (In neural network terms, it involves a "single hidden layer net.")

Our result was first announced in [14], where we considered a very special type of feedbacks for which (i) the saturations are exactly linear near 0 and (ii) when a system has pure imaginary eigenvalues, saturations with up to three different slopes may be needed. In this work, the saturations that we use are essentially arbitrary, subject only to the four conditions listed above in the abstract. So, mathematically, the results in this paper show that one can use analytic functions to implement feedback laws —a conclusion that would not follow from the results in [14] and [13]— and, from an engineering point of view, they insure that rather general components can be employed, subject only to mild conditions which are robustly satisfied. In the terminology of current "artificial neural networks" technology, our results allow the implementation of feedback controllers using very general types of activation (neuron characteristic) functions.

2. Statement of the Main Results

We first define $\mathcal{S}$ to be the class of all functions σ from $\mathbb{R}$ to $\mathbb{R}$ such that

- σ is locally Lipschitz,
- $s\sigma(s) > 0$ whenever $s \neq 0$,
- σ is differentiable at 0 and $\sigma'(0) > 0$,
- $\liminf_{|s|\to\infty} |\sigma(s)| > 0$.

For any finite sequence $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_k)$ of functions in $\mathcal{S}$, we define a set $\mathcal{F}_n(\boldsymbol{\sigma})$ of functions f from $\mathbb{R}^n$ to $\mathbb{R}$ inductively as follows:

- if $k = 0$ (i.e. if $\boldsymbol{\sigma}$ is the empty sequence), then $\mathcal{F}_n(\boldsymbol{\sigma})$ consists of one element, namely, the zero function from $\mathbb{R}^n$ to $\mathbb{R}$,
- $\mathcal{F}_n(\sigma_1)$ consists of all the functions $h : \mathbb{R}^n \to \mathbb{R}$ of the form $h(x) = \sigma_1(g(x))$, where $g : \mathbb{R}^n \to \mathbb{R}$ is linear,
- for every $k > 1$, $\mathcal{F}_n(\sigma_1, \cdots, \sigma_k)$ is the set of all functions $h : \mathbb{R}^n \to \mathbb{R}$ that are of the form $h(x) = \sigma_k(f(x) + cg(x))$, with f linear, $g \in \mathcal{F}_n(\sigma_1, \cdots, \sigma_{k-1})$, and $c \geq 0$.

We also define $\mathcal{G}_n(\boldsymbol{\sigma})$ to be the class of functions $h : \mathbb{R}^n \to \mathbb{R}$ given by

$$h(x) = a_1\sigma_1(f_1(x)) + a_2\sigma_2(f_2(x)) + \cdots + a_k\sigma_k(f_k(x)),$$

where $f_1, \cdots, f_k$ are linear functions and $a_1, \cdots, a_k$ are nonnegative constants such that $a_1 + \cdots + a_k \leq 1$.

Next, for an m-tuple $\boldsymbol{l} = (l_1, \cdots, l_m)$ of nonnegative integers, define $|\boldsymbol{l}| = l_1 + \cdots + l_m$. For a finite sequence $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_{|\boldsymbol{l}|}) = (\sigma_1^1, \cdots, \sigma_{l_1}^1, \cdots, \sigma_1^m, \cdots, \sigma_{l_m}^m)$ of functions in $\mathcal{S}$, we let $\mathcal{F}_n^{\boldsymbol{l}}(\boldsymbol{\sigma})$ (respectively $\mathcal{G}_n^{\boldsymbol{l}}(\boldsymbol{\sigma})$) denote the set of all functions $h : \mathbb{R}^n \to \mathbb{R}^m$ that are of the form $(h_1, \cdots, h_m)$, where $h_i \in \mathcal{F}_n(\sigma_1^i, \cdots, \sigma_{l_i}^i)$ (respectively $h_i \in \mathcal{G}_n(\sigma_1^i, \cdots, \sigma_{l_i}^i)$) for $i = 1, 2, \cdots, m$. (It is clear that $\mathcal{F}_n^{\boldsymbol{l}}(\boldsymbol{\sigma}) = \mathcal{F}_n(\boldsymbol{\sigma})$, $\mathcal{G}_n^{\boldsymbol{l}}(\boldsymbol{\sigma}) = \mathcal{G}_n(\boldsymbol{\sigma})$ if $m = 1$.)

Let $\delta > 0$. We say that a function $f : [0, +\infty) \to \mathbb{R}^n$ is *eventually bounded by* δ (and write $|f| \leq_{\text{ev}} \delta$), if there exists $T > 0$ such that $|f(t)| \leq \delta$ for all $t \geq T$. Given an n-dimensional system $\mathcal{E} : \dot{x} = f(x)$, we say that $\mathcal{E}$ is *SISS* (small-input small-state) if for every $\varepsilon > 0$ there is a $\delta > 0$ such that, if $e : [0, +\infty) \to \mathbb{R}^n$ is bounded, measurable, and eventually bounded by δ, then every solution $t \to x(t)$ of $\dot{x} = f(x) + e(t)$ is eventually bounded by ε. For $\Delta > 0, N > 0$, we say that $\mathcal{E}$ is $SISS_L(\Delta, N)$ if, whenever $0 < \delta \leq \Delta$, it follows that, if $e : [0, +\infty) \to \mathbb{R}^n$ is bounded, measurable, and eventually bounded by δ, then every solution of $\dot{x} = f(x) + e(t)$ is eventually bounded by $N\delta$. The system is $SISS_L$ if it is $SISS_L(\Delta, N)$ for some $\Delta > 0, N > 0$. For a system $\dot{x} = f(x, u)$, we say that a feedback $u = k(x)$ is *stabilizing* if 0 is a globally asymptotically stable equilibrium of the closed-loop system $\dot{x} = f(x, k(x))$. If, in addition, this closed-loop system is $SISS_L$, then we will say that k is $SISS_L$*-stabilizing.*

