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A Continuation Method for Nonholonomic Path-finding problems *

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Abstract

We study the mathematical theory of an algorithm for nonholonomic point-to-point path finding, based on a "continuation" or "deformation" method, in which one starts with an admissible trajectory that goes from the desired initial point $\bar{p}$ to some point $\bar{q}_0$, and then tries to construct a one-parameter family of trajectories, whose terminal points $\bar{q}_s$ describe, as s varies, a path that joins $\bar{q}_0$ to the desired terminal point $\bar{q}$.

1. Introduction

We investigate the mathematical properties of a continuation algorithm of the kind proposed by S. Seereeram and J. Wen in [4], who carried out extensive simulations and reported very good results. Independently, a similar proposal was made in [7], where a proof was outlined, for a very simple case, of the fact that the differential equation describing the path deformation (the "path-lifting equation") is well defined. The theoretical problem of the global existence of solutions of this equation for general nonholonomic systems without drift is still open and appears to be very hard. Our purpose here is to prove global existence for the class —introduced by R. Strichartz in [6]— of "strongly bracket generating systems." The proof shows how the properties of the path-lifting equation are related to the so-called "abnormal extremals," that have recently attracted considerable interest for different reasons (cf. [1], [2], [3]). (Work is now in progress on more general, not strongly bracket-generating, systems.)

The purpose of the algorithm is to compute, for a given pair $(\bar{p}, \bar{q})$ of points in the state space M of the given nonholonomic system Σ, a trajectory ξ of Σ that goes from $\bar{p}$ to $\bar{q}$. If we let $\mathcal{U}$ denote the space of admissible open-loop controls defined on $[a, b]$, and use $\mathcal{E}_{\bar{p}}$ to denote the *endpoint map* corresponding to the starting point $\bar{p}$ (i.e. the map that assigns, to each control $\eta \in \mathcal{U}$, the terminal point $\xi_{\eta,\bar{p}}(b)$ of the trajectory $\xi_{\eta,\bar{p}} : [a, b] \to M$ of Σ that corresponds to η and is such that $\xi_{\eta,\bar{p}}(a) = \bar{p}$), then our problem is to find η that solves the equation $\mathcal{E}_{\bar{p}}(\eta) = \bar{q}$. This is one particular instance of the general problem of solving a nonlinear equation $F(X) = Y$, where F is a map from a (possibly infinite-dimensional) manifold N_1 to another manifold N_2, Y is given, and X is the unknown. A general strategy for solving equations of this kind is the *continuation method*: (i) start with a pair $(\bar{X}, \bar{Y})$ such that $F(\bar{X}) = \bar{Y}$; (ii) find a path $\pi : [0, 1] \to N_2$ that goes from $\bar{Y}$ to Y, and (iii) try to lift π to a path $\Pi : [0, 1] \to N_1$ such that $\Pi(0) = \bar{X}$ and $F(\Pi(s)) = \pi(s)$ for all s. Then we can take $X = \Pi(1)$.

In our situation, the resulting algorithm proceeds as follows: one starts with a trajectory $t \to \xi_0(t)$ of Σ, defined on an interval $[a, b]$, that goes from $\bar{p}$ to some point $\bar{q}_0 = \xi_0(b)$, and then tries to deform ξ_0 by constructing a one-parameter family $\{\xi_s : 0 \le s \le 1\}$ of trajectories $\xi_s : [a, b] \to M$ such that $\xi_s(0) = \bar{p}$, in such a way that the terminal points $\xi_s(b)$ follow a given path $s \to \pi(s)$ in M. If $\pi : [0, 1] \to M$ is chosen so that $\pi(0) = \bar{q}_0$ and $\pi(1) = \bar{q}$, then the trajectory ξ_1 will solve our problem.

If the starting trajectory ξ_0 corresponds to an open-loop control η_0, then $\bar{q}_0$ is equal to $\mathcal{E}_{\bar{p}}(\eta_0)$. Since the path $\pi : [0, 1] \to M$ is such that $\pi(0) = \bar{q}_0$, finding trajectories $\xi_s : [a, b] \to M$ such that $\xi_s(0) = \bar{p}$ and $\xi_s(b) = \pi(s)$ for $s \in [0, 1]$ is tantamount to finding controls $\eta_s \in \mathcal{U}$ such that $\mathcal{E}_{\bar{p}}(\eta_s) = \pi(s)$. If we write $\Pi(s) = \eta_s$, then our problem is precisely that of "lifting" —with respect to $\mathcal{E}_{\bar{p}}$— the path $\pi : [0, 1] \to M$ to a path $\Pi : [0, 1] \to \mathcal{U}$, i.e. of constructing a path $\Pi : [0, 1] \to \mathcal{U}$ such that $\Pi(0) = \eta_0$ and $\mathcal{E}_{\bar{p}}(\Pi(s)) = \pi(s)$. The lifting is found by solving an ordinary differential equation —the "Path-lifting Equation," abbreviated PLE— in $\mathcal{U}$. The PLE expresses the lifting property of the path on the infinitesimal level. Intuitively, if we have found a lifting Π on the interval $[0, s]$, and want to extend this lifting a little further, say to $[0, s + \varepsilon]$, then we should find a direction $v \in \mathcal{U}$ such that $\mathcal{E}_{\bar{p}}(\Pi(s) + hv) \sim \pi(s + h)$ (i.e. $\mathcal{E}_{\bar{p}}(\Pi(s) + hv) \sim \pi(s) + h\dot{\pi}(s)$) as $h \to 0$, and then make sure that $\Pi(s + h) \sim \Pi(s) + hv$. (This will at least yield a first-order approximation to the desired lifting.) Since $\mathcal{E}_{\bar{p}}(\Pi(s) + hv) \sim \pi(s) + hd\mathcal{E}_{\bar{p}}(\Pi(s)) \cdot v$ (where $d\mathcal{E}_{\bar{p}}(\Pi(s))$ denotes the differential of $\mathcal{E}_{\bar{p}}$ at $\Pi(s)$), the condition that must be satisfied by the vector $v = \dot{\Pi}(s)$ is:

$$d\mathcal{E}_{\bar{p}}(\Pi(s)) \cdot v = \dot{\pi}(s) \ . \tag{1}$$

It is then natural to ask whether, once the lifting has been constructed up to a time s, a v can be found that satisfies (1). An obvious sufficient condition for this to be possible is that $d\mathcal{E}_{\bar{p}}(\Pi(s))$ be surjective. In that case, a particular solution v of (1) is $v = P(\Pi(s))$, where $P(\eta)$ is the *Moore-Penrose pseudoinverse* of $d\mathcal{E}_{\bar{p}}(\eta)$, given by $P(\eta) = d\mathcal{E}_{\bar{p}}(\eta)^* \left(d\mathcal{E}_{\bar{p}}(\eta) d\mathcal{E}_{\bar{p}}(\eta)^*\right)^{-1}$. With this choice of v, the fact that $v = \dot{\Pi}(s)$ implies that

$$\dot{\Pi}(s) = P(\Pi(s)) \cdot \dot{\pi}(s) \ . \tag{2}$$

Equation (2) is the PLE. Clearly, the right-hand side of the PLE is well defined if $d\mathcal{E}_{\bar{p}}$ is onto, i.e. if $\mathcal{E}_{\bar{p}}$ is

