Uniqueness results for the value function via direct trajectory-construction methods *

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1. Introduction

The purpose of this note is to present some new results, together with a number of particularly simple and user-friendly versions of results obtained in recent years by the author and M. Malisoff, on the uniqueness of solutions of the Hamilton-Jacobi-Bellman equation (HJBE) for deterministic finite-dimensional optimal control problems under non-standard hypotheses. Our approach is completely control-theoretic and totally self-contained, using the systematic construction of special trajectories of various kinds, and not involving any PDE methods. We will *not* assume that the Lagrangian is positive, or that the dynamics is Lipschitz-continuous.

We will consider autonomous Lagrangian optimization problems involving a state variable x which takes values in an open subset Ω of $\mathbb{R}^n$, a control variable u taking values in a control space U, and a target set $\mathcal{T}$, which is a closed subset of the closure of Ω disjoint from Ω itself. The dynamics is given by an ordinary differential equation

$$\dot{x} = f(x, u)\,, \tag{1}$$

the cost functional to be minimized is

$$J = \int_{\tau_-(\xi)}^{\tau_+(\xi)} L(\xi(t), \eta(t))\, dt\,, \tag{2}$$

(where $\tau_-(\xi)$, $\tau_+(\xi)$ are, respectively, the initial and terminal times of the trajectory ξ), and the minimization is supposed to be, for each initial state $x \in \Omega$, over the set $\mathcal{A}^{\hat{\Sigma}}_{x,\mathcal{T}}$ of all pairs $\Xi = (\xi, \xi_0)$ such that

(i) Ξ consists of a trajectory ξ of (1) (i.e., a locally absolutely continuous function ξ that satisfies $\dot{\xi}(t) = f(\xi(t), \eta(t))$ for almost all t) corresponding to some U-valued control η, and a "running cost" function ξ_0 corresponding to ξ and η (i.e., a locally absolutely continuous function ξ_0 such that $\dot{\xi}_0(t) = L(\xi(t), \eta(t))$ for almost all t);

(ii) ξ starts at x, and "ends at the target" in a sense to be defined precisely later.

We will refer to a pair $\Xi = (\xi, \xi_0)$ for which (i) above holds as an "augmented trajectory" of our system, because it really is a trajectory of the "augmented control system"

$$\dot{x} = f(x, u)\,, \qquad \dot{x}_0 = L(x, u) \tag{3}$$

obtained from (1) by "adding the cost as an extra variable" in a well known way. We will write $J(\Xi)$, rather than just J, for the left-hand side of (2), because it is easy to see that the natural arguments for our cost functional J are really augmented trajectories, since (i) one cannot just regard J as a functional of ξ only, because the integral of (2) involves the control η as well as ξ, but on the other hand (ii) the cost is completely determined once we know ξ and a running cost ξ_0, because in that case $L(\xi(t), \eta(t)) = \dot{\xi}_0(t)$.

The infimum $V(x)$ of the costs $J(\Xi)$ of all augmented trajectories $\Xi \in \mathcal{A}^{\hat{\Sigma}}_{x,\mathcal{T}}$ is the *value* of our problem at x. (If the set $\mathcal{A}^{\hat{\Sigma}}_{x,\mathcal{T}}$ is itself empty, then of course $V(x) = +\infty$.) The function $V : \Omega \mapsto \mathbb{R} \cup \{-\infty, +\infty\}$ is the *value function* of our problem.

Our goal is to prove uniqueness theorems, showing that a viscosity solution of the HJBE that satisfies an appropriate boundary condition is necessarily the value function. "Uniqueness" is to be understood as "uniqueness within a class defined by some additional properties," such as the class of all functions that are continuous and bounded below.

We will work with a class of systems which is sufficiently general to capture some interesting phenomena not commonly addressed in the literature, and at the same time restricted enough to make it possible to prove strong theorems. In particular, we will asume that the sets $F_{\hat{\Sigma}}(x, U) = \{(f(x, u), L(x, u)) : u \in U\}$ are closed and convex, but will not require them to be compact, and will instead impose a "local coerciveness" condition, according to which, locally, an inequality of the form $\|f(x, u)\|^r \leq L(x, u) + C$, with $C > 0$ and $r > 1$, holds uniformly with respect to u. We will also require $f(x, u)$ and $L(x, u)$ to be continuous with respect to x, with the continuity being uniform with respect to u locally in x.

On the other hand, we will most definitely *not* require that the dynamics $f(x, u)$ be Lipschitz-continuous with respect to x, since one of the main purposes of this work is to clarify the exact role of the Lipschitz-continuity assumptions often made in the viscosity literature. The answer we will propose is as follows:

(a) Without any Lipschitz-continuity hypotheses, one can prove, for continuous viscosity solutions V of the HJBE, an *existence theorem for trajectories*, asserting that, starting at every point of x of Ω, there is a maximally defined "augmented trajectory of steepest descent," that is, a maximally defined pair $\Xi = (\xi, \xi_0)$ defined on a interval I such that $0 = \min I$, and having the property that the inequality

$$V(\xi(t)) + \xi_0(t) \geq V(\xi(s)) + \xi_0(s) \tag{4}$$

*The author was supported in part by NSF Grant DMS01-03901. He wrote the final version of the paper while he was a guest of the Institut Mittag-Leffler, in Djursholm, Sweden, and is very grateful to the Institute for its kind hospitality. He particularly thanks Anders Lindquist for encouraging him to resume work on this topic and give the lectures that eventually resulted is this written presentation.

holds whenever $s, t \in I$ and $s \le t$. (This is Theorem 6.1 below.)

(b) As a trivial corollary of the existence of steepest descent trajectories (applied to $-V$ and $-L$), we get the existence of "DPI trajectories" (where "DPI" is an acronym for "Dynamic Programming Inequality"), i.e., augmented trajectories $\Xi = (\xi, \xi_0)$ along which the inequality

$$V(\xi(t)) + \xi_0(t) \le V(\xi(s)) + \xi_0(s) \tag{5}$$

(that is, the exact opposite of (4)) is satisfied.

(c) The existence result of (b) says that for every "sufficiently nice" (e.g., piecewise constant) control η *there exists a trajectory of η with the given initial condition along which the DPI holds.* This is almost, but not quite, what is needed to prove that V is bounded above by the value function.

(d) The gap between the existence result for DPI trajectories and what would actually be needed to prove that V is bounded above by the value function is that to achieve the latter goal one needs the DPI to hold for *all* trajectories, and it is not enough to have just one DPI trajectory for every intial condition and every control.

(e) The gap described in (d) clearly does not exist when there is uniqueness of trajectories for every given control and initial condition.

(f) In particular, the gap does not exist when, for every admissible control η, the corresponding time-varying vector field $(x, t) \mapsto f(x, \eta(t))$ satisfies a Lipschitz-Carathéodory condition that guarantees uniqueness of trajectories.

(g) Naturally, the Lipschitz-Carathéodory condition can be replaced by weaker conditions that guarantee uniqueness, such as the requirement that a bound

$$\langle f(x, \eta(t)) - f(y, \eta(t)), x - y \rangle \le k(t)\|x - y\|^2 \,, \tag{6}$$

with k integrable, hold locally.

(h) Even more generally, the only property that really matters is that, if we pick a sequence $\{\eta_j\}_{j=1}$ of piecewise constant controls such that our augmented trajectory Ξ can be approximated by augmented trajectories Ξ_j corresponding to the η_j—with, say, the same initial condition—then the Ξ_j converge to Ξ uniformly *no matter how the Ξ_j are chosen.* We call such trajectories "uniquely limiting," and use this concept in the statement of our main theorem.

The following important issues will not be discussed here:

(1) Whether the value function itself satisfies the conditions of our main theorem, i.e., whether it is a continuous viscosity solution of the HJBE and whether it is bounded below.

(2) What happens when the sets $F(x, U)$ are not closed and convex. (This would require considering relaxed controls, and using trajectories for which the steepest descent property holds approximately rather than exactly. It turns out to be possible to extend our results under fairly general conditions, provided our system has appropriate local controllability properties.)

2. The main theorem

If n is a positive integer, an *n-dimensional control system* is a triple $\Sigma = (\Omega, U, f)$ such that Ω (the *state space* of Σ) is an open subset of $\mathbb{R}^n$, U (the *control space* of Σ) is a set, and f (the *dynamics* of Σ) is a map $\Omega \times U \ni (x, u) \mapsto f(x, u) \in \mathbb{R}^n$.

An *n-dimensional augmented control system* is a 4-tuple $\hat{\Sigma} = (\Omega, U, f, L)$ such that $\Sigma = (\Omega, U, f, L)$ is an n-dimensional control system and L (the *Lagrangian* of $\hat{\Sigma}$ is a map $\Omega \times U \ni (x, u) \mapsto L(x, u) \in \mathbb{R}$. (In that case, the state space, control space, and dynamics of Σ are also called the state space, control space, and dynamics of $\hat{\Sigma}$.)

An augmented control system $\hat{\Sigma} = (\Omega, U, f, L)$ is *continuous* if the maps $\Omega \ni x \mapsto f(x, u) \in \mathbb{R}^n$ and $\Omega \ni x \mapsto L(x, u) \in \mathbb{R}$ are continuous for each fixed u. We call $\hat{\Sigma}$ *uniformly continuous* on a subset S of Ω if there exists a function $\omega :]0, \infty[\mapsto [0, \infty]$ such that $\lim_{s \downarrow 0} \omega(s) = 0$, having the property that

$$\|f(x, u) - f(y, u)\| + |L(x, u) - L(y, u)| \le \omega(\|x - y\|)$$

whenever $x, y \in S$ and $u \in U$. We call $\hat{\Sigma}$ *locally uniformly continuous* if it is uniformly continuous on every compact subset of Ω, and *globally uniformly continuous* if it is uniformly continuous on Ω. We say that $\hat{\Sigma}$ is *Lipschitz continuous* if the maps $\Omega \ni x \mapsto f(x, u) \in \mathbb{R}^n$ and $\Omega \ni x \mapsto L(x, u) \in \mathbb{R}$ are Lipschitz continuous for each fixed u. We call $\hat{\Sigma}$ *uniformly Lipschitz continuous* on a subset S of Ω if there exists a positive constant C such that

$$\|f(x, u) - f(y, u)\| + |L(x, u) - L(y, u)| \le C\|x - y\|$$

whenever $x, y \in S$ and $u \in U$. We call $\hat{\Sigma}$ *locally uniformly Lipschitz continuous* if it is uniformly Lipschitz continuous on every compact subset of Ω, and *globally uniformly Lipschitz continuous* if it is uniformly Lipschitz continuous on Ω.

Remark 2.1 Naturally, the concepts of continuity, uniform continuity, Lipschitz continuity, and uniform Lipschitz continuity, also make sense for a control system $\Sigma = (\Omega, U, f)$, by taking the same definitions given above and omitting the parts that refer to L. In order to avoid having to make a similar remark for other concepts to be introduced in the future, we adopt the convention that any concept X that we define for an augmented control system $\hat{\Sigma} = (\Omega, U, f, L)$ is automatically understood to apply to a control system $\Sigma = (\Omega, U, f)$, in the sense that "X of Σ" means "X of the augmented system $\hat{\Sigma} = (\Omega, U, f, 0)$." $\diamondsuit$

The augmented control system $\hat{\Sigma} = (\Omega, U, f, L)$ is *coercive* on a subset S of Ω if there exist real constants

r, A, C, such that $A > 0$, $C > 0$, $r > 1$, and

$$\|f(x,u)\|^r \le AL(x,u) + C \text{ for all } x \in S\,,\ u \in U\,. \quad (7)$$

We call $\hat{\Sigma}$ *locally coercive* if it is coercive on every compact subset of Ω, and *globally coercive* if it is coercive on Ω.