For a square matrix A, we let $\mu(A)$ denote the number of eigenvalues z of A such that $\operatorname{Re} z = 0$ and $\operatorname{Im} z \geq 0$, counting multiplicities. Our main result is as follows:

Theorem 1 *Let Σ be a linear system $\dot{x} = Ax + Bu$ with state space $\mathbb{R}^n$ and input space $\mathbb{R}^m$. Assume that Σ is asymptotically null-controllable, i.e. that all the eigenvalues of A have nonpositive real parts and all the eigenvalues of the uncontrollable part of A have strictly negative real parts. Let $\mu = \mu(A)$. Let $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_\mu)$ be an arbitrary sequence of bounded functions belonging to $\mathcal{S}$. Then there exists an m-tuple $\boldsymbol{l} = (l_1, \cdots, l_m)$ of nonnegative integers such that $|\boldsymbol{l}| = \mu$, for which there are $SISS_L$-stabilizing feedback*

$$u = -k_{\mathcal{F}}(x) \tag{4}$$
$$u = -k_{\mathcal{G}}(x) \tag{5}$$

such that (a) $k_{\mathcal{F}} \in \mathcal{F}_n^{\boldsymbol{l}}(\boldsymbol{\sigma})$, $k_{\mathcal{G}} \in \mathcal{G}_n^{\boldsymbol{l}}(\boldsymbol{\sigma})$ and (b) the linearizations of the closed-loop systems are asymptotically stable.

We will say that (4), (5) are feedbacks of Type $\mathcal{F}$, $\mathcal{G}$, respectively.

3. Discrete Time

There is also a discrete-time version of our theorem. As an illustration, we quote a result from [15]. Let $\mathcal{S}$ be now the subclass of the previous class $\mathcal{S}$ consisting just of the functions of the type $\sigma(s) = \delta\,\mathrm{sat}(s/\delta)$, where $\mathrm{sat}(s) = \mathrm{sign}(s)\min\{|s|, 1\}$. We define sets of functions $\mathcal{F}_n(\boldsymbol{\sigma})$ and $\mathcal{G}_n(\boldsymbol{\sigma})$ in a manner entirely analogous to what was done in continuous-time. Let $\nu(A)$ be the number of eigenvalues z of A such that $|z| = 1$ and $\mathrm{Im}\, z \geq 0$, counting multiplicities. Then we have the following (see [15], Chapter 6, where a stronger result, which involves a notion of "converging-input converging-state" is proved):

Theorem 2 *Let Σ be a linear system $x(t+1) = Ax(t) + Bu(t)$ with state space $\mathbb{R}^n$ and input space $\mathbb{R}^m$. Assume that Σ is asymptotically null-controllable, i.e. that all the eigenvalues of A have magnitude ≤ 1 and all the eigenvalues of the uncontrollable part of A have magnitude < 1. Let $\nu = \nu(A)$. Then, for each $\epsilon > 0$, there exists a sequence $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_\nu)$ of functions belonging to $\mathcal{S}$, with $\|\boldsymbol{\sigma}\| \leq \epsilon$, and an m-tuple $\boldsymbol{l} = (l_1, \cdots, l_m)$ of nonnegative integers such that $|\boldsymbol{l}| = \nu$, for which there are globally stabilizing feedbacks $u = -k_{\mathcal{F}}(x)$ and $u = -k_{\mathcal{G}}(x)$ such that $k_{\mathcal{F}} \in \mathcal{F}_n^{\boldsymbol{l}}(\boldsymbol{\sigma})$ and $k_{\mathcal{G}} \in \mathcal{G}_n^{\boldsymbol{l}}(\boldsymbol{\sigma})$.*

4. Preliminaries

In this section we present two technical lemmas that will be needed for the proofs of our main results.

Lemma 4.1 *Consider an n-dimensional linear single-input system*

$$\Sigma : \qquad \dot{x} = Ax + bu \; . \tag{6}$$

Suppose that (A, b) is a controllable pair and that all the eigenvalues of A have zero real part. Fix a $\nu > 0$. Then:

(i) If 0 is an eigenvalue of A, then there is a linear change of coordinates $Tx = (y_1, \cdots, y_n)' = (\bar{y}', y_n)'$ of $\mathbb{R}^n$ that puts Σ in the form

$$\begin{aligned} \dot{\bar{y}} &= A_1\bar{y} + b_1(y_n + \nu u) \; , \\ \dot{y}_n &= u \; , \end{aligned} \tag{7}$$

where the pair (A_1, b_1) is controllable and y_n is a scalar variable.