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a submersion. It turns out that the controls η where $\mathcal{E}_{\bar{p}}(\eta)$ fails to be a submersion correspond precisely to certain exceptional trajectories (referred to in the literature by various names, such as "abnormal extremals" or "singular trajectories"), and the PLE yields local lifts of paths wherever these abnormal extremals can be avoided. But, even after this particular issue is resolved, the possibility remains that the PLE might have explosions, and therefore fail to have a global solution $\Pi : [0,1] \to \mathcal{U}$. Our goal is to investigate in detail the properties of the PLE, and in particular to show that *the same conditions that imply the nonexistence of abnormal trajectories also suffice to guarantee that the PLE has no explosions.* The natural condition that guarantees nonexistence of abnormal extremals is the so-called *Strong Bracket-Generating (SBG) Condition*, and the main result of this paper is Theorem 5, which says precisely that under the SBG condition the PLE has global existence of solutions. The proof of Theorem 5 is based on a somewhat delicate estimate, proved in Theorem 4.

2. Basic definitions

We consider driftless control systems of the form

$$\Sigma : \quad \dot{x} = \sum_{i=1}^{m} u_i f_i(x) \tag{3}$$

where the state variable x takes values in a smooth (i.e. C^∞), connected n-dimensional manifold M, the control $u = (u_1, \ldots, u_m)$ takes values in $\mathbb{R}^m$, and $f_1, \ldots, f_m$ are smooth vector fields on M.

Regarding manifolds, we follow the usual conventions of Differential Geometry, and agree that manifolds are, by definition, Hausdorff and paracompact. This implies that every manifold M admits a complete Riemannian metric, a fact that will be used below. For $x \in M$, we use T_xM, T_x^*M, TM, T^*M, $T^\#M$ to denote, respectively, the tangent and cotangent spaces of M at x, the tangent and cotangent bundles of M, and the tangent bundle of M with the zero section removed. For $y \in T_xM$ and $z \in T_x^*M$, we write the duality product $z(y)$ as $\langle z, y\rangle$ and also as zy. If M is equipped with a Riemannian metric $\langle\cdot,\cdot\rangle$, then we write $\langle y_1, y_2\rangle$ for the inner product of two tangent vectors at a point $x \in M$. Using the identification between T_xM and T_x^*M induced by the metric, $\langle z_1, z_2\rangle$ is also well defined when $z_1, z_2 \in T_x^*M$. We also write $||y||^2 = \langle y, y\rangle$ and $||z||^2 = \langle z, z\rangle$.

We fix once and for all a time interval $[a, b]$, and take $\mathcal{U}$ —the space of open-loop controls— to be $L^2([a,b], \mathbb{R}^m)$. Then $\mathcal{U}$ is a real Hilbert space, with the inner product of two elements $\eta = (\eta_1, ..., \eta_m)$, $\tilde{\eta} = (\tilde{\eta}_1, ..., \tilde{\eta}_m)$ of $\mathcal{U}$ given by $\langle \eta, \tilde{\eta}\rangle = \sum_{i=1}^m \int_a^b \eta_i(t)\tilde{\eta}_i(t)\,dt$. If $\eta = (\eta_1, ..., \eta_m) \in \mathcal{U}$, a *trajectory of* Σ *for* η is an absolutely continuous curve $\xi : [a,b] \to M$ such that $\dot{\xi}(t) = \sum_{i=1}^m \eta_i(t) f_i(\xi(t))$ for a.e. $t \in [a,b]$. A *controlled trajectory* of Σ is a pair (ξ, η), where $\eta \in \mathcal{U}$ and ξ is a trajectory of Σ corresponding to η. (In many cases —e.g. if the vectors $f_1(x), \ldots, f_m(x)$ are linearly independent for every x— a trajectory ξ cannot arise from more than one control η, so it is not necessary to distinguish between "trajectories" and "controlled trajectories.") Carathéodory's Existence and Uniqueness Theorem guarantees that, if $\eta \in \mathcal{U}$ and $p \in M$, then: (i) there exists, on some nontrivial subinterval I of $[a,b]$ such that $a \in I$, a trajectory ξ for the restriction of η to I such that $\xi(a) = p$; (ii) if $\xi_i : I_i \to M$, $i = 1, 2$ are two such trajectories, then $\xi_1 = \xi_2$ on $I_1 \cap I_2$. *We will assume that the solution actually exists on the whole interval* $[a, b]$, i.e. that the following *No Explosions Condition* holds:

(NE) For every $\eta \in \mathcal{U}$, $p \in M$, there exists a trajectory $\xi : [a,b] \to M$ of Σ for η such that $\xi(a) = p$.

We will use $\xi_{\eta,p}$ to denote the —obviously unique— trajectory ξ of (NE).

We also assume that the vector fields f_i satisfy the following *Lie Algebra Rank Condition*:

(LARC) Let L_Σ denote the Lie algebra of vector fields on M generated by $f_1, \ldots, f_m$. For $x \in M$, define $L_\Sigma(x) = \{X(x) : X \in L_\Sigma\}$. Then $L_\Sigma(x) = T_xM$ for every $x \in M$.