Remark 2.2 If $\hat{\Sigma}$ is coercive on a set S, then it is always possible to choose C, r, such that $C > 0$, $r > 1$, and (7) holds with $A = 1$. Indeed, let A, C, r be such that $A > 0$, $C > 0$, $r > 1$, and (7) holds. Pick ρ such that $1 < \rho < r$, and let $s = \frac{\rho}{r-\rho}$. Let $x \in \Omega$, $u \in U$. Then, if $\|f(x,u)\|^{r-\rho} > A$, we have

$$\begin{aligned}
\|f(x,u)\|^\rho &= \frac{\|f(x,u)\|^r}{\|f(x,u)\|^{r-\rho}} \\
&\le A^{-1}(AL(x,u)+C) \\
&= L(x,u) + \frac{C}{A} \\
&\le L(x,u) + \frac{C}{A} + A^s\,,
\end{aligned}$$

while on the other hand, if $\|f(x,u)\|^{r-\rho} \le A$, we find that

$$\begin{aligned}
\|f(x,u)\|^\rho &= (\|f(x,u)\|^{r-\rho})^s \\
&\le A^s \\
&\le A^{-1}\|f(x,u)\|^r + A^s \\
&\le A^{-1}(AL(x,u)+C) + A^s \\
&= L(x,u) + \frac{C}{A} + A^s.
\end{aligned}$$

Hence (7) holds if r, C, A are replaced by ρ, $\frac{C}{A} + A^s$, and 1. $\diamondsuit$

For an augmented control system $\hat{\Sigma} = (\Omega, U, f, L)$, we define a map $F_{\hat{\Sigma}} : \Omega \times U \mapsto \mathbb{R}^{n+1}$ (called the *augmented dynamics* of $\hat{\Sigma}$) by letting $F_{\hat{\Sigma}}(x,u) = (f(x,u), L(x,u))$ for $x \in \Omega$, $u \in U$. We say that $\hat{\Sigma}$ *satisfies the convexity and upper semicontinuity condition* if, for each $x \in \Omega$,

$$F_{\hat{\Sigma}}(x,U) = \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\Big(\bigcup\{F_{\hat{\Sigma}}(x',U) : \|x'-x\| \le \varepsilon\}\Big)\,, \quad (8)$$

where "$\overline{\mathrm{co}}$" stands for "closed convex hull," and $F_{\hat{\Sigma}}(x,U) \stackrel{\text{def}}{=} \{(f(x,u), L(x,u)) : u \in U\}$.

Remark 2.3 If $\hat{\Sigma} = (\Omega, U, f, L)$ is locally uniformly continuous, then $\hat{\Sigma}$ satisfies the convexity and upper semicontinuity condition if and only if the set $F_{\hat{\Sigma}}(x,U)$ is closed and convex for every $x \in \Omega$. Indeed, the "only if" assertion is trivial, since the right-hand side of (8) is obviously closed and convex. To prove the "if" part, we fix $x \in \Omega$ and assume that $F_{\hat{\Sigma}}(x,U)$ is closed and convex. We choose a δ such that $\delta > 0$ and $B_\delta(x) \stackrel{\text{def}}{=} \{x' \in \mathbb{R}^n : \|x'-x\| \le \delta\} \subseteq \Omega$ and a function $\omega :]0,\infty[\mapsto [0,\infty]$ such that $\lim_{s\downarrow 0} \omega(s) = 0$ and $\|F_{\hat{\Sigma}}(y,u) - F_{\hat{\Sigma}}(z,u)\| \le \omega(\|y-z\|)$ whenever $u \in U$ and $y, z \in B_\delta(x)$. We then pick

$$v \in \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\Big(\bigcup\{F_{\hat{\Sigma}}(x',U) : \|x'-x\| \le \varepsilon\}\Big)$$

and prove that $v \in F_{\hat{\Sigma}}(x,U)$. We let $\varepsilon_k = 2^{-k}$, and use the fact that $v \in \overline{\mathrm{co}}\Big(\bigcup\{F_{\hat{\Sigma}}(',U) : \|x'-x\| \le \varepsilon_k\}\Big)$ to find, for each sufficiently large k, a member v_k of the convex hull of $\bigcup\{F_{\hat{\Sigma}}(x',U) : \|x'-x\| \le \varepsilon_k\}$ such that $\|v - v_k\| < \varepsilon_k$. We then write $v_k = \sum_{j=0}^{n+1} \alpha_{k,j} F_{\hat{\Sigma}}(x_{k,j}, u_{k,j})$, where the $x_{k,j}$ belong to Ω and satisfy $\|x_{k,j} - x\| \le \varepsilon_k$, the $u_{k,j}$ belong to U, and the $\alpha_{k,j}$ are nonnegative and satisfy $\sum_{j=0}^{n+1} \alpha_{k,j} = 1$. Then $\|F_{\hat{\Sigma}}(x, u_{k,j}) - F_{\hat{\Sigma}}(x, u_{k,j})\| \le \omega(\varepsilon_k)$. Therefore, if we let $w_k = \sum_{j=0}^{n+1} \alpha_{k,j} F_{\hat{\Sigma}}(x, u_{k,j})$, we have $\|w_k - v_k\| \le \omega(\varepsilon_k)$. Hence $w_k \to v$ as $k \to \infty$. Since $F_{\hat{\Sigma}}(x,U)$ is convex, the w_k belong to $F_{\hat{\Sigma}}(x,U)$. Since $F_{\hat{\Sigma}}(x,U)$ is closed, $v \in F_{\hat{\Sigma}}(x,U)$, and the proof is complete. $\diamondsuit$

A *target* for an augmented control system $\hat{\Sigma} = (\Omega, U, f, L)$ is a closed subset $\mathcal{T}$ of $\mathbb{R}^n$ such that $\mathcal{T} \subseteq \mathrm{Closure}\,\Omega$ and $\mathcal{T} \cap \Omega = \emptyset$.

A *trajectory* of $\hat{\Sigma} = (\Omega, U, f, L)$ is a locally absolutely continuous curve

$$I \ni t \mapsto \xi(t) \in \Omega\,, \quad (9)$$

defined on a nonempty subinterval I of $\mathbb{R}$, having the property that $\dot{\xi}(t) \in f(\xi(t), U)$ for almost every $t \in I$. An *augmented trajectory* of $\hat{\Sigma}$ is a locally absolutely continuous curve

$$I \ni t \mapsto \Xi(t) = (\xi(t), \xi_0(t)) \in \Omega \times \mathbb{R}\,, \quad (10)$$

defined on a subinterval I of $\mathbb{R}$, having the property that $\dot{\Xi}(t) \in F_{\hat{\Sigma}}(\xi(t), U)$ for almost every $t \in I$.

The *initial time*, or *starting time* of a trajectory ξ (resp. an augmented trajectory $\Xi = (\xi, \xi_0)$) with domain I is the time $\tau_-(\xi) \stackrel{\text{def}}{=} \min I$ (resp. $\tau_-(\Xi) \stackrel{\text{def}}{=} \min I$)), if the minimum exists, i.e., if I is bounded below and its infimum belongs to I. If the initial time of ξ (resp. Ξ) exists, then (i) the point $x_-(\xi) \stackrel{\text{def}}{=} \xi(\tau_-(\xi))$ (resp. $x_-(\Xi) \stackrel{\text{def}}{=} \xi(\tau_-(\Xi))$) is the *starting point*, or *initial point*, of ξ (resp. Ξ), and (ii) the ordered pair $\partial_-(\xi) \stackrel{\text{def}}{=} (\tau_-(\xi), x_-(\xi))$ (resp. $\partial_-(\Xi) \stackrel{\text{def}}{=} (\tau_-(\Xi), x_-(\Xi))$) is the *initial condition* of ξ (resp. Ξ). If $\partial_-(\xi) = (t,x)$ (resp. $\partial_-(\Xi) = (t,x)$), we say that ξ (resp. Ξ) *starts at x at time t*.

If $\mathcal{T}$ is a target for $\hat{\Sigma} = (\Omega, U, f, L)$, then a trajectory ξ or augmented trajectory $\Xi = (\xi, \xi_0)$ with domain I *ends at $\mathcal{T}$* if the limit

$$\xi(\uparrow) \stackrel{\text{def}}{=} \lim_{t\uparrow \sup I} \xi(t)$$

exists and belongs to $\mathcal{T}$.

For each $x \in \Omega$, we let $\mathcal{A}^{\hat{\Sigma}}_{x,\mathcal{T}}$ be the set of all augmented trajectories $\Xi = (\xi, \xi_0)$ of $\hat{\Sigma}$ such that

(i) $\partial_-(\Xi) = (0, x)$,

(ii) $\xi_0(0) = 0$,

(iii) Ξ ends at the target,

(iv) the limit $\xi_0(\uparrow) \stackrel{\text{def}}{=} \lim_{t\uparrow \sup \text{domain}\,\Xi} \xi_0(t)$ exists.

If $\Xi = (\xi, \xi_0) \in \mathcal{A}^{\hat{\Sigma}}_{x,\mathcal{T}}$ then the *cost* of Ξ is the number

$$J(\Xi) \stackrel{\text{def}}{=} \xi_0(\uparrow)\,. \quad (11)$$

The *value function* of the optimal control problem defined by $\hat{\Sigma}$ and the target $\mathcal{T}$ is the function $\mathcal{V}_{\mathcal{T}}^{\hat{\Sigma}} : \Omega \cup \mathcal{T} \mapsto \mathbb{R} \cup \{-\infty, +\infty\}$ given by

$$\mathcal{V}_{\mathcal{T}}^{\hat{\Sigma}}(x) = \begin{cases} \inf\{J(\Xi) : \Xi \in \mathcal{A}_{x,\mathcal{T}}^{\hat{\Sigma}}\} & \text{if} \quad x \in \Omega \\ 0 & \text{if} \quad x \in \mathcal{T}\,. \end{cases}$$

If $V : \Omega \mapsto \mathbb{R}$ is a function, then an *augmented trajectory of $\hat{\Sigma}$ of steepest descent with respect to V* is an augmented trajectory $\Xi = (\xi, \xi_0)$ of $\hat{\Sigma}$ such that

$$\begin{array}{rcl} \xi_0(s) + V(\xi(s)) & \geq & \xi_0(t) + V(\xi(t)) \\ & & \text{whenever } s, t \in \operatorname{domain}\Xi \text{ and } s \leq t\,. \end{array} \tag{12}$$

We use $\mathcal{SD}_{\hat{\Sigma},V}$ to denote the set of all augmented trajectories of $\hat{\Sigma}$ of steepest descent with respect to V, and $\mathcal{SD}_{\hat{\Sigma},V,x}$ to denote the set of all $\Xi \in \mathcal{SD}_{\hat{\Sigma},V}$ such that $\partial_-(\Xi) = (0, x)$.

If $x \in \Omega$, a *maximal augmented trajectory of $\hat{\Sigma}$ from x of steepest descent with respect to V* is a $\Xi \in \mathcal{SD}_{\hat{\Sigma},V,x}$ that cannot be extended to a $\tilde{\Xi} \in \mathcal{SD}_{\hat{\Sigma},V,x}$ defined on an interval which is strictly larger that the domain of Ξ. We use $\mathcal{MSD}_{\hat{\Sigma},V,x}$ to denote the set of all maximal augmented trajectories of $\hat{\Sigma}$ from x of steepest descent with respect to V.

Remark 2.4 We have set up our definitions in such a way that the notion of "maximal augmented trajectory of steepest descent" always means "maximal within the class $\mathcal{SD}_{\hat{\Sigma},V,x}$ of augmented trajectories Ξ of steepest descent with initial condition $\partial_-(\Xi) = (0, x)$, for a fixed $x \in \Omega$." (In other words, there is no such thing as a "maximal augmented trajectory;" there are only "maximal augmented trajectories *from a given point* x.") Since all the members of $\mathcal{SD}_{\hat{\Sigma},V,x}$ start at time 0 at the point x, the only way that a $\Xi \in \mathcal{SD}_{\hat{\Sigma},V,x}$, defined on an interval I (that necessarily starts at time 0), could fail to belong to $\mathcal{MSD}_{\hat{\Sigma},V,x}$, would be for Ξ to be "extendable to the right," that is, such that there exists an extension $\tilde{\Xi} \in \mathcal{SD}_{\hat{\Sigma},V,x}$ which is defined on an interval $\tilde{I}$ that also starts at 0 and is strictly larger than I. Naturally, it may also happen that Ξ can be extended on the left, to an augmented trajectory $\hat{\Xi}$ of steepest descent defined on an interval $\hat{I} =]\text{-}\varepsilon, 0] \cup I$ for some positive ε. But such an extension will of course no longer start at time 0, so its existence does not affect the possibility that Ξ might belong to $\mathcal{MSD}_{\hat{\Sigma},V,x}$. $\diamondsuit$

Definition 2.5 . If Ω is an open subset of $\mathbb{R}^n$, and $\xi : I \mapsto \Omega$ is a curve, we say that ξ is *right-unbounded* if

(i) the interval I is open on the right (that is, if $\tau = \sup I$, then either (a) $\tau = +\infty$ or (b) τ is finite and does not belong to I),

and

(ii) if τ is finite, then for every compact subset K of Ω there exists a τ_K such that $0 \leq \tau_K < \tau$ and $\xi(t) \notin K$ whenever $\tau_K < t < \tau$.

(Equivalently, condition (ii) asserts that $\lim_{t\uparrow\tau} \xi(t) = \infty_\Omega$, where ∞_Ω is the point at infinity of the one-point compactification of Ω.) $\diamondsuit$

The following observation is completely trivial given our definitions, but we state it explicitly as a separate result for future reference. We emphasize that this trivial result is valid *under no technical hypotheses whatsoever on $\hat{\Sigma}$ or V*. The reader is warned that the result is *not* a true "existence theorem" for trajectories of steepest descent, even though at first sight it may appear to be, because the member Ξ of $\mathcal{MSD}_{\hat{\Sigma},V,x}$ whose existence it asserts could very well turn out to be the trivial trajectory Ξ_x^{triv}, where Ξ_x^{triv} is the map $\Xi : \{0\} \mapsto \Omega \times \mathbb{R}$ such that $\Xi(0) = (x, 0)$. The true "existence theorem," yielding the existence of a *nontrivial* maximal trajectory of steepest descent and, in fact, asserting the stronger conclusion that *every* maximal trajectory of steepest descent is right-unbounded in the sense of Definition 2.5. This will be proved later (cf. Theorem (6.1) and, naturally, will depend on our technical hypotheses on $\hat{\Sigma}$ and V.