(ii) If A has an eigenvalue of the form $i\omega$, with $\omega > 0$, then there is a linear change of coordinates $Tx = (y_1, \cdots, y_n)' = (\bar{y}', y_{n-1}, y_n)'$ of $\mathbb{R}^n$ that puts Σ in the form:

$$\begin{aligned} \dot{\bar{y}} &= A_1\bar{y} + b_1(y_n + \nu u) \; , \\ \dot{y}_{n-1} &= \omega y_n \; , \\ \dot{y}_n &= -\omega y_{n-1} + u \; , \end{aligned} \tag{8}$$

where the pair (A_1, b_1) is controllable and y_{n-1}, y_n are scalar variables.

Lemma 4.2 *Let $\omega > 0$. Then for every $\sigma \in \mathcal{S}$ there exist functions $\varepsilon \to \delta_0(\varepsilon), \delta_1(\varepsilon), \delta_2(\varepsilon)$ from $(0, \infty)$ to $(0, \infty)$ such that*

(I) whenever e_0, e_1, e_2 are bounded measurable real-valued functions on $[0, \infty)$ such that

$$\limsup_{t \to +\infty} |e_i(t)| < \delta_i(\varepsilon) \quad \text{for } i = 0, 1, 2 \; , \tag{9}$$

then, if $\gamma = (x(\cdot), y(\cdot)) : [0, \infty) \to \mathbb{R}^2$ is any solution of the system

$$\begin{aligned} \dot{x} &= \omega y + e_1(t) \; , \\ \dot{y} &= -\omega x - \sigma(y - e_0(t)) + e_2(t) \; , \end{aligned} \tag{10}$$

it follows that

$$\limsup_{t \to +\infty} \|\gamma(t)\| < \varepsilon \; ; \tag{11}$$

and

(II) there exist an $\bar{\varepsilon} > 0$ and $\nu_0, \nu_1, \nu_2 > 0$ such that $\delta_i(\varepsilon) = \frac{\varepsilon}{\nu_i}$ for $i = 0, 1, 2$ and $0 < \varepsilon \leq \bar{\varepsilon}$.

The proof of Lemma 4.1 is included in [14]. Here, we provide a sketch of the proof of Lemma 4.2.

It clearly suffices to find $\delta_0(\varepsilon), \delta_1(\varepsilon), \delta_2(\varepsilon)$ such that

(I) Inequality (11) holds whenever $\gamma : [0, \infty) \to \mathbb{R}^2$ is a solution of (10) for some triple of functions e_0, e_1, e_2 such that $\|e_i\|_{L^\infty} < \delta_i(\varepsilon)$ for $i = 0, 1, 2$.

The hypotheses on σ imply that $|\sigma|$ is bounded away from 0 on the complement of every interval $(-\theta, \theta)$, $\theta > 0$. Define

$$\varphi(\theta) = \inf\{|\sigma(s)| : |s| \geq \theta\} \; , \tag{12}$$

$$\psi(\theta) = \sup\{|\sigma(s)| : |s| \leq 3\theta\} \; . \tag{13}$$

Then $\psi(\theta) \geq \varphi(\theta) > 0$.

For each triple $(\delta_0, \delta_1, \delta_2)$ with $\delta_i > 0$, let $\Gamma(\delta_0, \delta_1, \delta_2)$ denote the set of all $\gamma : [0, \infty) \to \mathbb{R}^2$ that are solutions of (10) for some triple (e_0, e_1, e_2) of functions from $[0, \infty)$ to $\mathbb{R}$ such that $\|e_i\|_{L^\infty} < \delta_i$ for $i = 0, 1, 2$.

Pick positive numbers $\alpha, \beta, \theta, \delta_0, \delta_1, \delta_2, A, B, C$ such that

$$\delta_0 \leq \theta \; , \tag{14}$$

$$\delta_1 \leq 2\theta \; , \tag{15}$$

$$\delta_1 \leq \omega\beta \; , \tag{16}$$

$$A\delta_2 + \alpha\omega \leq \varphi(\theta) \; , \tag{17}$$

$$\beta < 2\theta \; , \tag{18}$$

$$\alpha(\delta_1 + \beta\omega) < (2\theta - \beta)B\delta_2 \; , \tag{19}$$

$$\alpha < C\delta_2 \; , \tag{20}$$

and

$$A - 1 \geq \max\{B, C\} \; . \tag{21}$$

Define the functions

$$V_{1,\alpha,\beta}(x, y) = (x + \alpha)^2 + (y - \beta)^2 \; , \tag{22}$$

$$V_{2,\alpha,\beta}(x, y) = (x + \alpha)^2 + (y + \beta)^2 \; . \tag{23}$$

Then we have

(II) $V_{1,\alpha,\beta}$ is strictly decreasing along every trajectory $\gamma \in \Gamma(\delta_0, \delta_1, \delta_2)$ which is contained in the region

$$\mathcal{R}_1(\theta) = \{(x, y) : x \leq 0, y \geq 2\theta\} \; , \tag{24}$$

and $V_{2,\alpha,\beta}$ is strictly decreasing along every $\gamma \in \Gamma(\delta_0, \delta_1, \delta_2)$ which is contained in