It is well known that **(LARC)** implies that Σ is *completely controllable* (CC), i.e. that given any two points p, q of M there exists a trajectory $\xi : [a,b] \to M$ of Σ such that $\xi(a) = p$ and $\xi(b) = q$. (A CC system is often called a *nonholonomic system.*)

Remark 1 It is not hard to see that, if **(LARC)** holds for the vector fields f_i, $i = 1, \ldots, m$, then it still holds for the new system $\tilde{\Sigma}_\varphi$ in which each f_i is replaced by $\tilde{f}_i = \varphi f_i$, $i = 1, \ldots, m$, where φ is an arbitrary C^∞ real-valued strictly positive function. On the other hand, the trajectories of the two systems are the same, up to a reparametrization of time, so solving the path-finding problem for $\tilde{\Sigma}_\varphi$ is equivalent to solving it for Σ. It is also easy to see that, if M is an *arbitrary* connected manifold, and Σ is a system of the form (3) that satisfies **(LARC)** but not **(NE)**, then we can find φ such that $\tilde{\Sigma}_\varphi$ satisfies **(NE)**. (Indeed, since M is paracompact, it admits a complete Riemannian metric. One can then take φ to be any function such that the resulting $\tilde{f}_i$ are bounded, e.g. $\varphi = (1 + \sum_{i=1}^m ||f_i||^2)^{-1/2}$.) For example, if we start with a system (3) in $\mathbb{R}^n$, and then are interested in path-finding with obstacle avoidance, the "obstacle" being a closed subset C of $\mathbb{R}^n$, then we can take M to be $\mathbb{R}^n \backslash C$, if this set is connected. We can then choose φ to be a smooth function on M such that $\varphi(x)$ goes to zero sufficiently fast as x approaches C, and also as $||x|| \to \infty$. ∎

3. The continuation algorithm

Fix $\bar{p} \in M$. As explained above, we define the *endpoint map* $\mathcal{E}_{\bar{p}}$ by $\mathcal{E}_{\bar{p}}(\eta) = \xi_{\eta,\bar{p}}(b)$. Since **(NE)** holds, $\mathcal{E}_{\bar{p}}(\eta)$ is defined for every $\eta \in \mathcal{U}$. The complete controllability condition implies that $\mathcal{E}_{\bar{p}}$ is surjective. Our problem now becomes that of computing, for given $\bar{p}$, $\bar{q}$ in M, an $\eta \in \mathcal{U}$ that satisfies $\mathcal{E}_{\bar{p}}(\eta) = \bar{q}$. The continuation procedure requires that we start with an $\eta_0 \in \mathcal{U}$ and a path $\pi : [0,1] \to M$ such that $\pi(0) = \bar{q}_0$, $\pi(1) = \bar{q}$, where $\bar{q}_0 = \mathcal{E}_{\bar{p}}(\eta_0)$. We then try to find a path $\Pi : [0,1] \to \mathcal{U}$

such that $\Pi(0) = \eta_0$ and $\mathcal{E}_{\bar{p}}(\Pi(s)) = \pi(s)$. To find this lifting, we use the PLE, which is an ordinary differential equation on $\mathcal{U}$. We now describe the PLE in detail. For this purpose, we assume that *the path π is of class C^1.*

In general, if N_1 and N_2 are two smooth manifolds (with N_1 possibly infinite-dimensional, but N_2 finite-dimensional), and $F : N_1 \to N_2$ is a surjective map of class C^1, then one can lift C^1 paths $\pi : [0,1] \to N_2$ to paths $\Pi : [0,1] \to N_1$ as follows. Suppose we had, for each point w of N_1, a linear map $P(w)$ from $T_{F(w)}N_2$ to $T_w N_1$ which is a right inverse of $dF(w)$, i.e. is such that $dF(w) \circ P(w)$ is the identity map. (Here $dF(w)$ denotes the differential of F at w.) Consider first the special case when $N_2 = \mathbb{R}^n$. For a given path $\pi : [0,1] \to N_2$, consider the differential equation

$$\dot{w}(s) = P(w(s)) \cdot \dot{\pi}(s) \ , \tag{4}$$

which has local existence and uniqueness of solutions if the map P is locally Lipschitz with respect to some Riemannian metric on N_1. It is easy to see that, if $\omega : [0,1] \to N_1$ is a solution of (4), and $F(\omega(0)) = \pi(0)$, then $F(\omega(s)) = \pi(s)$ for all s. (To see this, just observe that the derivative of $F(\omega(s))$ with respect to s is $dF(\omega(s)) \cdot P(\omega(s)) \cdot \dot{\pi}(s)$, i.e. $\dot{\pi}(s)$.) Therefore the path Π given by $\Pi(s) = \omega(s)$ is a lifting of π. Now consider the general case when N_2 is a finite-dimensional manifold. An ordinary differential equation on N_1 is an equation of the form $\dot{w} = X(w,s)$, where $X(w,s)$ is defined for all $w \in N_1$ and all s in some interval. Equation (4) is not of that form, because the quantity $P(w) \cdot \dot{\pi}(s)$ (which is the obvious candidate for the role of $X(w,s)$) only makes sense if $F(w) = \pi(s)$, since the domain of $P(w)$ is $T_{F(w)}N_2$ and $\dot{\pi}(s) \in T_{\pi(s)}N_2$. We can get around this problem by finding an s-dependent vector field $(x,s) \to h(x,s)$ on N_2 (i.e. a map $(x,s) \to h(x,s)$ such that, for each fixed s, $h(\cdot,s)$ is a vector field on N_2) such that $h(\pi(s),s) = \dot{\pi}(s)$. We can then consider the equation

$$\dot{w}(s) = P(w(s)) \cdot h(F(w(s)),s) \ , \tag{5}$$

corresponding to the choice $X(w,s) = P(w) \cdot h(F(w),s)$, which is now perfectly meaningful, since $h(F(w),s)$ belongs to $T_{F(w)}N_2$, which is precisely the domain of $P(w)$. If P is locally Lipschitz as before, and in addition $h(x,s)$ is continuous with respect to (x,s), and locally Lipschitz with respect to x, locally uniformly in s, then (5) also has local existence and uniqueness of solutions. Once again, one verifies that a solution $\omega : [0,1] \to N_1$ of (5) for which $F(\omega(0)) = \pi(0)$ necessarily satisfies $F(\omega(s)) = \pi(s)$ for all s. (Indeed, the curve $s \to \xi(s) = F(\omega(s))$ is a solution of $\dot{x} = h(x,s)$, and satisfies $\xi(0) = \pi(0)$. Our hypotheses on h imply uniqueness of solutions for $\dot{x} = h(x,s)$, so $\xi(s) \equiv \pi(s)$.)

Equation (5) is the *Path-lifting Equation* (PLE) for F. The PLE can be written provided that a map P with the above properties exists, and in that case it depends on the choice of P. (It is easy to see that an h with the above properties always exists, and that the set of solutions $\omega(\cdot)$ of (5) that satisfy $F(\omega(0)) = \pi(0)$ is in fact independent of the choice of h.)