Proposition 2.6 *If $\hat{\Sigma} = (\Omega, U, f, L)$ is an augmented control system, $V : \Omega \mapsto \mathbb{R}$ is a function, and $x \in \Omega$, then the set $\mathcal{MSD}_{\hat{\Sigma},V,x}$ of maximal augmented trajectories of $\hat{\Sigma}$ from x of steepest descent with respect to V is nonempty.*

Proof. Fix x. Let $\mathcal{Z}$ be the set of all pairs (I, Ξ) such that I is a subinterval of $[0, \infty[$, $0 \in I$, and $\Xi = (\xi, \xi_0) : I \mapsto \Omega \times \mathbb{R}$ is an augmented trajectory of $\hat{\Sigma}$ which is of steepest descent with respect to V and such that $\xi(0) = x$. We partially order $\mathcal{Z}$ by stipulating that, if $(I_i, \Xi_i) \in \mathcal{Z}$ for $i = 1, 2$, then $(I_1, \Xi_1) \preceq (I_2, \Xi_2)$ iff $I_1 \subseteq I_2$ and Ξ_1 is the restriction of Ξ_2 to I_1.

It is clear that $\mathcal{Z} \neq \emptyset$, because the pair $(\{0\}, \Xi_x^{triv})$—where Ξ_x^{triv} is the map defined above, in the paragraph preceding the statement of our proposition—belongs to $\mathcal{Z}$. If Z is a totally ordered subset of $\mathcal{Z}$, we show that Z has an upper bound (I_*, Ξ_*) in $\mathcal{Z}$. This conclusion is trivial if $Z = \emptyset$, for in that case we can take $(I_*, \Xi_*) = (\{0\}, \Xi_x^{triv})$. Assume that $Z \neq \emptyset$. Let I_* be the union of the intervals I for all the members (I, Ξ) of Z. Then I_* is obviously a subinterval of $[0, \infty[$, and $0 \in I$. If $t \in I_*$, we define $\Xi_*(t) = \Xi(t)$, where (I, Ξ) is any member of Z such that $t \in I$. Write $\Xi_* = (\xi_*, \xi_{0,*})$. Then Ξ_* is obviously well defined, and is an augmented trajectory of $\hat{\Sigma}$ such that $\xi_*(0) = x$. If $t \in I_*$, then we can pick $(I, \Xi) \in Z$ such that $t \in I$, and then $\Xi_*(s) = \Xi(s)$ for all $s \in I$, and in particular for all $s \in [0, t]$, since $[0, t] \subseteq I$. Furthemore, if we write $\Xi = (\xi, \xi_0)$, then the fact that $\Xi \in \mathcal{SD}_{\hat{\Sigma},V}$ implies that $V(x) \geq \xi_0(t) + V(\xi(t)) = \xi_{0,*}(t) + V(\xi_*(t))$. Since t is an arbitrary member of I_*, we have shown that $\Xi_* \in \mathcal{SD}_{\hat{\Sigma},V}$. Therefore $(I_*, \xi_*) \in \mathcal{Z}$. Furthermore, it is clear that (I_*, Ξ_*) is an upper bound for Z. So we have shown that every totally ordered subset of $\mathcal{Z}$ has an upper bound, and that $\mathcal{Z} \neq \emptyset$. Therefore Zorn's Lemma implies that $\mathcal{Z}$ has a maximal element (I_*, Ξ_*). Clearly, such a maximal element is a member of $\mathcal{MSD}_{\hat{\Sigma},V,x}$, and our proof is complete. $\diamondsuit$

An *augmented arc* is an augmented trajectory whose domain is a compact interval. If $\Xi = (\xi, \xi_0)$ is an augmented arc with domain $[a, b]$, then an *improvement* of Ξ is an augmented arc $\Xi' = (\xi', \xi_0')$, with domain $[a', b']$, such that $\xi'(a') = \xi(a)$, $\xi'(b') = \xi(b)$, and $\xi_0'(b') - \xi_0'(a') \leq \xi_0(b) - \xi_0(a)$.

If $\hat{\Sigma} = (\Omega, U, f, L)$ is an augmented control system, then an augmented arc $\Xi = (\xi, \xi_0)$ of $\hat{\Sigma}$ with domain $[a, b]$ is *uniquely limiting* if there exists a sequence $\{\eta_j\}_{j=1}^{\infty}$ of piecewise constant functions $\eta_j : [a, b] \mapsto U$ such that

(*) *if $\{\Xi_j\}_{j=1}^{\infty}$ is an arbitrary sequence of maximally defined augmented trajectories of $\hat{\Sigma}$ such that $a \in \mathrm{domain}(\Xi_j)$ and $\Xi_j(a) = \Xi(a)$ for every j, then $[a, b] \subseteq \mathrm{domain}(\Xi_j)$ if j is large enough, and $\Xi_j \to \Xi$ uniformly on $[a, b]$ as $j \to \infty$.*

Example 2.7 Suppose $\Xi = (\xi, \xi_0)$ is an augmented arc of $\hat{\Sigma}$ with domain $[a, b]$ such that

(#) *there exist a positive number δ, a function $\eta : [a, b] \mapsto U$, and a function $\varphi : [a, b] \mapsto [0, \infty]$, such that*

(i) $\dot{\Xi}(t) = \Big(f(x, \eta(t)), L(x, \eta(t))\Big)$ *for almost every $t \in [a, b]$,*

(ii) the map $t \mapsto \Big(f(x, \eta(t)), L(x, \eta(t))\Big)$, *on the compact set*
$$I_x \stackrel{\mathrm{def}}{=} \{t : a \le t \le b \wedge \|x - \xi(t)\| \le \delta\},$$
is measurable for each $x \in \Omega$,

(iii) the map $x \mapsto \Big(f(x, \eta(t)), L(x, \eta(t))\Big)$, *on the compact set*
$$I^t \stackrel{\mathrm{def}}{=} \{x \in \Omega : \|x - \xi(t)\| \le \delta\},$$
is measurable for each $t \in [a, b]$,

(iv) φ is integrable,

(v) the inequality
$$\Big\langle f(x, \eta(t)) - f(x', \eta(t)), x - x' \Big\rangle \le \varphi(t)\|x - x'\|^2$$
holds whenever $t \in [a, b]$, $\|x - \xi(t)\| \le \delta$, and $\|x' - \xi(t)\| \le \delta$.

(vi) the inequality $|L(x, \eta(t))! \le \varphi(t)$ holds whenever $t \in [a, b]$ and $\|x - \xi(t)\| \le \delta$

Then Ξ is uniquely limiting. The proof is essentially as follows. By dividing $[a, b]$ into small intervals, we can assume that there is a fixed compact ball B such that $B \subseteq \Omega$, ξ is entirely contained in the interior of B, and the bound of *(v)* holds whenever $t \in [a, b]$ and $x, x' \in B$. We then write $F^u(x) = (f(x, u, L(x, u))$ for each $x \in B$, $u \in U$, and observe that the set $\mathcal{F} = \{F^{\eta(t)} : t \in [a, b]\}$ is a subset of the separable Banach space $C^0(B, \mathbb{R}^{n+1})$. Then $[a, b] \ni t \mapsto F^{\eta(t)}$ is an L^1 $C^0(B, \mathbb{R}^{n+1})$-valued map. Therefore one can approximate this map in L^1 by piecewise constant $\mathcal{F}$-valued maps. In other words, one can find a sequence $\{\eta_j\}_{j=1}^{\infty}$ of piecewise constant $\mathcal{G}$-valued functions (where $\mathcal{G} = \{\eta(t) : t \in [a, b]\}$), and integrable functions $k_j : [a, b] \mapsto [0, +\infty]$, such that
$$\|f(x, \eta(t)) - f(x, \eta_j(t))\| + |L(x, \eta(t)) - L(x, \eta_j(t))| \le k_j(t)$$
whenever $a \le t \le b$, $x \in B$, and $j \in \mathbb{N}$, and $\lim_{j \to \infty} \int_a^b k_j(t)\, dt = 0$. Then, if $a \le c \le d \le b$, and $\zeta : [c, d] \mapsto B$, $\theta : [c, d] \mapsto B$, are trajectories of η, η_j, respectively, Gronwall's inequality yields the bound
$$\|\zeta(t) - \theta(t)\| \le e^{\int_a^b \varphi(s) ds}\big(\|k_j\|_{L^1} + \|\zeta(c) - \theta(c)\|\big)$$
if $t \in [c, d]$. If we apply this with $c = a$ and $\zeta = \xi$, letting θ be any trajectory ξ_j of η_j starting at $\xi(a)$ at time a, and defined on some subinterval $[a, d]$ of $[a, b]$, we see that, as long as j is large enough, the maximum of the $\|\xi(t) - \xi_j(t)\|$ is bounded by a small constant. This guarantees that ξ_j actually exists on the whole interval $[a, b]$, and then the Gronwall bound implies that $\xi_j \to \xi$ uniformly as $j \to \infty$. Then the integrals $\int_a^t L(\xi_j(s), \eta_j(s))\, ds$ differ from $\int_a^t L(\xi_j(s), \eta(s))\, ds$ by less than $\|k_j\|_{L^1}$, in view of the bound
$$|L(x, \eta(t)) - L(x, \eta_j(t))| \le k_j(t)\,,$$
and $\int_a^t L(\xi_j(s), \eta(s))\, ds \to \int_a^t L(\xi(s), \eta(s))\, ds$ as $j \to \infty$, because $L(\xi_j(s), \eta(s)) \to L(\xi(s), \eta(s))$ for each s, and $|L(\xi_j(s), \eta(s))| \le \varphi(s)$. $\diamondsuit$

An augmented trajectory $\Xi = (\xi, \xi_0)$ with domain I is *locally uniquely limiting* if for every compact subinterval I' of I the restriction of Ξ to I' is uniquely limiting.

An augmented trajectory $\Xi = (\xi, \xi_0)$ with domain I is *almost uniquely limiting* if there exists a finite subset B of I such that the restriction of Ξ to every subinterval of $I \backslash B$ is locally uniquely limiting.

If Ω is an open subset of $\mathbb{R}^n$, we say that a function $V : \Omega \mapsto \mathbb{R}$ *satisfies the inequality*
$$\sup\{-\langle \nabla V(x), f(x, u)\rangle - L(x, u) : u \in U\} \ge 0 \quad (13)$$
on Ω *in the viscosity sense* if

(V_+) *whenever $x \in \Omega$ and $p \in \mathbb{R}^n$ is a subdifferential of V at x, it follows that*
$$\sup\{-\langle p, f(x, u)\rangle - L(x, u) : u \in U\} \ge 0\,.$$

(We recall that, if Ω is open in $\mathbb{R}^n$, then a *subdifferential* of a function $V : \Omega \mapsto \mathbb{R}$ at a point $\bar{x} \in \Omega$ is a vector $p \in \mathbb{R}^n$ such that
$$\liminf_{x \to \bar{x}} \frac{V(x) - V(\bar{x}) - p \cdot (x - \bar{x})}{\|x - \bar{x}\|} \ge 0\,.)$$

Similarly, we say that V *satisfies the inequality*
$$\sup\{-\langle \nabla V(x), f(x, u)\rangle - L(x, u) : u \in U\} \le 0 \quad (14)$$
on Ω *in the viscosity sense* if

(V_-) *whenever $x \in \Omega$ and $p \in \mathbb{R}^n$ is a superdifferential of V at x, it follows that*
$$\sup\{-\langle p, f(x, u)\rangle - L(x, u) : u \in U\} \le 0\,.$$

(A *superdifferential* of V at x_* is a vector p such that $-p$ is a subdifferential of $-V$ at x_*.)

We say that V *satisfies the equation*
$$\sup\{-\langle \nabla V(x), f(x, u)\rangle - L(x, u) : u \in U\} = 0 \quad (15)$$

on Ω *in the viscosity sense* if it satisfies (13) and (14) in the viscosity sense.

Remark 2.8 The definition of "viscosity solution" given here is known to be equivalent to the more common one involving test functions, cf. [1]. $\diamondsuit$

Our main result is the following theorem:

Theorem 2.9 *Let $\hat{\Sigma} = (\Omega, U, f, L)$ be an augmented control system, let $\mathcal{T}$ be a target for $\hat{\Sigma}$, and let $V : \Omega \cup \mathcal{T} \mapsto \mathbb{R}$ be a function. Assume that*

(1) $\hat{\Sigma}$ is locally uniformly continuous, locally coercive, and such that $F_{\hat{\Sigma}}(x, U)$ is closed and convex for every $x \in \Omega$.