$$\mathcal{R}_2(\theta) = \{(x, y) : x \geq 0, y \geq 2\theta\} \; . \tag{25}$$

For each $L > 2\alpha$, we define a closed contour $\mathcal{C}(L)$ as follows. We let $P_0(L)$ be the point $(-L, 2\theta)$. Let $\tilde{C}_1(L)$ be the circle through $P_0(L)$ with center $(-\alpha, \beta)$. Then define $C_1(L)$ to be the arc of $\tilde{C}_1(L)$ obtained by moving clockwise along $C_1(L)$ from $P_0(L)$ to the point $P_1(L)$ where $\tilde{C}_1(L)$ intersects the half-line $\{(x,y) : x = 0, y \geq 2\theta\}$. Then let $\tilde{C}_2(L)$ be the circle through $P_1(L)$ with center $(-\alpha, -\beta)$, and define $C_2(L)$ to be the arc of $\tilde{C}_2(L)$ obtained by moving clockwise along $C_2(L)$ from $P_1(L)$ to the point $P_2(L)$ where $\tilde{C}_2(L)$ intersects the half-line $\{(x,y) : x \geq 0, y = 2\theta\}$. Then let $C_3(L)$ be the straight-line segment from $P_2(L)$ to $P_3(L)$, where $P_3(L) = (L, -2\theta)$. The concatenation $\mathcal{C}^+(L) = C_1(L) \cup C_2(L) \cup C_3(L)$ is an arc from $P_0(L)$ to $P_3(L)$. By reflecting with respect to the origin, we get another arc $\mathcal{C}^-(L) = C_4(L) \cup C_5(L) \cup C_6(L)$ from $P_3(L)$ to $P_0(L)$. The concatenation $\mathcal{C}^+(L) \cup \mathcal{C}^-(L)$ is our closed contour $\mathcal{C}(L)$. We let $\mathcal{K}(L)$ be the closed region bounded by $\mathcal{C}(L)$, so $\mathcal{K}(L)$ is compact.

We now let $\Gamma_c(\delta_0, \delta_1, \delta_2)$ be the set of those solutions γ of (10) that correspond to continuous functions $e_i : [0, \infty) \to \mathbb{R}$ such that $\|e_i\|_{L^\infty} < \delta_i$ for $i = 0, 1, 2$. If $\gamma \in \Gamma_c(\delta_0, \delta_1, \delta_2)$ then $\dot{\gamma}(t)$ exists for all t. Then from (II), it is easy to prove

(III) if $\gamma \in \Gamma_c(\delta_0, \delta_1, \delta_2)$ and $\gamma(t) \in \Big(C_1(L) \cup C_2(L)\Big) - \{P_0(L), P_2(L)\}$, then $\gamma(s) \in \mathrm{Int}(\mathcal{K}(L))$ for $s > t$, s near t.

The segment $C_3(L)$ is a level curve of the function $V_{3,\xi}$, given by

$$V_{3,\xi}(x, y) = \xi x + y\,. \tag{26}$$

We now impose the further requirement that

$$\beta \leq \alpha/4\,, \tag{27}$$

and let L_{crit} (the "critical" L) be defined by

$$L_{\mathrm{crit}} = \frac{8\theta^2}{\alpha} + \frac{4\theta\delta_1}{\alpha\omega} + 2\alpha + \frac{\delta_2}{\omega} + \frac{\psi(\theta)}{\omega}\,. \tag{28}$$

Then, if $L > L_{\mathrm{crit}}$, we have

$$\dot{V}_{3,\xi} < 0\,. \tag{29}$$

Therefore,

(IV) if $L > L_{\mathrm{crit}}$, and $\gamma \in \Gamma_c(\delta_0, \delta_1, \delta_2)$ is such that $\gamma(t) \in \mathcal{K}(L)$ for some $t \in [0, \infty)$, it follows that $\gamma(s) \in \mathrm{Int}(\mathcal{K}(L))$ for $s > t$, s near t.

Using (IV), it is easy to strengthen the conclusion to "$\gamma(s) \in \mathrm{Int}(\mathcal{K}(L))$ for all $s > t$," and we can even drop the restriction that $\gamma \in \Gamma_c(\delta_0, \delta_1, \delta_2)$, and allow arbitrary $\gamma \in \Gamma(\delta_0, \delta_1, \delta_2)$. With further analysis, we can show that

(V) if $L > L_{\mathrm{crit}}$ and $\gamma \in \Gamma(\delta_0, \delta_1, \delta_2)$, then there exists a $T \geq 0$ such that $\gamma(t) \in \mathcal{K}(L)$ for all $t \geq T$.

We now observe that $\mathcal{K}(L)$ is contained in the disk $\mathcal{D}(R) = \{p \in \mathbb{R}^2 : \|p\| \leq R\}$, provided that $R \geq L + 2\theta$. Therefore (V) implies:

(VI) if $\alpha, \beta, \theta, \delta_0, \delta_1, \delta_2, A, B, C$ are positive numbers such that (14)-(21) and (27) hold, L_{crit} is defined by (28), and $R > L_{\mathrm{crit}} + 2\theta$, then every $\gamma \in \Gamma(\delta_0, \delta_1, \delta_2)$ satisfies

$$\limsup_{t \to +\infty} \|\gamma(t)\| < R\,. \tag{30}$$

To conclude our proof, it suffices to point out that, for every $\varepsilon > 0$, one can always choose α, β, θ, δ_0, δ_1, δ_2, A, B, C so as to satisfy (14)-(21) and (27), and such that $L_{\mathrm{crit}} + 2\theta < \varepsilon$. □