We now turn to the problem of how to choose P. An obvious necessary condition for the existence of P is that F be a submersion, i.e. that $dF(w)$ be surjective for every $w \in N_1$. If this holds, and N_1 is equipped with a Riemannian metric such that every space $T_w N_1$ is a Hilbert space, then we can define $P(w) : T_{F(w)}N_2 \to T_w N_1$ by letting $P(w) \cdot y$ be, for each $y \in T_{F(w)}N_2$, the shortest vector $v \in T_w N_1$ such that $dF(w) \cdot v = y$. The linear map $P(w) : T_{F(w)}N_2 \to T_w N_1$ so defined is the *Moore-Penrose pseudo-inverse* (MPPI) (or generalized pseudo-inverse) of $dF(w)$. We now recall some general properties of the MPPI.

Let H_1 and H_2 be two Hilbert spaces. Assume that $\dim H_2 < \infty$. Let $L : H_1 \to H_2$ be a continuous linear mapping and let $L^* : H_2 \to H_1$ be its adjoint. Let $W = LL^*$, so $W \geq 0$. It is clear that L is onto iff $W > 0$. In that case, if we define $L^\# : H_2 \to H_1$ by $L^\# = L^* W^{-1}$, then $L^\#$ is a right inverse of L. By definition, the operator $L^\#$ is the MPPI of L. If $x \in H_2$, then $L^\# x$ is the vector $v \in H_1$ of smallest length such that $Lv = x$. (In particular,the operator $L^\#$ does not depend on the inner product of H_2.) Clearly, $||L^\# x||^2 = \langle W^{-1}x, x\rangle$ for every $x \in H_2$, so $||L^\#||^2 = ||W^{-1}||$. On the other hand, $||W^{-1}||$ is the inverse of $\sigma_{\min}(W)$, the smallest eigenvalue of W. Since $\sigma_{\min}(W)$ is the infimum of $\langle z, Wz\rangle$, over $z \in H_2$ such that $||z|| = 1$, we conclude that

$$||L^\#||^2 = \left(\inf_{\substack{z \in H_2 \\ ||z||=1}} \langle z, LL^* z\rangle \right)^{-1} \ . \tag{6}$$

Returning now to the abstract path-lifting situation considered above, of two manifolds N_1 and N_2, and a surjective submersion $F : N_1 \to N_2$, let us specialize even further by assuming that N_1 is actually an open subset of a Hilbert space $\mathcal{H}$ and $F \in C^2$. In that case, we can define $P(w)$, for each $w \in N_1$, to be the MPPI of the map $dF(w) : \mathcal{H} \to T_{F(w)}N_2$, and it is clear that $P \in C^1$. So the PLE will have local solutions for any C^1 path $\pi : [0,1] \to N_2$ and any $w(0)$ such that $F(w(0)) = \pi(0)$. If, moreover, P satisfies a *linear growth estimate*:

(LGE) For every compact subset K of N_2 there exists a $C > 0$ such that $||P(w)|| \leq C(1 + ||w||)$ for all $w \in N_1$ for which $F(w) \in K$.

then Gronwall's inequality implies that the PLE has global solutions for any C^1 path $\pi : [0,1] \to N_2$, provided that F satisfies:

(CL) For every compact subset K of N_2, the set $F^{-1}(K)$ is closed in $\mathcal{H}$.

Remark 2 The LGE involves the operator norm of the linear map $P(w) : T_{F(w)}N_2 \to \mathcal{H}$. For this to be well defined, we need in principle to specify norms in the domain and the range of $P(w)$. The norm on $\mathcal{H}$ poses no problem, since $\mathcal{H}$ is a Hilbert space. To have a norm on $T_{F(w)}N_2$ we can equip N_2 with a Riemannian metric. Since any two Riemannian metrics are necessarily equivalent on a compact subset of N_2, *the validity of Condition* **(LGE)** *is in fact independent of the metric.* A similar remark applies to the equivalent conditions **(LGE')** and **(LGE")** introduced below. ∎

It will be convenient to use Formula (6) to rewrite **(LGE)**. Since $P(w) = dF(w)^{\#}$, the bound $||P(w)|| \leq C(1 + ||w||)$ is equivalent to the requirement that

$$\inf_{\substack{z \in H_2 \\ ||z||=1}} \langle z, dF(w)dF(w)^* z\rangle \geq \frac{c}{(1+||w||)^2} \tag{7}$$

(where $c = C^{-2}$), so **(LGE)** is equivalent to

(LGE') For every compact subset K of N_2 there exists a $c > 0$ such that $(1 + ||w||^2)||dF(w)(z)||^2 \geq c||z||^2$ whenever $w \in N_1$, $z \in T_{F(w)}N_2$, and $F(w) \in K$.

Summarizing, we have shown:

Theorem 1 *Let N_1 be an open subset of a real Hilbert space $\mathcal{H}$. Let N_2 be a finite-dimensional manifold, and let $F : N_2 \to N_1$ be a surjective submersion of class C^2 such that Condition* **(CL)** *holds. For $w \in N_1$, let $P(w)$ be the Moore-Penrose pseudoinverse of $dF(w)$. Assume that F satisfies* **(LGE')**. *Then, for each C^1 path $\pi : [0,1] \to N_2$ and each $\omega(0)$ such that $F(\omega(0)) = \pi(0)$, the solution of the corresponding PLE exists on the whole interval $[0,1]$, and yields a lifting of π.* ∎

Let us now return to our original path finding problem for a nonholonomic system Σ. We start by taking $N_1^0 = \mathcal{U}$, $N_2^0 = M$, and try to apply the preceding considerations to the map $F = \mathcal{E}_{\bar{p}}$. The complete controllability condition implies that $\mathcal{E}_{\bar{p}}$ is surjective. Moreover, $\mathcal{E}_{\bar{p}}$ is of class C^∞. To decide whether $\mathcal{E}_{\bar{p}}$ is a submersion we have to compute its differential.

It turns out that, for $\eta \in \mathcal{U}$, the differential $d\mathcal{E}_{\bar{p}}(\eta)$ is given in terms of the solutions of the *variational equation* along the trajectory $\xi_{\eta,\bar{p}}$. To make this precise, let $v = (v_1, \ldots, v_m) \in \mathcal{U}$. Let $[a,b] \ni t \to y_{\eta,\bar{p},v}(t) \in T_{\xi_{\eta,\bar{p}}(t)}M$ be the solution of the *variational equation*

$$\dot{y}(t) = (\eta(t) \cdot Df)(\xi_{\eta,\bar{p}}(t)) \cdot y(t) + (v(t) \cdot f)(\xi_u(t)) \tag{8}$$

with initial condition $y(0) = 0$. (The expressions $\eta(t) \cdot Df$, $v(t) \cdot f$ are abbreviations for $\sum_{i=1}^m \eta_i(t) Df_i$, $\sum_{i=1}^m v_i(t) f_i$.) Here, if X is any smooth vector-valued function on an open subset of $\mathbb{R}^n$, $DX(x)$ denotes the Jacobian matrix of X at x. Equation (8) makes sense in coordinates, but it is easy to see that, if $y(t)$ is thought of as a *tangent vector at* $\xi_{\eta,\bar{p}}(t)$, then the property that $y(\cdot)$ is a solution of (8) is intrinsic. Having defined $y_{\eta,\bar{p},v}$, the map $d\mathcal{E}_{\bar{p}}(\eta) : \mathcal{U} \to T_qM$ (where $q = \mathcal{E}_{\bar{p}}(\eta)$) is simply given by $d\mathcal{E}_{\bar{p}}(\eta)(v) = y_{\eta,\bar{p},v}(1)$. Notice that (8) is the *linearization* of our system at the reference trajectory $\xi_{\eta,\bar{p}}$. Hence $d\mathcal{E}_{\bar{p}}(\eta)$ is surjective iff if the linearized (time-varying) system (8) is controllable.