(2) V is continuous.

(3) V satisfies (15) on Ω in the viscosity sense.

(4) V vanishes on $\mathcal{T}$.

(5) Every augmented arc has an almost locally uniquely limiting improvement.

(6) Whenever $x \in \Omega$, $\Xi = (\xi, \xi_0) \in \mathcal{MSD}_{\hat{\Sigma},V,x}$, and ξ is right-unbounded, it follows that $\Xi \in \mathcal{A}^{\hat{\Sigma}}_{x,\mathcal{T}}$.

Then $V \equiv \mathcal{V}^{\hat{\Sigma}}_{\mathcal{T}}$.

3. Examples

Example 3.1 *(Linear-quadratic optimal control)* Consider the standard linear-quadratic optimal control problem, in which x, u take values in $\mathbb{R}^n$, $\mathbb{R}^m$, respectively, the dynamical law is

$$\dot{x} = Ax + Bu\,,$$

the Lagrangian is given by

$$L(x, u) = x^\dagger Rx + u^\dagger Su\,,$$

the square matrices R, S are strictly positive definite, and the pair (A, B) is stabilizable. We take the target set $\mathcal{T}$ to consists of the origin of $\mathbb{R}^n$. The crucial technical issue here is the fact that the Lagrangian is not bounded away from zero. The hypotheses of our main theorem (including the coerciveness, which follows from the positive definiteness of S) are easily verified as long as V is bounded below. The only nontrivial point is the verification of condition (6). To prove that this holds, let $\Xi : [0, \tau[\mapsto \mathbb{R}^n \times \mathbb{R}$ be a right-unbounded maximal trajectory of steepest descent with respect to V that does not end at the target, and write $\Xi = (\xi, \xi_0)$ in the usual way. Then τ has to be infinite, because if τ was finite then the boundedness of the cost (arising from the fact that $\Xi = (\xi, \xi_0)$ is of steepest descent and V is bounded below) would trivially imply an L^2 bound on the control, from which it would follow that Ξ can be extended to the closed interval $[0, \tau]$, and then the assumption that Ξ does not end at the target would enable us to use Proposition 2.6 and Theorem 6.1 (applied with $\Omega = \mathbb{R}^n \backslash \{0\}$) to extend ξ even further, contradicting maximality. So τ is infinite. On the other hand, the fact that V is bounded below and Ξ is of steepest descent implies that the integral

$$\int_0^\infty \Big(\xi(t)^\dagger R\xi(t) + \eta(t)^\dagger S\eta(t)\Big)dt$$

is finite, if η is an open-loop control that generates Ξ. But then $\xi(\cdot)$ and η are square-integrable, so the condition that $\dot{\xi} = A\xi + B\eta$ implies that ξ is square-integrable and has a square-integrable derivative, and then Barbalat's lemma implies that ξ ends at the target, as desired. $\diamondsuit$

Example 3.2 *(Fuller's problem, cf., e.g., [9])* This is the optimal control problem for the dynamical law

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= u\,, \end{aligned}$$

with control constraint $-1 \leq u \leq 1$. The target set $\mathcal{T}$ consists of the origin of $\mathbb{R}^2$. The Lagragian is $L(x, y, u) = x^2$. The crucial technical issue here is the fact that the Lagrangian is not bounded away from zero, and in fact has a whole line of zeros. The hypotheses of our main theorem are easily verified as long as V is bounded below. The only nontrivial point is the verification of condition (6). To prove that this holds, let $\Xi : [0, \tau[\mapsto \mathbb{R}^3$ be a right-unbounded maximal trajectory of steepest descent with respect to V that does not end at the target, and write $\Xi = (\xi, \xi_0)$ in the usual way. Then τ has to be infinite, because if τ was finite then the boundedness of the control would trivially imply that Ξ can be extended to the closed interval $[0, \tau]$, and then the assumption that Ξ does not end at the target enables us to use Proposition 2.6 and Theorem 6.1 (applied with $\Omega = \mathbb{R}^2 \backslash \{(0, 0)\}$) to extend ξ even further, contradicting maximality. So τ is infinite. On the other hand, the fact that V is bounded below and Ξ is of steepest descent implies that the integral $\int_0^\infty x(t)^2 dt$ is finite, if we write $\xi(t) = (x(t), y(t))$. But then $x(\cdot)$ is a square-integrable function on $[0, \infty[$ whose second derivative is bounded. By a straightforward generalization of Barbalat's lemma, this implies that both $x(\cdot)$ and $y(\cdot)$ go to zero, i.e., that ξ ends at the target, as desired. $\diamondsuit$

Example 3.3 *(The reflected brachistochrone problem.)* This is the minimum time problem for the dynamical law

$$\begin{aligned} \dot{x} &= u\sqrt{|y|}\,, \\ \dot{y} &= v\sqrt{|y|}\,, \end{aligned}$$

with control constraint $u^2 + v^2 \leq 1$. The target set $\mathcal{T}$ consists of a single point $B \in \mathbb{R}^2$. The crucial technical issue here is the fact that the dynamical law is not Lipschitz-continuous with respect to the state. The hypotheses of our main theorem are easily verified. The only nontrivial point is the verification of condition (5). To prove that this holds, we pick an arbitrary integral arc $\xi : [a, b] \mapsto \mathbb{R}^2$, and observe that either (i) $\xi(t)$ never belongs to the x axis $X = \{(x, y) \in \mathbb{R}^2 : y = 0\}$ or (ii) there exist $t_-, t_+ \in [a, b]$ ¡such that $t_- \leq t_+$, $\xi(t) \notin X$ whenever $a \leq t < t_-$ or $t_+ > t \leq b$, $\xi(t_-) \in X$, and $\xi(t_+) \in X$. If (i) holds, then ξ satisfies the conditions of Example 2.7, so ξ is uniquely limiting. If (ii) holds and $t_+ = t_-$, then the restriction of ξ to each of the intervals $[a, t_-[+,, \;]t_+, b]$, is locally uniquely limiting, so ξ is almost locally uniquely limiting. Finally, if (ii) holds and $t_+ < t_-$, then the restriction $\tilde{\xi}$ of ξ to the interval $[t_-, t_+]$ is such that the set $S = \{t \in [a, b] : \xi(t) \notin X\}$ is the union of a finite or countable infinite collection $\mathcal{I}$ of pairwise disjoint relatively open subintervals of $[a, b]$. If $I \in \mathcal{I}$, then the restriction $\tilde{\xi}_I$ of $\tilde{\xi}$ to I is entirely contained in the open upper half-plane or in the open lower half-plane. By reflecting $\tilde{\xi}_I$ with respect to X, if necessary, we may assume that $\tilde{\xi}_I$ is entirely contained in the open upper half-plane for every $I \in \mathcal{I}$. Then $\tilde{\xi}$ is a trajectory of our system entirely contained in the closed upper half-plane. It is well known that the problem in the closed upper half-plane H_+ is the famous "brachistochrone problem," whose time-optimal trajectories $\zeta : [\alpha, \beta] \mapsto H_+$ are cycloids such that $\zeta(t)$ is an interior point of H_+ whenever $\alpha < t < \beta$. It follows that we can always replace $\tilde{\xi}$ by a cycloid ζ, thereby obtaining an almost locally uniquely limiting improvement of ξ. $\diamondsuit$

4. The main technical lemma

Let $\hat{\Sigma} = (\Omega, U, f, L)$ be an augmented control system. For every $x \in \Omega$ and every positive number δ such that $\mathrm{dist}(x, \mathbb{R}^n \backslash \Omega) > \delta$, we let $\Phi_{\delta,\hat{\Sigma}}(x)$ be the closed convex hull of all the vectors $F_{\hat{\Sigma}}(x', u)$, for all pairs (x', u) such that $x' \in \Omega$, $\|x' - x\| \leq \delta$, and $u \in U$. Then $\Phi_{\delta,\hat{\Sigma}}(x)$ is a closed convex subset of $\mathbb{R}^{n+1}$. Clearly, $\Phi_{\delta,\hat{\Sigma}}(x) \subseteq \Phi_{\delta',\hat{\Sigma}}(x)$ whenever $0 < \delta \leq \delta'$.

Let $V : \Omega \mapsto \bar{\mathbb{R}}$ be a real-valued function, and let $x_* \in \Omega$. We say that V *satisfies the infinitesimal steepest descent condition at* x_* if

(ISD) *there exist sequences* $\{x_j\}_{j=1}^\infty$, $\{v_j\}_{j=1}^\infty$, $\{\lambda_j\}_{j=1}^\infty$, $\{h_j\}_{j=1}^\infty$, $\{\gamma_j\}_{j=1}^\infty$ *in* Ω, $\mathbb{R}^n$, $\mathbb{R}$, $\mathbb{R}$, *and* $\mathbb{R}$, *respectively, such that*

- *(1)* $h_j > 0$, $\gamma_j > 0$, $(v_j, \lambda_j) \in F_{\hat{\Sigma}}(x_*, U)$, *and* $\|x_j - x_* - h_j v_j\| \leq h_j \gamma_j$ *for all* j,
- *(2)* $h_j \downarrow 0$, $x_j \to x_*$, *and* $\gamma_j \downarrow 0$ *as* $j \to \infty$,

and

- (3) $V(x_j) \leq V(x_*) - h_j \lambda_j + h_j \gamma_j$ for all j.

Theorem 4.1 *Let $\hat{\Sigma} = (\Omega, U, f, L)$ be an n-dimensional augmented control system. Suppose that $\hat{\Sigma}$ is locally uniformly continuous, locally coercive, and such that $F_{\hat{\Sigma}}(x)$ is closed and convex for every $x \in \Omega$. Let $V : \Omega \mapsto \mathbb{R}$ be a continuous function. Then V satisfies (13) on Ω in the viscosity sense if and only if V satisfies the infinitesimal steepest descent condition at every $x \in \Omega$.*

Proof. First, we assume that V satisfies the infinitesimal steepest descent condition at every $x \in \Omega$, and prove that V satisfies (13) on Ω in the viscosity sense. For this purpose, we pick $x_* \in \Omega$ and a subdifferential p of V at x_*, and prove that

$$\sup\{-p \cdot f(x_*, u) - L(x_*, u) : u \in U\} \geq 0\,. \qquad (16)$$

Using (ISD), we pick sequences $\{x_j\}_{j=1}^\infty$, $\{v_j\}_{j=1}^\infty$, $\{\lambda_j\}_{j=1}^\infty$, $\{h_j\}_{j=1}^\infty$, $\{\gamma_j\}_{j=1}^\infty$ in Ω, $\mathbb{R}^n$, $\mathbb{R}$, $\mathbb{R}$, and $\mathbb{R}$, respectively, such that $h_j > 0$, $\gamma_j > 0$, $(v_j, \lambda_j) \in F_{\hat{\Sigma}}(x_*)$, $\|x_j - x_* - h_j v_j\| \leq h_j \gamma_j$, and $V(x_j) \leq V(x_*) - h_j \lambda_j + h_j \gamma_j$ for all j, and $h_j \downarrow 0$, $x_j \to x_*$, and $\gamma_j \downarrow 0$ as $j \to \infty$. Let $J = \{j : x_j = x_*\}$. Then for $j \in J$ the inequality $\|x_j - x_* - h_j v_j\| \leq h_j \gamma_j$ implies $\|v_j\| \leq \gamma_j$, so $\lim_{j\to\infty, j\in J} v_j = 0$. On the other hand, the inequality $V(x_j) \leq V(x_*) - h_j \lambda_j + h_j \gamma_j$ implies $h_j \lambda_j \leq h_j \gamma_j$, i.e., $\lambda_j \leq \gamma_j$. Since the sequence $\{\lambda_j\}_{j=1}^\infty$ is bounded below (for example, because the local coercivity implies a bound $\lambda_j \geq \|v_j\|^r - C$), we may assume, after replacing J by a smaller infinite set, if necessary, that $\lambda = \lim_{j\to\infty, j\in J} \lambda_j$ exists. Since $\lambda_j \leq \gamma_j$, λ must be ≤ 0. Furthermore, the vector $(0, \lambda)$ is a limit of vectors $(v_j, \lambda_j) \in F_{\hat{\Sigma}}(x_*)$, so $(0, \lambda) \in F_{\hat{\Sigma}}(x_*)$, since $F_{\hat{\Sigma}}$ is closed. Hence there exists $\bar{u} \in U$ such that $f(x_*, \bar{u}) = 0$ and $L(x_*, \bar{u}) \leq 0$. But then $-p \cdot f(x_*, \bar{u}) - L(x_*, \bar{u}) \geq 0$, so (16) holds.