The following corollary is an immediate consequence of Lemma 4.2:

Corollary 4.3 *For $n = 1$ or 2, let J be an $n \times n$ skew-symmetric matrix. Let $b = 1$ if $n = 1$, and $b = (0,1)'$ if $n = 2$. Assume that (J, b) is a controllable pair. Then for every $\sigma \in \mathcal{S}$ there exist functions $\varepsilon \to \delta_1(\varepsilon), \delta_2(\varepsilon)$ from $(0, +\infty)$ to $(0, +\infty)$ such that*

(I) whenever $e_1 : [0, +\infty) \to \mathbb{R}$ and $e_2 : [0, +\infty) \to \mathbb{R}^n$ are bounded measurable functions such that

$$\limsup_{t \to +\infty} |e_i(t)| < \delta_i(\varepsilon) \quad \text{for } i = 1, 2\,,$$

then, if $\gamma : [0, \infty) \to \mathbb{R}^n$ is any solution of the system

$$\dot{x} = Jx - \sigma(x_n - e_1(t))b + e_2(t)\,, \tag{31}$$

where $x \in \mathbb{R}^n$, it follows that

$$\limsup_{t \to +\infty} \|\gamma(t)\| < \varepsilon\,;$$

and

(II) there exist an $\bar{\varepsilon} > 0$ and $\nu_1, \nu_2 > 0$ such that $\delta_i(\varepsilon) = \frac{\varepsilon}{\nu_i}$ for $i = 1, 2$ and $0 < \varepsilon \leq \bar{\varepsilon}$. □

5. The Proof of Theorem 1

First, we notice that under the conditions of the theorem there exists a linear change of coordinates of the state space that transforms Σ into the block form

$$\Sigma: \quad \begin{cases} \dot{x}_1 &= A_1 x_1 + B_1 u, \quad x_1 \in \mathbb{R}^{n_1}\,, \\ \dot{x}_2 &= A_2 x_2 + B_2 u, \quad x_2 \in \mathbb{R}^{n_2}\,, \end{cases}$$

where (i) $n_1 + n_2 = n$, (ii) all the eigenvalues of A_1 have zero real part, (iii) all the eigenvalues of A_2 have negative real part, and (iv) (A_1, B_1) is a controllable pair. Suppose that we find a $SISS_L$-stabilizing feedback $u = k(x_1)$ of Type $\mathcal{F}$ or Type $\mathcal{G}$ for the system $\dot{x}_1 = A_1 x_1 + B_1 u$ such that the linearization of the resulting closed-loop system is asymptotically stable. Then it is clear that this same feedback law will work for Σ as well. Thus, in order to stabilize Σ, it is enough to stabilize the "critical subsystem" $\dot{x}_1 = A_1 x_1 + B_1 u$. Without loss of generality, in our proof of the theorem we will suppose that Σ is already in this form.

We start with the single-input case, and use by induction on the dimension of the system. As discussed earlier, we may assume that all the eigenvalues of A have zero real part and the pair (A, B) is controllable.

For dimension zero, there is nothing to prove. Now assume given a single-input n-dimensional system, $n \geq 1$, and suppose that Theorem 1 has been established for all single-input systems of dimension $\leq n - 1$. We consider separately the two possibilities: (i) zero is an eigenvalue of A, and (ii) zero is not an eigenvalue of A. Recall that $\mu = \mu(A)$. We want to prove that for any finite sequence $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_\mu)$ of bounded functions in $\mathcal{S}$ there are $SISS_L$-stabilizing feedbacks $u = -k_{\mathcal{F}}(x)$ and $u = -k_{\mathcal{G}}(x)$ such that $k_{\mathcal{F}} \in \mathcal{F}_n(\boldsymbol{\sigma})$, $k_{\mathcal{G}} \in \mathcal{G}_n(\boldsymbol{\sigma})$, and the linearizations of the resulting closed-loop systems are asymptotically stable.

Assume that $\sigma_\mu'(0) = \kappa > 0$. In Case (i), we apply Part (i) of Lemma 4.1 and rewrite our system in the form

$$\begin{array}{rcl} \dot{\bar{y}} & = & A_1\bar{y} + (y_n + \frac{1}{\kappa}u)b_1 \,, \\ \dot{y}_n & = & u \,, \end{array} \tag{32}$$

where $\bar{y} = (y_1, \cdots, y_{n-1})'$. (Note that if $n = 1$, only the second equation occurs.) In Case (ii), since $n > 0$, A has an eigenvalue of the form $i\omega$, with $\omega > 0$. So we apply Part (ii) of Lemma 4.1 and make a linear transformation that puts Σ in the form

$$\begin{array}{rcl} \dot{\bar{y}} & = & A_1\bar{y} + (y_n + \frac{1}{\kappa}u)b_1 \,, \\ \dot{y}_{n-1} & = & \omega y_n \,, \\ \dot{y}_n & = & -\omega y_{n-1} + u \,, \end{array} \tag{33}$$

where $\bar{y} = (y_1, y_2, \cdots, y_{n-2})'$. (Naturally, in the special case when $n = 2$, the first equation will be missing.) So, in any case, by using Lemma 4.1, we can rewrite our system in the form

$$\begin{array}{rl} \dot{\bar{y}} & = A_1\bar{y} + (y_n + \frac{1}{\kappa}u)b_1 \,, \\ \dot{\tilde{y}} & = J\tilde{y} + ub_0 \,, \end{array} \tag{34}$$

where J is a skew-symmetric matrix, (J, b_0) is a controllable pair, and in Case (i) we have $\tilde{y} = y_n, b_0 = 1$, in Case (ii) we have $\tilde{y} = (y_{n-1}, y_n)', b_0 = (0, 1)'$.