Naturally, $d\mathcal{E}_{\bar{p}}(\eta)$ is surjective iff if its adjoint $d\mathcal{E}_{\bar{p}}(\eta)^*$ is one-to-one. It will be convenient to determine $d\mathcal{E}_{\bar{p}}(\eta)^*$ explicitly. As expected, this requires that we consider the *adjoint variational equation along* $\xi_{\eta,\bar{p}}$.

For a smooth vector field X on M, the *variational covector field of* X —also known as the *Hamiltonian lift of* X— is the vector field X^* on T^*M whose expression in coordinates is given by $X^*(x,z) = (X(x), -z \cdot DX(x))$. The integral curves of X^* are the solutions $t \to (\xi(t), \zeta(t))$ of the system

$$\dot{x} = \frac{\partial h}{\partial z} \quad , \quad \dot{z} = -\frac{\partial h}{\partial x} \ , \tag{9}$$

that corresponds to the Hamiltonian function $h : T^*M \to \mathbb{R}$ given by $h(x,z) = \langle z, X(x)\rangle$ for $z \in T_x^*M$. It is well known that, if h is any smooth function on T^*M, then the vector field whose trajectories are given in coordinates by (9) is in fact intrinsically defined on T^*M.

Similarly, Σ^* will denote the *Hamiltonian lift of* Σ, i.e. the system

$$\Sigma^* : \quad \dot{x}^* = \sum_{i=1}^m u_i f_i^*(x^*) \ , \tag{10}$$

where $x^* = (x,z)$ evolves in T^*M. A trajectory $t \to \xi^*(t)$ of Σ^* for $\eta = (\eta_1, \cdots, \eta_m) \in \mathcal{U}$ is a pair $\xi^* = (\xi, \zeta)$ where ξ is a trajectory of Σ for η and the map $[a,b] \ni t \to \zeta(t) \in T^*_{\xi(t)}M$ satisfies, in coordinates, the *adjoint equation*

$$\dot{\zeta} = -\zeta\big(\sum_{i=1}^m \eta_i(t)(Df_i)(\xi(t))\big) \qquad \text{for a.e. } t \in [a,b] \ . \tag{11}$$

(A map ζ that picks, for each t, a covector $\zeta(t) \in T^*_{\xi(t)}M$, is called a *covector along* ξ. If, in addition, ζ satisfies (11), then ζ is called an *adjoint vector* along ξ.) Since (11) is linear in ζ, the solution of (11) is well-defined on $[a,b]$ for any terminal condition $\zeta(b) = \bar{z} \in T^*_{\xi(b)}M$. Then $\xi^* = (\xi, \zeta)$ is a trajectory of Σ^* for $\eta = (\eta_1, \cdots, \eta_m)$ iff $\dot{\xi}^* = \sum_{i=1}^m u_i f_i^*(\xi)$ for a.e. $t \in [a,b]$, i.e. iff

$$\dot{\xi} = \frac{\partial H}{\partial z}(\xi, \zeta, \eta) \, , \, \dot{\zeta} = -\frac{\partial H}{\partial x}(\xi, \zeta, \eta) \, , \tag{12}$$

where H is the *Hamiltonian* of Σ, given by $H(x,z,u) = \sum_{i=1}^m u_i \langle z, f_i(x)\rangle$. With these definitions, we can finally write a formula for $d\mathcal{E}_{\bar{p}}(\eta)^*$. As before, let $q = \mathcal{E}_{\bar{p}}(\eta)$, and pick $z \in T_q^*M$. Let $\zeta_{\eta,\bar{p},z}$ be the solution of (11) that satisfies the terminal condition $\zeta(b) = z$. Define the *switching functions* $\varphi_{\eta,\bar{p},z,i}$ by $\varphi_{\eta,\bar{p},z,i}(t) = \langle \zeta(t), f_i(\xi_{\eta,\bar{p}}(t))\rangle$ for $i = 1, \ldots, m$. We can then associate, to each $\eta, \bar{p}, z$, the m-tuple of functions $\Phi_{\eta,\bar{p},z} = (\varphi_{\eta,\bar{p},z,1}, \ldots, \varphi_{\eta,\bar{p},z,m})$. Then

$$d\mathcal{E}_{\bar{p}}(\eta)^*(z) = \Phi_{\eta,\bar{p},z} \quad \text{for } z \in T_q^*M \ , \ q = \mathcal{E}_{\bar{p}}(\eta) \ . \tag{13}$$

In particular, $d\mathcal{E}_{\bar{p}}(\eta)^*$ is one-to-one if and only if there does not exist any nonzero covector $z \in T_q^*M$ for which the switching functions $\varphi_{\eta,\bar{p},z,i}$ vanish identically for $i = 1, \ldots, m$. Equivalently, $d\mathcal{E}_{\bar{p}}(\eta)^*$ fails to be one-to-one if and only if there exists a nontrivial solution of the corresponding adjoint equation, for which the switching functions $\varphi_{\eta,\bar{p},z,i}$ vanish identically for $i = 1, \ldots, m$.

A controlled trajectory (ξ, η) for which there exists a nontrivial solution of the adjoint equation whose associated switching functions vanish identically is called an *abnormal extremal of* Σ. We have therefore shown:

Theorem 2 *The map $d\mathcal{E}_{\bar{p}}(\eta)^*$ is one-to-one if and only if $(\xi_{\eta,\bar{p}}, \eta)$ is not an abnormal extremal.* ∎

Now consider an open subset Ω of Msuch that

(NAE) there exist no abnormal extremals going from $\bar{p}$ to a point in Ω.