We now consider the case when the set J is finite. In this case, after passing to a subsequence, if necessary, we may assume that J is empty, i.e., that $x_j \neq x_*$ for all j. Since p is a subdifferential of V at x_*, we have

$$\lim\inf_{x\to x_*, x\neq x_*} \frac{V(x) - V(x_*) - p \cdot (x - x_*)}{\|x - x_*\|} \geq 0\,. \quad (17)$$

Since the x_j converge to x_* and are different from x_*, (17) implies

$$\liminf_{j\to\infty} \frac{V(x_j) - V(x_*) - p \cdot (x_j - x_*)}{\|x_j - x_*\|} \geq 0\,. \qquad (18)$$

Since $V(x_j) \leq V(x_*) - h_j \lambda_j + h_j \gamma_j$, (18) implies

$$\liminf_{j\to\infty} \frac{-h_j \lambda_j + h_j \gamma_j - p \cdot (x_j - x_*)}{\|x_j - x_*\|} \geq 0\,. \qquad (19)$$

Now, $x_j - x_* = x_j - x_* - h_j v_j + h_j v_j = h_j(w_j + v_j)$, where $w_j = h_j^{-1}(x_j - x_* - h_j v_j)$. Hence

$$\liminf_{j\to\infty} \frac{-\lambda_j + \gamma_j - p \cdot (w_j + v_j)}{\|w_j + v_j\|} \geq 0\,. \qquad (20)$$

Hence, given a positive ε there exists a $j(\varepsilon)$ such that $-\lambda_j + \gamma_j - p \cdot (w_j + v_j) \geq -\varepsilon \|w_j + v_j\|$ whenever $j \geq j(\varepsilon)$. Therefore

$-\lambda_j - p \cdot v_j \geq -\varepsilon \|v_j\| - \varepsilon \|w_j\| + p \cdot w_j - \gamma_j$ if $j \geq j(\varepsilon)$.

Since $w_j \to 0$, there is—for each ε—a $j'(\varepsilon)$ such that

$$-\lambda_j - p \cdot v_j \geq -\varepsilon \|v_j\| - \varepsilon \text{ whenever } j \geq j'(\varepsilon)\,. \quad (21)$$

The coercivity bound yields $\|v_j\|^r \leq \lambda_j + C$, so $-\lambda_j \leq -\|v_j\|^r + C$. Hence

$-\|v_j\|^r + C - p \cdot v_j \geq -\varepsilon \|v_j\| - \varepsilon$ whenever $j \geq j'(\varepsilon)$,

so

$$-\|v_j\|^r + \varepsilon \|v_j\| + \|p\|.\|v_j\| \geq -\varepsilon - C \text{ if } j \geq j'(\varepsilon)\,. \quad (22)$$

Now, if the sequence $\{\|v_j\|\}_{j=1}^\infty$ was unbounded, then we could pick an infinite subset J of $\mathbb{N}$ such that $\|v_j\| \to +\infty$ as $j \to \infty$ via values in J. But then, taking for example $\varepsilon = 1$, we would contradict (22), because the number $-\|v_j\|^r + \|v_j\| + \|p\|.\|v_j\|$ is equal to $-\|v_j\|^r \Big(1 - (1 + \|p\|)\|v_j\|^{1-r}\Big)$, which goes to $-\infty$ as $j \to \infty$ via values in J. Therefore the sequence $\{\|v_j\|\}_{j=1}^\infty$ is bounded. Pick a constant K such that $\|v_j\| \leq K$ for all j. Then, for each ε, if u_ε is such that $v_{j'(\varepsilon)} = f(x_*, u_\varepsilon)$ and $\lambda_{j'(\varepsilon)} = L(x_*, u_\varepsilon)$, (21) implies

$$-p \cdot f(x_*, u_\varepsilon) - L(x_*, u_\varepsilon) \geq -\varepsilon(K + 1)\,.$$

Hence

$$\sup\{-p \cdot f(x_*, u) - L(x_*, u) : u \in U\} \geq -\varepsilon(K + 1)\,.$$

Since ε is arbitrary, we see that (16) holds.

This concludes the proof of the "if" part of our statement. We now proceed to proving the "only if" part. We assume that V satisfies (13) on Ω in the viscosity sense, pick an $x_* \in \Omega$, and prove that V satisfies (ISD).

We will actually prove that the following condition, which is in principle weaker, holds.

(ISD') *there exist sequences* $\{x_j\}_{j=1}^\infty$, $\{v_j\}_{j=1}^\infty$, $\{\lambda_j\}_{j=1}^\infty$, $\{h_j\}_{j=1}^\infty$, $\{\delta_j\}_{j=1}^\infty$, $\{\gamma_j\}_{j=1}^\infty$ *in* Ω, $\mathbb{R}^n$, $\mathbb{R}$, $\mathbb{R}$, $\mathbb{R}$, *and* $\mathbb{R}$, *respectively, such that*

(1) $h_j > 0$, $\delta_j > 0$, $\gamma_j > 0$, $(v_j, \lambda_j) \in \Phi_{\delta_j}(x_*)$, $\|x_j - x_*\| \le \delta_j$, *and* $\|x_j - x_* - h_j v_j\| \le h_j\gamma_j$ *for all* j,

(2) $h_j \downarrow 0$, $\delta_j \downarrow 0$, *and* $\gamma_j \downarrow 0$ *as* $j \to \infty$,

and

(3) $V(x_j) \le V(x_*) - h_j\lambda_j + h_j\gamma_j$ for all j.

We first show that if (ISD') holds then (ISD) holds as well. To do this, we pick sequences $\{x_j\}_{j=1}^\infty$, $\{v_j\}_{j=1}^\infty$, $\{\lambda_j\}_{j=1}^\infty$, $\{h_j\}_{j=1}^\infty$, $\{\delta_j\}_{j=1}^\infty$, $\{\gamma_j\}_{j=1}^\infty$ with the properties specified in (ISD'). We pick δ such that

$$B_\delta(x_*) = \{x \in \mathbb{R}^n : \|x - x_*\| \le \delta\} \subseteq \Omega\,,$$

and a function $\omega\ :\,]0, +\infty[\ \mapsto\ [0, +\infty]$ such that $\lim_{s\downarrow 0}\omega(s) = 0$ and

$$\|f(x,u) - f(x',u)\| + |L(x,u) - L(x',u)| \le \omega(\|x - x'\|)$$

whenever $x, x' \in B_\delta(x_*)$. We then pass to a subsequence, if necessary, and assume that $\delta_j \le \delta$ for all j. We use the fact that $(v_j, \lambda_j) \in \Phi_{\delta_j\hat{\Sigma}}(x_*)$ to find a (w_j, ℓ_j) of the form $\sum_{k=0}^{n+1}\alpha_{j,k}F_{\hat{\Sigma}}(x_{j,k}, u_{j,k})$ such that $\alpha_{j,k} \ge 0$, $\sum_{k=0}^{n+1}\alpha_{j,k} = 1$, $\|x_{j,k} - x_*\| \le \delta_j$, and $\|w_j - v_j\| + |\ell_j - \lambda_j| \le 2^{-j}$. We then define

$$\begin{aligned}\tilde{v}_j &= \sum_{k=0}^{n+1}\alpha_{j,k} f(x_*, u_{j,k})\,,\\ \tilde{\lambda}_j &= \sum_{k=0}^{n+1}\alpha_{j,k} L(x_*, u_{j,k})\,,\end{aligned}$$

and conclude that $\|\tilde{v}_j - w_j\| \le \omega(\delta_j)$ and $\|\tilde{\lambda}_j - \ell_j\| \le \omega(\delta_j)$. We then let

$$\tilde{\gamma}_j = \gamma_j + 2^{-j} + \omega(\delta_j)\,,$$

so $\tilde{\gamma}_j \to 0$ as $j \to \infty$. It is then clear that $\|x_j = x_* - h_j\tilde{v}_j\| \le h_j\tilde{\gamma}_j$ and $V(x_j) \le V(x_*) - h\tilde{\lambda}_j + h_j\tilde{\gamma}_j$ for all j. Finally, we have $(\tilde{v}_j, \tilde{\lambda}_j) = \sum_{k=0}^{n+1}\alpha_{j,k}F_{\hat{\Sigma}}(x_*, u_{j,k})$, so $(\tilde{v}_j, \tilde{\lambda}_j) \in F_{\hat{\Sigma}}(x_*, U)$, since $F_{\hat{\Sigma}}(x_*, U)$ is convex. This completes the proof that (ISD') implies (ISD).

We now prove (ISD'), assuming of course that V satisfies (13) in the viscosity sense. We do this by assuming that the sequences whose existence is asserted by (ISD') do not exist and deriving a contradiction. We will assume, as we clearly may without loss of generality, that $x^* = 0$ and $V(x^*) = 0$, i.e., $V(0) = 0$.

Since V does not satisfy (ISD'), there must exist a $\bar{\gamma}$ such that $0 < \bar{\gamma} < \mathrm{dist}(0, \mathbb{R}^n\backslash\Omega)$ and

$$V(h(v + v')) + h\lambda > h\bar{\gamma} \tag{23}$$

whenever $0 < h \le \bar{\gamma}$, $(v, \lambda) \in \Phi_{\bar{\gamma},\hat{\Sigma}}(0)$, $h\|v + v'\| \le \bar{\gamma}$, and $\|v'\| \le \bar{\gamma}$. (Indeed, if $\bar{\gamma}$ did not exist, then for each sufficiently natural number j we could define $\gamma_j = 2^{-j}$, and find h_j, v_j, λ_j, v'_j, x_j, such that $0 < h_j \le \gamma_j$, $(v_j, \lambda_j) \in \Phi_{\gamma_j,\hat{\Sigma}}(0)$, $h_j\|v_j + v'_j\| \le \gamma_j$, $\|v'_j\| \le \gamma_j$, and $V(x_j) + h_j\lambda_j \le \gamma_j h_j$. But then, if we take $x_j = h_j(v_j + v'_j)$, $\delta_j = \gamma_j$, the sequence $\{(x_j, v_j, \lambda_j, h_j, \delta_j, \gamma_j)\}_{j=1}^\infty$ satisfies conditions (1), (2) and (3) of (ISD'), contradicting the fact that V does not satisfy (ISD').)

By making $\bar{\gamma}$ smaller, if necessary, we can assume that there exist real numbers C, r, such that $C > 0$, $r > 1$ and $\|f(x,u)\|^r \le L(x,u) + C$ whenever $\|x\| \le \bar{\gamma}$ and $u \in U$. It then follows, if we let

$$\psi(v, \lambda) = \|v\|^r - \lambda - C \text{ for } (v, \lambda) \in \mathbb{R}^n \times \mathbb{R}\,,$$

that

$$\psi(w) \le 0 \text{ whenever } w = F(x,u)\,, \|x\| \le \bar{\gamma}, u \in U\,. \tag{24}$$

Since ψ is convex, the inequality $\psi(w) \le 0$ holds whenever $w \in \Phi_{\bar{\gamma},\hat{\Sigma}}(0)$, and then

$$\|v\|^r \le \lambda + C \quad \text{whenever } (v, \lambda) \in \Phi_{\bar{\gamma},\hat{\Sigma}}(0)\,. \tag{25}$$

If $(v, \lambda) \in \Phi_{\bar{\gamma},\hat{\Sigma}}(0)$, $v' \in \mathbb{R}^n$, and $\|v'\| \le \bar{\gamma}$, then the inequality $(\alpha + \beta)^r \le 2^r(\alpha^r + \beta^r)$, valid for nonnegative α, β, implies

$$\begin{aligned}\|v + v'\|^r &\le (\|v\| + \|v'\|)^r\\ &\le 2^r(\|v\|^r + \|v'\|^r)\\ &\le 2^r(\lambda + C + \bar{\gamma}^r)\,.\end{aligned}$$

Pick $\tilde{r}$ such that $1 < \tilde{r} < r$. Let $A = 2^{\frac{r}{r-\tilde{r}}}$ Then

$$\begin{aligned}\|v + v'\|^{\tilde{r}} &= \|v + v'\|^{\tilde{r}-r}\cdot\|v + v'\|^r\\ &\le A^{\tilde{r}-r}\|v + v'\|^r\\ &\le A^{\tilde{r}-r}2^r(\lambda + C + \bar{\gamma}^r)\\ &= 2^{-r}2^r(\lambda + C + \bar{\gamma}^r)\\ &= \lambda + C + \bar{\gamma}^r\end{aligned}$$

if $\|v + v'\|^{\tilde{r}} \ge A$. On the other hand, if $\|v + v'\|^{\tilde{r}} \le A$, then $\|v + v'\|^{\tilde{r}} \le A \le \|v\|^r + A \le \lambda + C + A$. Therefore, if we let $\tilde{C} = C + \max(A, \bar{\gamma}^r)$, and then relabel $\tilde{r}$, $\tilde{C}$ as our new r and C, we have shown that

$$\Big((v, \lambda) \in \Phi_{\bar{\gamma},\hat{\Sigma}}(0) \wedge v' \in \mathbb{R}^n \wedge \|v'\| \le \bar{\gamma}\Big) \Rightarrow \|v + v'\|^r \le \lambda + C\,. \tag{26}$$