Let

$$u = -\sigma_\mu(y_n - \xi v) - \eta v \,, \tag{35}$$

where ξ and η are constants such that $\xi\eta = 0, \xi + \eta = 1$, and v is to be chosen further. Define $g(s) = \sigma_\mu(s) - \kappa s$, where $\kappa = \sigma'(0)$. Then

$$y_n + \frac{1}{\kappa}u = (\xi - \frac{1}{\kappa}\eta)v - \frac{1}{\kappa}g(y_n - \xi v) \,.$$

Therefore the first equation of (34) is written as

$$\dot{\bar{y}} = A_1\bar{y} + vb_2 - g(y_n - \xi v)b_3 \,, \tag{36}$$

where $b_2 = (\xi - \frac{1}{\kappa}\eta)b_1$, $b_3 = \frac{1}{\kappa}b_1$. Notice that $\xi - \frac{1}{\kappa}\eta \neq 0$, therefore (A_1, b_2) is still a controllable pair. From the inductive hypothesis we know that there exist $\bar{k}_{\mathcal{F}} \in \mathcal{F}_n(\sigma_1, \cdots, \sigma_{\mu-1})$ and $\bar{k}_{\mathcal{G}} \in \mathcal{G}_n(\sigma_1, \cdots, \sigma_{\mu-1})$, for simplicity we use $\bar{k}$ for both cases, such that the system $\dot{\bar{y}} = A_1\bar{y} - \bar{k}(\bar{y})b_2$ is $SISS_L(\bar{\Delta}, \bar{N})$ for some $\bar{\Delta}, \bar{N} > 0$, and the linearization of the closed-loop system is asymptotically stable. (For instance, in Case (ii), there exists a $SISS_L$-stabilizing feedback $u = -\bar{k}_{\mathcal{F}}$ for $\dot{\bar{y}} = A_1\bar{y} + b_2 v$ such that $\bar{k}_{\mathcal{F}} \in \mathcal{F}_{n-2}(\sigma_1, \cdots, \sigma_{\mu-1})$. With the natural extension, $\bar{k}_{\mathcal{F}}$ can be also viewed as a function from $\mathbb{R}^n$ to $\mathbb{R}$. Thus $\bar{k} \in \mathcal{F}_n(\sigma_1, \cdots, \sigma_{\mu-1})$.) Let $\lambda > 0$. Then $\dot{\bar{y}} = A_1\bar{y} - \lambda\bar{k}(\frac{\bar{y}}{\lambda})b_2$ is $SISS_L(\lambda\bar{\Delta}, \bar{N})$.

Besides, since $\bar{k}$ is bounded and locally Lipschitz, and $\bar{k}(0) = 0$, it follows that there exist positive numbers L, M such that $|\bar{k}(\bar{y})| \leq L|\bar{y}|$ and $|\bar{k}(\bar{y})| \leq M$ for all $\bar{y}$.

On the other hand, the second equation of (34) can be rewritten as

$$\dot{\tilde{y}} = J\tilde{y} - \sigma_\mu(y_n - \xi v)b_0 - \eta v b_0 \,. \tag{37}$$

From Corollary 4.3, we see that there exist $\Delta_1, \nu_1, \nu_2 > 0$ such that, whenever $\delta \leq \Delta_1$, and e_1, e_2 are bounded measurable functions such that $|e_i| \leq_{\text{ev}} \frac{\delta}{\nu_i}$, $i = 1, 2$, it follows that all the solutions of $\dot{\tilde{y}} = J\tilde{y} - \sigma_n(y_n - e_1) + e_2$ will be eventually bounded by δ.

Recall that σ_μ is differentiable at 0. So for any $\varepsilon > 0$ there exists $\Delta_2 > 0$ such that

$$|g(s)| \leq \varepsilon|s| \quad \text{for } |s| \leq \Delta_2 \,. \tag{38}$$

Now, fix a positive number $\alpha < 1$. (To get a feedback of Type $\mathcal{F}$, we choose α close to 1; while to get a feedback of Type $\mathcal{G}$, we choose α close to 0.) Choose $\varepsilon > 0$ so that

$$\varepsilon\Big(M(\xi\nu_1 + \eta\nu_2 + \xi) + \Delta_1\Big)||b_3|| < \alpha\bar{\Delta} \,, \tag{39}$$

and

$$\varepsilon\bar{N}L(\xi\nu_1 + \eta\nu_2 + \xi) < \frac{1}{4} \,. \tag{40}$$

Then there exists $\Delta_2 > 0$ such that (38) is satisfied. (If σ_μ is linear around 0, then we can take $\varepsilon = 0$. Therefore (39) and (40) are valid and there exists $\Delta_2 > 0$ such that (38) is still satisfied.)