If we now take $N_1 = \mathcal{E}_{\bar{p}}^{-1}(\Omega)$, $N_2 = \Omega$, then the restriction to N_1 of the map $\mathcal{E}_{\bar{p}}$ is a C^2 surjective submersion onto N_2. The PLE is then well defined for paths in Ω. Global solutions will exist if **(LGE')** holds. In our case, **(LGE')** will hold if we can find, for every compact subset K of Ω, a constant $c > 0$ such that $||d\mathcal{E}_{\bar{p}}(\eta)^*(z)||^2 \geq \frac{c||z||^2}{1+||\eta||^2}$ for all $\eta \in \mathcal{U}$ such that $\mathcal{E}_{\bar{p}}(\eta) \in \Omega$.

The map $G_{\eta,\bar{p}} \stackrel{\text{def}}{=} d\mathcal{E}_{\bar{p}}(\eta)d\mathcal{E}_{\bar{p}}(\eta)^*$ is the *controllability gramian* of the linearized system (8). Clearly,

$$\langle z, G_{\eta,\bar{p}}\, z\rangle = \sum_{i=1}^{m} \int_a^b \varphi_{\eta,\bar{p},z,i}(t)^2\, dt\ . \qquad (14)$$

Moreover, it is clear that $||d\mathcal{E}_{\bar{p}}(\eta)^*(z)||^2 = \langle z, G_{\eta,\bar{p}}\, z\rangle$. Hence the linear growth estimate is equivalent to:

(LGE") If $K \subseteq \Omega$ is compact, then there is a $c > 0$ such that $(1 + ||\eta||^2)\sum_{i=1}^m \int_a^b \varphi_{\eta,\bar{p},z,i}(t)^2\, dt \geq c||z||^2$ for all $\eta \in \mathcal{U}$ for which $\mathcal{E}_{\bar{p}}(\eta) \in K$, and all $z \in T_{\mathcal{E}_{\bar{p}}(\eta)}M$.

4. SBG systems

For a system Σ of the form (3) we define $\varphi_{\Sigma,i} : T^*M \to \mathbb{R}$, $\psi_{\Sigma,ij} : T^*M \to \mathbb{R}$, $\Phi_\Sigma : T^*M \to \mathbb{R}^m$, $\Psi_\Sigma : T^*M \to \mathbb{R}^{m\times m}$, by letting $\varphi_{\Sigma,i}(x,z) = \langle z, f_i(x)\rangle$, $\psi_{\Sigma,ij}(x,z) = \langle z, [f_i,f_j](x)\rangle$, $\Phi_\Sigma = (\varphi_{\Sigma,1},\ldots,\varphi_{\Sigma,m})$, $\Psi_\Sigma = (\psi_{\Sigma,ij})_{1\leq i,j\leq m}$. It is clear that, for every (x,z), the matrix $\Psi_\Sigma(x,z)$ is skew-symmetric.

A system Σ of the form (3) is said to be *Strong Bracket Generating* (SBG) if the matrix $\Psi_\Sigma(x,z)$ is nonsingular whenever $(x,z) \in T^\# M$ is such that $\Phi_\Sigma(x,z) = 0$,

The class of SBG systems is extremely restrictive. For example, a system for which m is odd can never be SBG unless $f_1(x),\ldots,f_m(x)$ span T_xM for all x, since an $m \times m$ skew-symmetric matrix cannot be nonsingular if m is odd. One interesting example of SBG systems is the following:

EXAMPLE. *Let f_1, f_2 be two smooth vector fields on a 3-dimensional manifold M, such that $f_1(x)$, $f_2(x)$ and $[f_1,f_2](x)$ span T_xM for each $x \in M$. Then the system $\dot{x} = u_1f_1(x) + u_2f_2(x)$ is SBG.* ∎

The SBG condition was introduced by R. Strichartz in [5] and [6] as the most natural assumption that excludes the possibility of abnormal extremals. Indeed, it is easy to show that

Theorem 3 *If a system of the form (3) is SBG, then the only abnormal extremals are the trajectories that correspond to a control $\eta(t) \equiv 0$.* ∎

It follows from Theorem 3 that, for an SBG system, if we fix a starting point $\bar{p}$, then the set $\Omega = M\backslash\{\bar{p}\}$ satisfies Condition **(NAE)**. Therefore the PLE is well defined for every path π that does not go through $\bar{p}$. In particular, to construct a trajectory of Σ from $\bar{p}$ to another point $\bar{q} \neq \bar{p}$, it suffices to find one control η_0 that steers $\bar{p}$ to some point $\bar{q}_0$ other than $\bar{p}$, and then find a path π in M that goes from $\bar{q}_0$ to $\bar{q}$ *and does not go through* $\bar{p}$. (Such a path always exists if $\dim M > 1$.) The PLE then enables us to lift π, at least locally. The only remaining obstruction to a global lift is the possibility that solutions of the PLE might explode. It turns out that this does not happen, as we now show.

5. Global liftings

Theorem 4 *Let Σ be a system of the form (3), and assume that Σ is SBG. Let $\bar{p} \in M$, and write $\Omega = M\backslash\{\bar{p}\}$. Let $a < b$, and let $\mathcal{U} = L^2([a,b],\mathbb{R}^m)$. Then for every compact subset K of Ω there exists a constant $c > 0$ such that, if $\eta \in \mathcal{U}$ is such that $\mathcal{E}_{\bar{p}}(\eta) \in K$, and $\hat{z}$ is any covector at $\mathcal{E}_{\bar{p}}(\eta)$ such that $||\hat{z}|| = 1$, then the inequality*

$$\Big(\int_a^b ||\eta||^2\Big)\Big(\int_a^b \big(\varphi^2_{\eta,\bar{p},\hat{z},1} + \ldots + \varphi^2_{\eta,\bar{p},\hat{z},m}\big)\Big) \geq c\ , \quad (15)$$

holds, where $\varphi_{\eta,\bar{p},\hat{z},i}(t) \stackrel{\text{def}}{=} \langle\zeta(t)\,,\ f_i(\xi_{\eta,\bar{p}}(t))\rangle$, and ζ is the solution of the adjoint equation along $(\xi_{\eta,\bar{p}},\eta)$ such that $\zeta(b) = \hat{z}$.