For $0 < \delta \le \bar{\gamma}$, we define

$$\Xi_\delta(0) \stackrel{\text{def}}{=} \Big\{(v + v', \lambda + \lambda') : (v, \lambda) \in \Phi_{\delta,\hat{\Sigma}}(0), v' \in \mathbb{R}^n\,, \|v'\| \le \bar{\gamma}\,, \lambda' \in \mathbb{R}\,, \lambda' \ge 0\Big\}\,.$$

Then

(a) *If* $0 < \delta \le \bar{\gamma}$, *then* $\Xi_\delta(0)$ *is a closed, convex, subset of* $\mathbb{R}^{n+1}$ *with nonempty interior.*

(The fact that $\Xi_\delta(0)$ has a nonempty interior follows because, if we pick any $(\hat{v}, \hat{\lambda}) \in \Phi_{\delta,\hat{\Sigma}}(0)$, then

$$\Big\{(\hat{v} + v', \lambda) : v' \in \mathbb{R}^n\,, \|v'\| < \bar{\gamma}\,, \lambda > \hat{\lambda}\Big\} \subseteq \Xi_\delta(0)\,.$$

The fact that $\Xi_\delta(0)$ is closed follows because, if a sequence $\{(v_j + v'_j, \lambda_j + \lambda'_j)\}_{j=1}^\infty$ with the property that $(v_j, \lambda_j) \in \Phi_{\delta,\hat{\Sigma}}(0)$, $v'_j \in \mathbb{R}^n$, $\|v'_j\| \le \bar{\gamma} \in \mathbb{R}^n$, $\lambda'_j \ge 0$ converges to a limit $(\hat{v}, \hat{\lambda})$, then the sequence $\{v'_j\}_{j=1}^\infty$ is bounded, so we may assume after passing to a subsequence that $\lim_{j\to\infty} v'_j = v'$ exists, and then of course $\|v'\| \leq \bar{\gamma} \in \mathbb{R}^n$, and $\lim_{j\to\infty} v_j = v$ exists as well, since $v_j + v'_j \to \hat{v}$, and then $\hat{v} = v + v'$. Furthermore, the bound $\|v_j\|^r \leq \lambda_j + C$ implies that $\lambda_j \geq \|v_j\|^r - C \geq -C$, so

$$\lambda'_j = (\lambda_j + \lambda'_j) - \lambda_j \leq (\lambda_j + \lambda'_j) + C\,.$$

Hence the sequence $\{\lambda'_j\}_{j=1}^{\infty}$ is bounded above, because $\{\lambda_j + \lambda'_j\}_{j=1}^{\infty}$ is convergent. Since $\lambda'_j \geq 0$, the sequence $\{\lambda'_j\}_{j=1}^{\infty}$ is bounded, so we may assume it is convergent to a limit λ', after passing to a subsequence. Clearly, then, $\lambda' \geq 0$, and the limit $\lim_{j\to\infty} \lambda_j = \lambda$ exists as well, and satisfies $\hat{\lambda} = \lambda + \lambda'$. Since $(v, \lambda) = \lim_{j\to\infty}(v_j, \lambda_j)$, $(v_j, \lambda_j) \in \Phi_{\delta,\hat{\Sigma}}(0)$, and $\Phi_{\delta,\hat{\Sigma}}(0)$ is closed, we see that $(v, \lambda) \in \Phi_{\delta,\hat{\Sigma}}(0)$. Since $\|v'\| \leq \bar{\gamma} \in \mathbb{R}^n$, and $\lambda' \geq 0$, we see that $(\hat{v}, \hat{\lambda}) \in \Xi_\delta(0)$. The convexity of $\Xi_\delta(0)$ is trivial.)

Let $\Lambda = \{0\} \times [\bar{\gamma}, +\infty\,[$, so $\Lambda \subseteq \mathbb{R}^{n+1}$. We let $\Psi_\delta(0)$ be the convex hull of $\Lambda \cup \Xi_\delta(0)$. We show that

(b) *If* $0 < \delta \leq \bar{\gamma}$ *then* $\Psi_\delta(0)$ *is a closed convex subset of* $\mathbb{R}^{n+1}$ *with nonempty interior.*

(c) *There exist real numbers* r, C *such that* $r > 1$ *and*

$$\|v\|^r \leq \lambda + C \quad \text{whenever } (v, \lambda) \in \Psi_{\bar{\gamma}}(0)\,. \tag{27}$$

(d) *The inequality*

$$V(hv) + h\lambda \geq h\bar{\gamma} \tag{28}$$

holds whenever $0 < h \leq \bar{\gamma}$, $(v, \lambda) \in \Psi_{\bar{\gamma}}(0)$, *and* $h\|v\| \leq \bar{\gamma}$.

(e) $(0, \ell) \notin \Psi_{\bar{\gamma}}(0)$ *whenever* $\ell < \bar{\gamma}$

We will prove the above assertions in order, except for the statement that $\Psi_\delta(0)$ is closed if $0 < \delta \leq \bar{\gamma}$, which will be proved last.

The fact that $\Psi_\delta(0)$ is convex is trivial, and the fact that $\Psi_\delta(0)$ has nonempty interior follows from (a), because $\Xi_\delta(0) \subseteq \Psi_\delta(0)$.

To prove (c), we choose r, C such that $r > 1$, $C > 0$, and (26) holds, and observe that (26) trivially implies that the inequality $\|v\|^r \leq \lambda + C$ is true whenever $(v, \lambda) \in \Xi_{\bar{\gamma}}(0)$. Since $C > 0$, the inequality is also true whenever $(v, \lambda) \in \Lambda$. Hence $\|v\|^r \leq \lambda + C$ whenever $(v, \lambda) \in \Xi_{\bar{\gamma}}(0) \cup \Lambda$, from which it follows that $\|v\|^r \leq \lambda + C$ whenever $(v, \lambda) \in \Psi_\delta(0)$, since the function $(v, \lambda) \mapsto \|v\|^r - \lambda - C$ is convex.

To prove (d), we observe that (23) trivially implies that $V(hv) + h\lambda > h\bar{\gamma}$ whenever $(v, \lambda) \in \Xi_{\bar{\gamma}}(0)$, $0 < h \leq \bar{h}$, and $h\|v\| \leq \bar{\gamma}$. (Indeed, if $(v, \lambda) \in \Xi_{\bar{\gamma}}(0)$, $0 < h \leq \bar{h}$, and $h\|v\| \leq \bar{\gamma}$, then $(v, \lambda) = (\bar{v}, \bar{\lambda}) + (v', \lambda')$, with $(\bar{v}, \bar{\lambda}) \in \Phi_{\bar{\gamma},\hat{\Sigma}}(0)$, $v' \in \mathbb{R}^n$, $\|v'\| \leq \bar{\gamma}$, $\lambda' \in \mathbb{R}$, and $\lambda' \geq 0$. Then we can apply (23), with $\bar{v}$, $\bar{\lambda}$ in the roles of v, λ, and conclude that $V(h(\bar{v} + v')) + h\bar{\lambda} > h\bar{\gamma}$, since $h\|\bar{v} + v'\| \leq \bar{\gamma}$. Therefore $V(hv) + h\bar{\lambda} > h\bar{\gamma}$, and then *a fortiori* $V(hv) + h\lambda > h\bar{\gamma}$, since $\lambda = \bar{\lambda} + \lambda'$ and $\lambda' \geq 0$.) Assume that $(v, \lambda) \in \Psi_{\bar{\gamma}}(0)$. Then we can write $(v, \lambda) = \alpha(v', \lambda') + (1 - \alpha)(0, \ell)$, with $(v', \lambda') \in \Xi_{\bar{\gamma}}(0)$, $\ell \geq \bar{\gamma}$, and $0 \leq \alpha \leq 1$. Let h be such that $0 < h \leq \bar{\gamma}$ and $h\|v\| \leq \bar{\gamma}$. If $\alpha = 0$, then $hv = 0$ and $\lambda = \ell \geq \bar{\gamma}$, so $V(hv) + h\lambda = V(0) + h\ell = h\ell \geq h\bar{\gamma}$. If $\alpha > 0$, define $\tilde{h} = \alpha h$. Then $0 < \tilde{h} \leq \bar{\gamma}$ and $\tilde{h}\|v'\| = \alpha h\|v'\| = h\|v\| \leq \bar{\gamma}$, since $v = \alpha v'$. Therefore $V(\tilde{h}v') + \tilde{h}\lambda' \geq \tilde{h}\bar{\gamma}$, since $(v', \lambda') \in \Psi_{\bar{\gamma}}(0)$. On the other hand, $(1 - \alpha)h\ell \geq (1 - \alpha)h\bar{\gamma}$, since $\ell \geq \bar{\gamma}$. Therefore

$$\begin{aligned}
V(hv) + h\lambda &= V(h\alpha v') + h\alpha\lambda' + h(1 - \alpha)\ell \\
&= V(\tilde{h}v') + \tilde{h}\lambda' + h(1 - \alpha)\ell \\
&\geq \tilde{h}\bar{\gamma} + h(1 - \alpha)\bar{\gamma} \\
&= h\alpha\bar{\gamma} + h(1 - \alpha)\bar{\gamma} \\
&= h\bar{\gamma}\,,
\end{aligned}$$

completing the proof of (d).

Statement (e) now follows easily: if $(0, \ell) \in \Psi_{\bar{\gamma}}(0)$, then we can apply (d) taking $h = \bar{\gamma}$, $v = 0$, $\lambda = \ell$, and conclude that $\bar{\gamma}\ell \geq \bar{\gamma}^2$, so that $\ell \geq \bar{\gamma}$.

We now prove that $\Psi_\delta(0)$ is closed if $0 < \delta \leq \bar{\gamma}$. Let $\{w_j\}_{j=1}^{\infty}$ be a sequence of points of $\Psi_\delta(0)$ that converges to a limit $w \in \mathbb{R}^{n+1}$. We will show that $w \in \Psi_\delta(0)$. Let $w_j = \alpha_j(v_j, \lambda_j) + (1 - \alpha_j)(0, \ell_j)$, where $(v_j, \lambda_j) \in \Xi_\delta(0)$, $\ell_j \geq \bar{\gamma}$, and $0 \leq \alpha_j \leq 1$. Let $w = (v, \lambda)$. By passing to a subsequence, if necessary, we may assume that the α_j converge to a limit $\hat{\alpha}$. If the sequence $\{(v_j, \lambda_j)\}$ is bounded, then we may pass to a subsequence and assume that $(\hat{v}, \hat{\lambda}) = \lim_{j\to\infty}(v_j, \lambda_j)$ exists. Then $(\hat{v}, \hat{\lambda}) \in \Xi_\delta(0)$. Therefore the limit $\lim_{j\to\infty} \alpha_j(v_j, \lambda_j)$ exists, so $\mu = \lim_{j\to\infty}(1 - \alpha_j)\ell_j$ exists as well, and $\mu \geq 0$. Clearly, $w = \hat{\alpha}(\hat{v}, \hat{\lambda}) + (0, \mu)$. If $\hat{\alpha} = 1$, then $w = (\hat{v}, \hat{\lambda}) + (0, \mu)$, so $w \in \Xi_\delta(0)$—and *a fortiori* $w \in \Psi_\delta(0)$—because $(\hat{v}, \hat{\lambda}) \in \Xi_\delta(0)$ and $\mu \geq 0$. If $\hat{\alpha} < 1$, then $\ell = \lim_{j\to\infty} \ell_j$ exists and satisfies $\ell = \frac{\mu}{1-\hat{\alpha}}$ and $\ell \geq \bar{\gamma}$. Then $w = \hat{\alpha}(\hat{v}, \hat{\lambda}) + (1 - \hat{\alpha})(0, \ell)$, and $(\hat{v}, \hat{\lambda})$, $(0, \ell)$, belong to $\Xi_\delta(0)$ and Λ, respectively, so $w \in \Psi_\delta(0)$. Now suppose that the sequence $\{(v_j, \lambda_j)\}$ is unbounded. Then (27) implies that $\{\lambda_j\}$ is unbounded. Since the λ_j are bounded below, we may assume, after passing to a subsequence, that $\lambda_j \to +\infty$ as $j \to \infty$. On the other hand, $\alpha_j v_j \to v$ and $\alpha_j\lambda_j + (1 - \alpha_j)\ell_j \to \lambda$. Since both sequences $\{\alpha_j\lambda_j\}$, $\{(1 - \alpha_j)\ell_j\}$ are bounded below, we pass to a subsequence and assume that the limits $\mu = \lim_{j\to\infty}(1 - \alpha_j)\ell_j$ and $\nu = \lim_{j\to\infty} \alpha_j\lambda_j$ exist. But then

$$\begin{aligned}
\alpha_j\|v_j\| \leq \alpha_j(\lambda_j + C)^{1/r} \quad &= \quad \alpha_j\lambda_j^{1/r}\Big(1 + \frac{C}{\lambda_j}\Big)^{1/r} \\
&= \quad \alpha_j\lambda_j\lambda_j^{1/r-1}\Big(1 + \frac{C}{\lambda_j}\Big)^{1/r} \\
&\underset{j\to\infty}{\longrightarrow} \quad 0\,,
\end{aligned}$$