Now, choose $\delta_0 > 0$ so that

$$\delta_0 \leq \min\{\Delta_1, \alpha\Delta_2\} \,, \tag{41}$$

and let $\lambda > 0$ satisfy

$$\lambda \leq 1 \,, \tag{42}$$

$$\lambda\xi M \leq (1 - \alpha)\Delta_2 \,, \tag{43}$$

$$\lambda(\xi\nu_1 + \eta\nu_2)M \leq \alpha\delta_0 \,. \tag{44}$$

Then define

$$\Delta = \min\{\delta_0, \frac{(1 - \alpha)\lambda\delta_0}{\nu_2}, (1 - \alpha)\lambda\bar{\Delta}\} \,.$$

Let $v = -\lambda\bar{k}(\frac{\bar{y}}{\lambda})$ in (36) and (37). Then it is enough to show that the resulting closed-loop system

$$\begin{array}{rl} \dot{\bar{y}} & = A_1\bar{y} - \lambda\bar{k}(\frac{\bar{y}}{\lambda})b_2 - g(y_n + \lambda\xi\bar{k}(\frac{\bar{y}}{\lambda}))b_3 \,, \\ \dot{\tilde{y}} & = J\tilde{y} - \sigma_\mu(y_n + \lambda\xi\bar{k}(\frac{\bar{y}}{\lambda}))b_0 + \lambda\eta\bar{k}(\frac{\bar{y}}{\lambda})b_0 \end{array} \tag{45}$$

is $SISS_L(\Delta, N)$ for some $N > 0$. Therefore, the conclusion of Theorem 1 for the single-input case is proved.

Indeed, note that the linearization Σ_L of (45) is obviously asymptotically stable because the $\tilde{y}$-part in Σ_L has all eigenvalues with negative real parts, and the $\bar{y}$-part in Σ_L is just the linearization of $\dot{\bar{y}} = A_1\bar{y} - \bar{k}(\bar{y})b_2$ and therefore is asymptotically stable by the inductive hypothesis. As a consequence, the local asymptotic stability of (45) is no problem. To get a feedback of the form of (4), we take $\xi = 1, \eta = 0$. Then the feedback $u = -\sigma_\mu(y_n + \lambda\bar{k}(\frac{\bar{y}}{\lambda}))$ is as desired. To get a feedback of the form of (5), we take $\xi = 0, \eta = 1$. Then, for any $r > 0$, the closed-loop system of (34), with

$$u = -r\sigma_\mu(\frac{y_n}{r}) - r\lambda\bar{k}(\frac{\bar{y}}{r\lambda}) \,, \tag{46}$$

is $SISS_L(r\Delta, N)$. Choosing r sufficiently small, we can make the sum of the coefficients of all saturations in (46) to be bounded by 1. It turns out that the feedback defined by (46) is as desired. So we only need to show that (45) is $SISS_L(\Delta, N)$ for some $N > 0$.

Given $\delta \leq \Delta$, let $\bar{e}(t), \tilde{e}(t)$ be of the same dimensions as $\bar{y}, \tilde{y}$ and such that $\bar{e}, \tilde{e}$ are bounded, measurable, and $\leq_{\text{ev}} \delta$. Consider the system:

$$\begin{array}{rl} \dot{\bar{y}} & = A_1\bar{y} - \lambda\bar{k}(\frac{\bar{y}}{\lambda})b_2 - g(y_n + \lambda\xi\bar{k}(\frac{\bar{y}}{\lambda}))b_3 + \bar{e} \,, \\ \dot{\tilde{y}} & = J\tilde{y} - \sigma_\mu(y_n + \lambda\xi\bar{k}(\frac{\bar{y}}{\lambda}))b_0 + \lambda\eta\bar{k}(\frac{\bar{y}}{\lambda})b_0 + \tilde{e} \,. \end{array} \tag{47}$$

From (44) we have

$$\lambda\xi|\bar{k}(\frac{\bar{y}}{\lambda})| \leq_{\text{ev}} \lambda\xi M \leq \frac{\delta_0}{\nu_1} \,, \tag{48}$$

and (because $\delta \leq \frac{(1-\alpha)\lambda\delta_0}{\nu_2}$)

$$|\lambda\eta\bar{k}(\frac{\bar{y}}{\lambda})b_0+\tilde{e}| \leq_{\rm ev} \lambda\eta M+\delta \leq \frac{\delta_0}{\nu_2}. \tag{49}$$

So applying Corollary 4.3 to the second equation in (47), we conclude that

$$|\tilde{y}| \leq_{\rm ev} \lambda M(\xi\nu_1+\eta\nu_2)+\delta\nu_2\,, \tag{50}$$

and therefore

$$|y_n+\lambda\xi\bar{k}(\frac{\bar{y}}{\lambda})| \leq_{\rm ev} \lambda M(\xi\nu_1+\eta\nu_2)+\delta\nu_2+\lambda\xi M\,. \tag{51}$$