PROOF. Let K be a compact subset of Ω. As explained in Remark 2, we can assume that M is equipped with a Riemannian metric. From now on, d denotes the distance with respect to this metric. Also, if v is a tangent vector or a covector, then $||v||$ is its norm relative to the metric. For $\alpha \geq 0$, define $K_\alpha = \{x \in M : d(x,K) \leq \alpha\}$. Then, if $\alpha > 0$ is sufficiently small, K_α is a compact subset of Ω. From now on, we fix an $\alpha > 0$ with this property. We let $K^* = \{(x,z) \in T^*M : x \in K\,, ||z|| = 1\}$, $K^*_\alpha = \{(x,z) \in T^*M : x \in K_\alpha\,, \frac{1}{2} \leq ||z|| \leq 2\}$. Notice that the closure of $T^*M\backslash K^*_\alpha$ is disjoint from K^*. So we can choose a smooth function $\theta : T^*M \to \mathbb{R}$ such that $\theta \equiv 0$ on K^* and $\theta > 1$ on $T^*M\backslash K^*_\alpha$. We let $\nu_i(x,z)$ denote the directional derivative $f_i^*\theta(x,z)$ of θ at (x,z) in the direction of the vector field f_i^*. We let $C_1 > 0$ be such that $\nu_1^2 + \ldots + \nu_m^2 \leq C_1^2$ throughout K^*_α. We write φ_i, ψ_{ij}, Φ, Ψ for $\varphi_{\Sigma,i}$, $\psi_{\Sigma,ij}$, Φ_Σ, Ψ_Σ. We let $\Delta : T^*M \to \mathbb{R}$ be the square of the determinant of Ψ. The SBG hypothesis then implies that the function $||\Phi||^2 + \Delta$ never vanishes on $T^\# M$. Let 2β be the infimum of this function on the compact subset K^*_α of $T^\# M$. Then $\beta > 0$ and, if $(x,z) \in K^*_\alpha$ is such that $||\Phi(x,z)|| \leq \sqrt{\beta}$, then we can conclude that $\Delta(x,z) \geq \beta$.

Now let $\eta \in \mathcal{U}$ be such that $q \stackrel{\text{def}}{=} \mathcal{E}_{\bar{p}}(\eta) \in K$. Use ξ to denote the trajectory $\xi_{\eta,\bar{p}}$. Fix a $\hat{z} \in T^*_qM$ such that $||\hat{z}|| = 1$. Let ζ be the solution of the adjoint equation along (ξ,η) such that $\zeta(b) = \hat{z}$. Let $\xi^*(t) = (\xi(t),\zeta(t))$, so that $\xi^* : [a,b] \to T^\# M$ is a trajectory of Σ^*. Then $\xi^*(b) \in K^*$, and $\xi^*(a) \notin K^*_\alpha$, since $\xi(a) = \bar{p} \notin K$. So $\theta(\xi^*(b)) = 0$, and $\theta(\xi^*(a)) > 1$. Therefore there exists a τ such that $a < \tau < b$, with the property that $\theta(\xi^*(\tau)) = 1$ and $\theta(\xi^*(t)) < 1$ for $\tau < t \leq b$. Then $\xi^*(t) \in K^*_\alpha$ for $\tau \leq t \leq b$.

From now on, whenever h is a smooth scalar- or vector-valued function on T^*M, $\underline{h}$ will denote h evaluated along ξ^*, i.e. $\underline{h}(t) = h(\xi^*(t))$. Then $\dot{\xi}^*(t) = \sum_{i=1}^m \eta_i(t)f_i^*(\xi^*(t))$ for a.e. $t \in [a,b]$, so $\dot{\underline{\theta}}(t) = \sum_{i=1}^m \eta_i(t)\underline{\nu_i}(t)$ for a.e. $t \in [a,b]$. Since $\underline{\theta}$ is absolutely continuous, $\underline{\theta}(b) = 0$, and $\underline{\theta}(\tau) = 1$, we easily conclude that

$$\left|\int_\tau^b \langle \eta(t), \underline{\nu}(t)\rangle\, dt\right| \geq 1\ , \tag{16}$$

where $\langle \eta(t), \underline{\nu}(t)\rangle \stackrel{\text{def}}{=} \sum_{i=1}^m \eta_i(t)\underline{\nu}_i(t)$. If we knew that $||\underline{\nu}|| \leq$ constant.$||\underline{\Phi}||$, then (16), together with the Cauchy-Schwarz inequality, would yield the estimate $\left(\int_\tau^b ||\eta(t)||^2\, dt\right)\left(\int_\tau^b ||\underline{\Phi}(t)||^2\, dt\right) \geq$ constant, which would prove our conclusion. On the other hand, $||\underline{\nu}||$ is clearly $\leq$ constant.$||\underline{\Phi}||$, as long as $||\underline{\Phi}||$ is "large." So the difficult case is when $||\underline{\Phi}||$ is "small."

To take care of this problem, we introduce a cut-off function $\sigma_0 : \mathbb{R} \to \mathbb{R}$ by letting $\sigma_0(s) = 0$ for $s \leq 0$, $\sigma_0(s) = \frac{s}{\rho}$ for $0 \leq s \leq \rho$, and $\sigma_0(s) = 1$ for $s \geq \rho$. Here $\rho > 0$ will be chosen later, and we will require that $\rho^2 \leq \beta$. Define $\sigma(x,z) = \sigma_0(|\varphi_1(x,z)| + \ldots + |\varphi_m(x,z)|)$. Notice that σ is Lipschitz. Write $\langle \eta(t), \underline{\nu}(t)\rangle = \underline{\sigma}(t)\langle \eta(t), \underline{\nu}(t)\rangle + (1-\underline{\sigma}(t))\langle \eta(t), \underline{\nu}(t)\rangle$, and express the integral of (16) as a sum

$$\int_\tau^b \langle \eta(t), \underline{\nu}(t)\rangle\, dt = I_1 + I_2\ , \tag{17}$$

where $I_1 = \int_\tau^b \underline{\sigma}\langle \eta, \underline{\nu}\rangle$, $I_2 = \int_\tau^b (1-\underline{\sigma})\langle \eta, \underline{\nu}\rangle$. It is clear that $\sigma_0(s) \leq \frac{s}{\rho}$ for all $s \in [0, \infty)$. So σ is pointwise bounded by $\frac{\sqrt{m}}{\rho}||\Phi||$, and then

$$|I_1| \leq \frac{C_1\sqrt{m}}{\rho}\left(\int_\tau^b ||\eta(t)||^2\, dt\right)\left(\int_\tau^b ||\underline{\Phi}(t)||^2\, dt\right)\ . \tag{18}$$