since $\lambda_j \underset{j\to\infty}{\longrightarrow} +\infty$, $\alpha_j\lambda_j \underset{j\to\infty}{\longrightarrow} \nu$, and $r > 1$. Then $v = \lim_{j\to\infty} \alpha_j v_j = 0$. This implies, in particular, that $\bar{\gamma}\alpha_j\|v_j\| \leq \bar{\gamma}$ if j is large enough. So we can apply (d) with $h = \bar{\gamma}\alpha_j$ and (v_j, λ_j) in the role of (v, λ), and conclude that $V(\bar{\gamma}\alpha_j v_j) + \bar{\gamma}\alpha_j\lambda_j \geq \bar{\gamma}\alpha_j\bar{\gamma}$. On the other hand, $(1 - \alpha_j)\ell_j \geq (1 - \alpha_j)\bar{\gamma}$, because $\ell_j \geq \bar{\gamma}$. Therefore

$$V(\bar{\gamma}\alpha_j v_j) + \bar{\gamma}\alpha_j\lambda_j + \bar{\gamma}(1 - \alpha_j)\ell_j \geq \bar{\gamma}\alpha_j\bar{\gamma} + \bar{\gamma}(1 - \alpha_j)\bar{\gamma} = \bar{\gamma}^2$$

for large enough j. If we let $j \to \infty$, and use the facts that $\alpha_j v_j \underset{j\to\infty}{\longrightarrow} 0$, V is continuous, and $V(0) = 0$, we find that $\bar{\gamma}\lambda = \lim_{j\to\infty} 0(\bar{\gamma}\alpha_j\lambda_j + \bar{\gamma}(1 - \alpha_j)\ell_j) \geq \bar{\gamma}^2$, so $\lambda \geq \bar{\gamma}$. Therefore $w = (0, \lambda)$ and $\lambda \geq \bar{\gamma}$, so $w \in \Lambda$ and then $w \in \Psi_\delta(0)$.

We have now completed the proofs of (b), (c), (d) and (e). Since $\Psi_{\bar{\gamma}}(0)$ is closed and convex, the fact that $(0, 0) \notin \Psi_{\bar{\gamma}}(0)$ implies that there exist a nonzero vector

$p^* \in \mathbb{R}^{n+1}$ and a real number $\bar{p}$ such that $\bar{p} > 0$ and $\langle p^*, w\rangle \geq \bar{p}$ whenever $w \in \Psi_{\bar{\gamma}}(0)$. We fix p^* and $\bar{p}$ from now on.

We now let Q be the set of all vectors $q \in \mathbb{R}^{n+1}$ such that $q = hw$ for some h, w such that $h \in [0, \infty[$ and $w \in \Psi_{\bar{\gamma}}(0)$, and write $\mathring{Q}$ to denote the interior of Q. We will show that

(f) *Q is a closed convex cone with nonempty interior.*

(g) *$\langle p^*, q\rangle \geq 0$ for all $q \in Q$ and $\langle p^*, q\rangle > 0$ for all $q \in \mathring{Q}$.*

(h) *$(0,0) \notin \mathring{Q}$.*

Indeed, Q is obviously a convex cone. The fact that $\mathring{Q} \neq \emptyset$ follows from (b). To show that Q is closed, we pick a sequence $\{q_j\}_{j=1}^{\infty}$ of points of Q that converges to a limit $q \in \mathbb{R}^{n+1}$, and show that $q \in Q$. Write $q_j = h_j w_j$, $h_j \geq 0$, $w_j \in \Psi_{\bar{\gamma}}(0)$. If $q = (0,0)$ then $q \in Q$, so we may assume that $q \neq (0,0)$ and that $q_j \neq (0,0)$ for all j. Then $h_j \neq 0$ as well. If the sequence $\{w_j\}_{j=1}^{\infty}$ is bounded, then we may pass to a subsequence and assume that the w_j converge to a limit w, which must belong to $\Psi_{\bar{\gamma}}(0)$ because $\Psi_{\bar{\gamma}}(0)$ is closed. In particular, w and the w_j are $\neq (0,0)$. But then $h_j = \frac{\|q_j\|}{\|w_j\|} \underset{j\to\infty}{\longrightarrow} \frac{\|q\|}{\|w\|} \stackrel{\text{def}}{=} h$. Therefore $q = hw$, so $q \in Q$. Now suppose that the sequence $\{w_j\}_{j=1}^{\infty}$ is unbounded. Write $w_j = (v_j, \lambda_j)$, and use (c) to conclude that $\|v_j\|^r \leq \lambda_j + C$ for all j. Then the sequence $\{\lambda_j\}_{j=1}^{\infty}$ is unbounded, and we may assume, after passing to a subsequence, that $\lambda_j \underset{j\to\infty}{\longrightarrow} +\infty$. Since $q = \lim_{j\to\infty}(h_j v_j, h_j \lambda_j)$, the sequence $\{h_j\lambda_j\}_{j=1}^{\infty}$ converges to a finite limit μ, so $h_j \underset{j\to\infty}{\longrightarrow} 0$. Then $h_j\|v_j\| \leq h_j(\lambda_j + C)^{1/r} = h_j \lambda_j \lambda_j^{1/r-1}\left(1 + \frac{C}{\lambda_j}\right)^{1/r} \underset{j\to\infty}{\longrightarrow} 0$. So $h_j v_j \underset{j\to\infty}{\longrightarrow} 0$. Therefore $q = (0, \mu)$, so $q = h(0, \bar{\gamma})$, where $h = \frac{\mu}{\bar{\gamma}}$. Since $\mu \geq 0$ and $(0, \bar{\gamma}) \in \Psi_{\bar{\gamma}}(0)$, it is now clear that $q \in Q$. This completes the proof of (f).

The fact that $\langle p^*, q\rangle \geq 0$ for all $q \in Q$ follows trivially from the definition of Q, since $\langle p^*, w\rangle \geq \bar{p}$ whenever $w \in \Psi_{\bar{\gamma}}(0)$. The fact that $\langle p^*, q\rangle > 0$ for all $q \in \mathring{Q}$ is then a consequence of the nontriviality of p^*. This proves (g), and then (h) follows.

Next, we define a function $\sigma : Q \mapsto \mathbb{R}$ by letting $\sigma(q)$ be, if $q \in Q$, the largest $h \in \mathbb{R}$ such that $q = hw$ for some $w \in \Psi_{\bar{\gamma}}(0)$. (The existence of such a largest h is trivial if $q = 0$, for in that case the only possible value of h is 0, since $(0,0) \notin \Psi_{\bar{\gamma}}(0)$. If $q \in Q$ and $q \neq 0$, let $H = \{h \in \mathbb{R} : h > 0,\ h^{-1}q \in \Psi_{\bar{\gamma}}(0)\}$. Then H must be bounded, for otherwise $(0,0)$ would be a limit of points of $\Psi_{\bar{\gamma}}(0)\}$, and then $(0,0)$ would have to belong to $\Psi_{\bar{\gamma}}(0)$. If $h = \sup H$, then the fact that $\Psi_{\bar{\gamma}}(0)$ is closed implies that $h \in H$, so $\sigma(q)$ exists and is equal to h.) We prove the following properties of σ.

(i) *σ is strictly positive on $Q\backslash\{0\}$.*
(j) *σ is positively homogeneous of degree 1 (that is, $\sigma(q) \geq 0$ and $\sigma(rq) = r\sigma(q)$ whenever $q \in Q$ and $r \geq 0$).*
(k) *σ is upper semicontinuous.*
(l) *There exists a constant $\kappa \in \mathbb{R}$ such that $\sigma(q) \leq \kappa\|q\|$ whenever $q \in Q$.*
(m) *σ is continuous at $(0,0)$.*
(n) *σ is concave on Q (that is, the inequality*

$$\sigma(\alpha q + (1-\alpha)q') \geq \alpha\sigma(q) + (1-\alpha)\sigma(q') \qquad (29)$$

holds whenever $q, q' \in Q$ and $0 \leq \alpha \leq 1$).
(o) *$\sigma(\alpha q + \alpha' q') \geq \alpha\sigma(q) + \alpha'\sigma(q')$ whenever $q, q' \in Q$ and $\alpha, \alpha' \in [0, \infty[$.*
(p) *σ is locally Lipschitz on $\mathring{Q}$.*

Statements (i) and (j) are immediate consequences of the definition of σ. To show that σ is upper semicontinuous, we assume that $\{q_j\}_{j=1}^{\infty}$ is a sequence of members of Q that converges to a limit $q \in Q$ and $\alpha \in \mathbb{R}$ is such that $\sigma(q_j) \geq \alpha$ for all j, and prove that $\sigma(q) \geq \alpha$. By passing to a subsequence, we may assume that the limit $L = \lim_{j\to\infty}\sigma(q_j)$ exists in $[0, \infty]$, and it is then clear that $L \geq \alpha$. The conclusion that $\sigma(q) \geq \alpha$ is trivial if $\alpha \leq 0$, so we may assume that $\alpha > 0$. Write $q_j = \sigma(q_j)w_j$, $w_j \in \Psi_{\bar{\gamma}}(0)$. Then $w_j = \frac{q_j}{\sigma(q_j)}$. If $L = +\infty$ then $\lim_{j\to\infty} w_j = 0$, so 0 (that is, the zero point $(0,0)$ of $\mathbb{R}^n \times \mathbb{R}$) would belong to $\Psi_{\bar{\gamma}}(0)$, since $\Psi_{\bar{\gamma}}(0)$ is closed. Since we know that $(0,0) \notin \Psi_{\bar{\gamma}}(0)$, L has to be finite. Since $L \geq \alpha$ and we are assuming that $\alpha > 0$, L must be strictly positive. Then, if we let $w = \frac{q}{L}$, we have $w = \lim_{j\to\infty}\frac{q_j}{\sigma(q_j)} = \lim_{j\to\infty} w_j$. Therefore $w \in \Psi_{\bar{\gamma}}(0)$. Since $q = Lw$, the definition of σ implies that $\sigma(q) \geq L \geq \alpha$, concluding the proof of (k).

Let $K = \{q \in Q : \|q\| = 1\}$. Then K is compact. Since σ is upper semicontinuous, σ is bounded above on K. Let κ be such that $\sigma(q) \leq \kappa$ whenever $q \in K$. Then the homogeneity of σ implies that $\sigma(q) \leq \kappa\|q\|$ whenever $q \in Q$. This proves (l), and then (m) follows immediately.

Let us prove the concavity of σ. We must show that $\sigma(\alpha q + (1-\alpha)q') \geq \alpha\sigma(q) + (1-\alpha)\sigma(q')$ whenever q, q' belong to Q and $0 \leq \alpha \leq 1$. The conclusion is trivial if $q = 0$, $q' = 0$, $r = 0$, or $r = 1$. So let us suppose that $q, q' \in Q$, $q \neq 0 \neq q'$, and $0 < r < 1$. Then $\sigma(q) > 0$, $\sigma(q') > 0$, $q = \sigma(q)w$ and $q' = \sigma(q')w'$ for some $w, w' \in \Psi_{\bar{\gamma}}(0)$. It follows that

$$\begin{aligned} \alpha q + (1-\alpha)q' &= \alpha\sigma(q)w + (1-\alpha)\sigma(q')w' \\ &= h\tilde{w}\,, \end{aligned}$$

where

$$h = \alpha\sigma(q) + (1-\alpha)\sigma(q')\,,\ \ \tilde{w} = \frac{\alpha\sigma(q)w + (1-\alpha)\sigma(q')w'}{h}\,.$$

The convexity of $\Psi_{\bar{\gamma}}(0)$ then implies that $\tilde{w} \in \Psi_{\bar{\gamma}}(0)$. It then follows from the definition of σ that $\sigma(\alpha q + (1-\alpha)q') \geq h$, as desired.

Finally, statement (o) follows trivially from the concavity and the homogeneity of σ, and statement (p) is a consequence of the well-known fact that a concave function on an open convex set is locally Lipschitz. This completes the proof of (f), (g), (h), (i), (j), (k), (l), (m), (n), (o), and (p).