Inequality (44) and $\delta\nu_2 \leq (1-\alpha)\delta_0$ then imply that the right-hand side of (50) is bounded by δ_0. Therefore, it follows from (41) that

$$|\tilde{y}| \leq_{\rm ev} \min\{\Delta_1, \alpha\Delta_2\}\,. \tag{52}$$

So, $|y_n+\lambda\xi\bar{k}(\frac{\bar{y}}{\lambda})| \leq_{\rm ev} \Delta_2$ follows from (52) and (43). From (38), we then get

$$|g(y_n+\lambda\xi\bar{k}(\frac{\bar{y}}{\lambda}))| \leq_{\rm ev} \varepsilon|y_n+\lambda\xi\bar{k}(\frac{\bar{y}}{\lambda})|\,. \tag{53}$$

In particular, since $\delta\nu_2 \leq (1-\alpha)\lambda\delta_0$ and $\delta_0 \leq \Delta_1$, using (51), we obtain

$$|y_n+\lambda\xi\bar{k}(\frac{\bar{y}}{\lambda})| \leq_{\rm ev} \lambda\Big(M(\xi\nu_1+\eta\nu_2+\xi)+(1-\alpha)\Delta_1\Big)\,.$$

Therefore

$$|g(y_n+\lambda\xi\bar{k}(\frac{\bar{y}}{\lambda}))b_3+\bar{e}| \leq_{\rm ev}$$
$$\lambda\varepsilon\Big(M(\xi\nu_1+\eta\nu_2+\xi)+(1-\alpha)\Delta_1\Big)||b_3||+\delta\,. \tag{54}$$

From (39) we see that the right-hand side of (54) is bounded by $\lambda\bar{\Delta}$. Since the system $\dot{\bar{y}}=A_1\bar{y}-\lambda\bar{k}(\frac{\bar{y}}{\lambda})b_2$ is $SISS_L(\lambda\bar{\Delta},\bar{N})$, it follows that

$$|\bar{y}| \leq_{\rm ev} \lambda\bar{N}\bar{\Delta}\,. \tag{55}$$

Now, suppose $\limsup_{t\to\infty}|\bar{y}(t)|=\rho>0$. Then $|\bar{y}| \leq_{\rm ev} 2\rho$. Since $|\bar{k}(y)| \leq L|y|$, we have

$$|\lambda\xi\bar{k}(\frac{\bar{y}}{\lambda})| \leq_{\rm ev} 2\xi L\rho \tag{56}$$

and

$$|\lambda\eta\bar{k}(\frac{\bar{y}}{\lambda})b_0+\tilde{e}| \leq_{\rm ev} 2\eta L\rho+\delta\,. \tag{57}$$

Applying Corollary 4.3 to the second equation in (47) and using (56) and (57), we obtain

$$|\tilde{y}| \leq_{\rm ev} 2(\xi\nu_1+\eta\nu_2)L\rho+\nu_2\delta\,. \tag{58}$$

(Note that if the right-hand side of (58) is $\geq \Delta_1$, then the inequality is trivial since we know from (52) that $|\tilde{y}| \leq_{\rm ev} \Delta_1$.) From (53), (56), and (58), we have

$$|g(y_n+\lambda\xi\bar{k}(\frac{\bar{y}}{\lambda}))| \leq_{\rm ev} \varepsilon\Big(2(\xi\nu_1+\eta\nu_2+\xi)L\rho+\nu_2\delta\Big)\,.$$

Now, if

$$\varepsilon\Big(2(\xi\nu_1+\eta\nu_2+\xi)L\rho+\nu_2\delta\Big)+\delta \leq \lambda\bar{\Delta}\,, \tag{59}$$

then, since the system $\dot{\bar{y}}=A_1\bar{y}-\lambda\bar{k}(\frac{\bar{y}}{\lambda})b_2$ is $SISS_L(\lambda\bar{\Delta},\bar{N})$, it follows that

$$|\bar{y}| \leq_{\rm ev} \varepsilon\bar{N}\Big(2(\xi\nu_1+\eta\nu_2+\xi)L\rho+\nu_2\delta\Big)+\bar{N}\delta\,. \tag{60}$$

If (59) is not satisfied, then $\varepsilon\bar{N}\Big(2(\xi\nu_1+\eta\nu_2+\xi)L\rho+\nu_2\delta\Big)+\bar{N}\delta > \lambda\bar{N}\bar{\Delta}$. However, from (55) we see that (60) still holds. So we established (60) in all cases. From (40) we then get

$$|\bar{y}| \leq_{\rm ev} \frac{1}{2}\rho+\bar{N}(\varepsilon\nu_2+1)\delta\,. \tag{61}$$

Taking the $\limsup_{t\to\infty}$ of the left-hand side of (61), we have $\rho \leq \frac{1}{2}\rho+\bar{N}(\varepsilon\nu_2+1)\delta$ i.e. $\rho \leq 2\bar{N}(\varepsilon\nu_2+1)\delta$. Substituting this into (58) and (61), we get $|\tilde{y}| \leq_{\rm ev} \Big(4(\xi\nu_1+\eta\nu_2)L\bar{N}(\varepsilon\nu_2+1)+\nu_2\Big)\delta$, and $|\bar{y}| \leq_{\rm ev} 2\bar{N}(\varepsilon\nu_2+1)\delta$. So, if we take $N=2\bar{N}(\varepsilon\nu_2+1)(1+2L(\xi\nu_1+\eta\nu_2))+\nu_2$, the conclusion follows.

To complete the proof, we need to deal with the general case of $m>1$ inputs. This is done by induction on m, as in the proof in [14], and will be omitted here. □

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