To estimate I_2, we first observe that the derivative of $\underline{\varphi}_i(t)$ is $\sum_j \underline{\psi}_{ij}(t)\eta_j(t)$. In vector notation, this says that $\dot{\underline{\Phi}} = \eta\underline{\Psi}$, where we are writing both Φ and η as row vectors. If we use $\Psi^\dagger$ to denote the matrix of complementary minors of Ψ, then we have $\underline{\Delta}\eta = \eta\underline{\Psi}\,\underline{\Psi}^\dagger$. We also notice that the integrand in the formula defining I_2 vanishes unless $|\underline{\varphi}_1| + \ldots + |\underline{\varphi}_m| \leq \rho$. Since $\xi^*(t) \in K_\alpha^*$ for $\tau \leq t \leq b$, and $\rho^2 \leq \beta$, the bound $|\underline{\varphi}_1| + \ldots + |\underline{\varphi}_m| \leq \rho$ implies $||\underline{\Phi}||^2 \leq \rho^2$, and therefore $\underline{\Delta} \geq \beta$. If, in the expression for I_2, we replace η by $\frac{1}{\underline{\Delta}}\eta\underline{\Psi}\,\underline{\Psi}^\dagger$, and use the fact that $\dot{\underline{\Phi}} = \eta\underline{\Psi}$, we end up with the equality:

$$I_2 = \int_\tau^b \underline{\Delta}(t)^{-1}\big(1-\underline{\sigma}(t)\big)\langle \dot{\underline{\Phi}}(t)\underline{\Psi}^\dagger(t), \underline{\nu}(t)\rangle\, dt\ \ , \tag{19}$$

and $\underline{\Delta} \geq \beta$ whenever the integrand is $\neq 0$. We now integrate by parts. Define $A(\mathbf{v}, t)$, where $\mathbf{v}$ is a m-dimensional row-vector variable, by $A(\mathbf{v}, t) = \left(\frac{1-\underline{\sigma}(t)}{\underline{\Delta}(t)}\right)\langle \mathbf{v}\underline{\Psi}^\dagger(t), \underline{\nu}(t)\rangle$. Then (19) says that $I_2 = \int_\tau^b A(\dot{\underline{\Phi}}(t), t)\, dt$. Since A is linear in $\mathbf{v}$, $\frac{d}{dt}\big[A(\underline{\Phi}(t), t)\big]$ is equal to $A(\dot{\underline{\Phi}}(t), t) + D_2A(\underline{\Phi}(t), t)$, where D_2A denotes the derivative of A with respect to the second component, i.e.

$$D_2A(\mathbf{v}, t) = -\underline{\Delta}(t)^{-1}\big(\dot{\underline{\sigma}}(t)\big)\langle \mathbf{v}\underline{\Psi}^\dagger(t), \underline{\nu}(t)\rangle$$

$$-\underline{\Delta}(t)^{-2}\dot{\underline{\Delta}}(t)\big(1-\underline{\sigma}(t)\big)\langle \mathbf{v}\underline{\Psi}^\dagger(t), \underline{\nu}(t)\rangle$$

$$+\underline{\Delta}(t)^{-1}\big(1-\underline{\sigma}(t)\big)\Big(\langle \mathbf{v}\dot{\underline{\Psi}}^\dagger(t), \underline{\nu}(t)\rangle + \langle \mathbf{v}\Psi^\dagger(t), \dot{\underline{\nu}}(t)\rangle\Big)\ .$$

Therefore $I_2 = I_{21} - I_{22}$, where $I_{21,\ 22}$ are equal, respectively, to $A(\underline{\Phi}(b), b) - A(\underline{\Phi}(\tau), \tau)$ and $\int_\tau^b D_2A(\underline{\Phi}(t), t)\, dt$. It is easily verified that $||D_2A(\mathbf{v}, t)||$ is bounded by a fixed constant C_2 times $||\mathbf{v}||$ times $||\eta(t)||$ times $\frac{1}{\rho}$. (To see this, first let Q be the set of those $(x,z) \in K_\alpha^*$ such that $\Delta(x,z) \geq \beta$. Observe that $1-\underline{\sigma}(t)$ and $\dot{\underline{\sigma}}(t)$ vanish unless $\xi^*(t) \in Q$. The functions ν, $\Psi^\dagger$, σ and $\frac{1}{\Delta}$ are bounded on Q. If h is any of these four functions (so h can be scalar- or vector- or matrix-valued), then $\dot{\underline{h}}(t)$ is equal to $\sum_i \eta_i(t)(f_i^* h)(\xi^*(t))$, and the functions $f_i^* h$ are bounded on Q. When $h = \sigma$, then h is the composite of the Lipschitz function $g = |\varphi_1| + \ldots + |\varphi_m|$ with σ_0. The $f_i^* g$ are of course bounded on K_α^*, and the derivative of σ_0 is bounded by $\frac{1}{\rho}$.) So

$$|I_{22}| \leq \rho^{-1}C_2\left(\int_\tau^b ||\eta(t)||^2\, dt\right)\left(\int_\tau^b ||\underline{\Phi}(t)||^2\, dt\right)\ . \tag{20}$$

On the other hand, $A(\mathbf{v}, t)$ is bounded by a fixed constant C_3 times $||\mathbf{v}||$, since $\underline{\Delta}(t) \geq \beta$ whenever $\underline{\sigma}(t) \neq 1$. But $\underline{\sigma}(t) \neq 1$ only if $|\underline{\varphi}_1| + \ldots + |\underline{\varphi}_m| \leq \rho$, and in that case $||\underline{\Phi}|| \leq \rho$. So $|A(\underline{\Phi}(t), t)| \leq C_3\rho$. Therefore $|I_{21}| \leq 2C_3\rho$. If we now combine this with (16), (17, (18), and (20), and let $C_4 = C_1\sqrt{m} + C_2$, we get the bound

$$1 \leq 2C_3\rho + \frac{C_4}{\rho}\left(\int_\tau^b ||\eta(t)||^2\, dt\right)\left(\int_\tau^b ||\underline{\Phi}(t)||^2\, dt\right)\ . \tag{21}$$

Next, choose ρ such that $2C_3\rho \leq \frac{1}{2}$ (as well as $\rho^2 \leq \beta$). Then $\left(\int_\tau^b ||\eta(t)||^2\, dt\right)\left(\int_\tau^b ||\underline{\Phi}(t)||^2\, dt\right) \geq c$, where $c = (2C_4)^{-1}\rho$. But then $\left(\int_a^b ||\eta(t)||^2\, dt\right)\left(\int_a^b ||\underline{\Phi}(t)||^2\, dt\right) \geq c$, and our proof is complete. ∎

Theorem 4 implies that Condition **(LGE")** holds. We have therefore proved:

Theorem 5 *Let Σ be a strongly bracket-generating system that satisfies the LARC and the No Explosions Condition. Then the path-lifting equation corresponding to an initial point $\bar{p}$ yields global liftings for any path π that does not go through $\bar{p}$.*

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