We now define a function $\theta : Q \mapsto \mathbb{R}$ by letting

$$\theta(q) = \bar{\gamma}\sigma(q)\,.$$

Then θ is nonnegative, positively homogeneous of degree 1, concave, upper semicontinuous, and continuous at $(0,0)$, and satisfies the bound $\theta(q) \leq \kappa\bar{\gamma}\|q\|$ for all $q \in Q$. In particular, (o) implies that

$$\begin{aligned} &\theta(\alpha q + \alpha' q') \ \geq\ \alpha\theta(q) + \alpha'\theta(q') \\ &\text{whenever} \quad q, q' \in Q \ \text{and}\ \alpha, \alpha' \in [0, \infty[\,. \end{aligned} \qquad (30)$$

Next, we define

$$\gamma^* \stackrel{\text{def}}{=} \frac{\bar{\gamma}}{\max(1,\kappa)}$$

$$Q^* \stackrel{\text{def}}{=} \Big\{q \in Q : \|q\| \le \gamma^*\Big\}.$$

It then follows immediately that the interior $\mathring{Q}^*$ of Q^* is given by

$$\mathring{Q}^* = \Big\{q \in \mathring{Q} : \|q\| < \gamma^*\Big\}.$$

Let us define a function $W : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ by letting $W(x, x_0) = V(x) + x_0$ for $(x, x_0) \in \Omega \times \mathbb{R}$. We then claim that

$$W(q) \ge \theta(q) \quad \text{whenever } q \in Q^*. \tag{31}$$

To prove (31), we first observe that the inequality is clearly true if $q = 0$. Let us pick $q \in Q^* \backslash \{0\}$ and write $q = hw$, where $h = \sigma(q)$ and $w = (v, \lambda) \in \Psi_{\bar{\gamma}}(0)$. Then $0 < h$, since $q \neq 0$, and $h \le \bar{\gamma}$, because $\sigma(q) \le \kappa\|q\| \le \kappa\gamma^* \le \bar{\gamma}$. Furthermore, $\|q\| \le \bar{\gamma}$ (because $\gamma^* \le \bar{\gamma}$), and $q = (hv, h\lambda)$, so $h\|v\| \le \bar{\gamma}$. It then follows from (23) that

$$W(q) = V(hv) + h\lambda \ge h\bar{\gamma} = \bar{\gamma}\sigma(q) = \theta(q),$$

completing the proof of (31).

We now fix a point $q^\# \in \mathring{Q}$ such that $\sigma(q^\#) = 1$. We then define, for each positive real number s, a subset Q_s of Q, by letting $Q_s = \{q + sq^\# : q \in Q\}$. Then Q_s is a closed convex set, because it is a translate of Q. Moreover, it is clear that $Q_s \subseteq \mathring{Q}$.

If $0 < a < b < c < \infty$, define a function $\eta_{a,b,c} : \mathbb{R} \mapsto \mathbb{R}$ by letting

$$\eta_{a,b,c}(t) = \begin{cases} t + b - a & \text{if } \ t \le a\,, \\ \frac{t(c-b)+c(b-a)}{c-a}) & \text{if } \ a \le t \le c\,, \\ t & \text{if } \ c \le t\,. \end{cases}$$

Then $\eta_{a,b,c}$ is strictly increasing, continuous, and piecewise linear. Furthermore,,

$$\frac{d\eta_{a,b,c}}{dt}(t) = \begin{cases} 1 & \text{if } \ t \le a \ \text{ or } \ t \ge c, \\ \frac{c-b}{c-a} & \text{if } \ a \le t \le c\,. \end{cases}$$

If s, ε, ρ, ζ are positive numbers such that $0 < \varepsilon < \rho$, we define a function $\theta_{s,\varepsilon,\rho,\zeta} : Q \mapsto \mathbb{R}$ by letting

$$\theta_{s,\varepsilon,\rho,\zeta}(q) = \begin{cases} \theta(q) & \text{if } \ q \in Q\backslash Q_s\,, \\ \max\Big(\theta(q), \eta_{0,\varepsilon,\rho}\big(\zeta.\langle p^*, q\rangle\big)\Big) & \text{if } \ q \in Q \cap Q_s\,, \end{cases}$$

where p^* is the vector introduced earlier. Then $\theta_{s,\varepsilon,\rho,\zeta}$ need not be continuous. However,

(q) *$\theta_{s,\varepsilon,\rho,\zeta}$ is upper semicontinuous on Q.*

(Indeed, if $\alpha \in \mathbb{R}$ and $q \in Q$, then $\theta_{s,\varepsilon,\rho,\zeta}(q) \ge \alpha$ if and only if either (i) $\theta(q) \ge \alpha$ or (ii) $q \in Q_s$ and $\eta_{0,\varepsilon,\rho}\big(\zeta.\langle p^*, q\rangle\big) \ge \alpha$. Suppose $q_j \in Q$, $q_j \to q$, and $\theta_{s,\varepsilon,\rho,\zeta}(q_j) \ge \alpha$. Let $J = \{j \in \mathbb{N} : \theta_{s,\varepsilon,\rho,\zeta}(q_j) = \theta(q_j)\}$. If J is infinite, then $\theta(q_j) \ge \alpha$ for $j \in J$, so $\theta(q) \ge \alpha$, and then $\theta_{s,\varepsilon,\rho,\zeta}(q) \ge \alpha$. If J is finite and $I = \mathbb{N}\backslash J$, then $q_j \in Q_s$ and $\eta_{0,\varepsilon,\rho}\big(\zeta.\langle p^*, q_j\rangle\big) = \theta_{s,\varepsilon,\rho,\zeta}(q_j) \ge \alpha$ for $j \in I$, so $q \in Q_s$ and $\eta_{0,\varepsilon,\rho}\big(\zeta.\langle p^*, q\rangle\big) \ge \alpha$, from which it follows that $\theta_{s,\varepsilon,\rho,\zeta}(q) \ge \alpha$.)

Next, we define $\tilde{\gamma} \stackrel{\text{def}}{=} \min\Big(\bar{\gamma}, \bar{p}\zeta\Big(1 - \frac{\varepsilon}{\rho}\Big)\Big)$, and prove that

(r) *$\theta_{s,\varepsilon,\rho}(q + uw) \ge \theta_{s,\varepsilon,\rho}(q) + u\tilde{\gamma}$ whenever $q \in Q$, $w \in \Psi_{\bar{\gamma}}(0)$, and $u \ge 0$.*

Fix $q \in Q$, $w \in \Psi_{\bar{\gamma}}(0)$ and $u \in [0, \infty[$. If both q and $q + uw$ belong to $Q\backslash Q_s$, then

$$\begin{aligned} \theta_{s,\varepsilon,\rho}(q + uw) &= \theta(q + uw) \\ &\ge \theta(q) + u\theta(w) \\ &= \theta(q) + u\sigma(w)\bar{\gamma} \\ &\ge \theta(q) + u\bar{\gamma} \\ &= \theta_{s,\varepsilon,\rho}(q) + u\bar{\gamma} \\ &\ge \theta_{s,\varepsilon,\rho}(q) + u\tilde{\gamma}\,, \end{aligned}$$

where we have used (30) and the trivial fact that $\sigma(w) \ge 1$ whenever $w \in \Psi_{\bar{\gamma}}(0)$.

Now suppose that both q and $q+uw$ belong to Q_s. If $\theta(q) \ge \eta_{0,\varepsilon,\rho}\big(\zeta.\langle p^*, q\rangle\big)$, then $\theta_{s,\varepsilon,\rho,\zeta}(q) = \theta(q)$, and then

$$\begin{aligned} \theta_{s,\varepsilon,\rho,\zeta}(q + uw) &\ge \theta(q + uw) \\ &\ge \theta(q) + u\bar{\gamma} \\ &= \theta_{s,\varepsilon,\rho,\zeta}(q) + u\bar{\gamma} \\ &\ge \theta_{s,\varepsilon,\rho,\zeta}(q) + u\tilde{\gamma}\,. \end{aligned}$$

If, on the other hand, $\eta_{0,\varepsilon,\rho}\big(\zeta.\langle p^*, q\rangle\big) \ge \theta(q)$, then

$$\begin{aligned} \theta_{s,\varepsilon,\rho,\zeta}(q + uw) &\ge \eta_{0,\varepsilon,\rho}\big(\zeta.\langle p^*, q + uw\rangle\big) \\ &\ge \eta_{0,\varepsilon,\rho}\big(\zeta.\langle p^*, q\rangle\big) + u\zeta.\langle p^*, w\rangle\frac{\rho - \varepsilon}{\rho} \\ &\ge \theta_{s,\varepsilon,\rho,\zeta}(q) + u\bar{p}\zeta\Big(1 - \frac{\varepsilon}{\rho}\Big) \\ &\ge \theta_{s,\varepsilon,\rho,\zeta}(q) + u\tilde{\gamma}\,, \end{aligned}$$

so $\theta_{s,\varepsilon,\rho,\zeta}(q + uw) \ge \theta_{s,\varepsilon,\rho,\zeta}(q) + u\tilde{\gamma}$.

Finally, we consider the case when one of the points q, $q + uw$ belongs to Q_s, while the other one is in $Q\backslash Q_s$. Then $q \in Q\backslash Q_s$ and $q + uw \in Q_s$, because the definition of Q_s clearly implies that $Q_s + Q \subseteq Q_s$, so that if $q \in Q_s$ it would follow that $q + uw \in Q_r$ as well. Then

$$\theta_{s,\varepsilon,\rho,\zeta}(q + uw) \ge \theta(q + uw) \ge \theta(q) + u\bar{\gamma} \ge \theta(q) + u\tilde{\gamma}\,,$$

completing the proof of (r).

We now observe that, if we let

$$\mu \stackrel{\text{def}}{=} \min\{W(q) : q \in Q\,,\ \|q\| = \gamma^*\}\,,$$

then μ exists (because W is continuous and the set $\{q \in Q : \|q\| = \gamma^*\}$ is compact) and $\mu > 0$ (because $W(q) \ge \theta(q) > 0$ whenever $q \in Q$ and $\|q\| = \gamma^*$). Also, we let

$$\nu = \max\{\langle p, q\rangle : q \in Q\,,\ \|q\| = \gamma^*\}\,.$$

We then choose ζ and ρ such that

$$\zeta\nu < \mu \quad \text{and} \quad \rho = \zeta\nu\,.$$

With these choices, we will prove that

(s) *$W(q) \ge \theta_{s,\varepsilon,\rho,\zeta}(q)$ whenever q belongs to the boundary of Q^*.*

Indeed, the conclusion is true if $q \in Q\backslash Q_s$, because in that case $\theta_{s,\varepsilon,\rho,\zeta}(q) = \theta(q)$. Assume that $q \in Q_s$. Then $\|q\| = \gamma^*$, so $\langle p^*, q\rangle \le \nu$, $\zeta.\langle p^*, q\rangle \le \zeta\nu = \rho$, and then

$$\eta_{0,\varepsilon,\rho}\big(\zeta.\langle p^*, q\rangle\big) \le \eta_{0,\varepsilon,\rho}(\rho) = \rho < \mu \le W(q)\,,$$

while on the other hand (31) tells us that $\theta(q) \le W(q)$ as well, and then

$$\theta_{s,\varepsilon,\rho,\zeta}(q) = \max\Big(\theta(q), \eta_{0,\varepsilon,\rho}\big(\zeta.\langle p^*, q\rangle\big)\Big) \le W(q)\,.$$

We now choose s such that

$$0 < s < \gamma^* \quad \text{and} \quad W(sq^\#) + s < \rho$$

(this is possible because $W(sq^\#) + s \to 0$ as $j \downarrow 0$), and then let

$$\varepsilon = W(sq^\#) + s\,.$$

Then $0 < \varepsilon < \rho$. We then let Θ be the function $\theta_{s,\varepsilon,\rho,\zeta}$ corresponding to our choices of s, ε, ρ and ζ. It then follows from (r) that

(t) $\Theta(q+uw) \geq \Theta(q)+u\tilde{\gamma}$ *whenever* $q \in Q$, $w \in \Psi_{\bar{\gamma}}(0)$, *and* $u \geq 0$.

We claim that

(u) *The function* $Q^* \ni q \mapsto \mathcal{W}(q) \stackrel{\rm def}{=} W(q)-\Theta(q)$ *attains a global minimum at some point* $q \in \overset{\circ}{Q}{}^*$.

(Indeed, $\mathcal{W}$ is lower semicontinuous on Q, since W is continuous and Θ is upper semicontinuous. So $\mathcal{W}$ attains a minimum value on the compact set Q^*. Furthermore,

$$\begin{aligned}
&\mathcal{W}(sq^\#)\\
=\;& W(sq^\#) - \max\Big(\theta(sq^\#), \eta_{0,\varepsilon,\rho}\big(\zeta.\langle p^*, sq^\#\rangle\big)\Big)\\
=\;& \min\Big(W(sq^\#) - \theta(sq^\#), W(sq^\#)\\
& -\eta_{0,\varepsilon,\rho}\big(\zeta.\langle p^*, sq^\#\rangle\big)\Big)\\
\leq\;& W(sq^\#) - \eta_{0,\varepsilon,\rho}\big(\zeta.\langle p^*, sq^\#\rangle\big)\\
\leq\;& W(sq^\#) - \eta_{0,\varepsilon,\rho}(0)\\
=\;& W(sq^\#) - \varepsilon\\
=\;& -s\,.
\end{aligned}$$

Therefore $\mathcal{W}(sq^\#) < 0$. In view of (s), the minimum value of $\mathcal{W}$ must be attained at an interior point of Q^*, and the proof of (u) is complete.)

Our next step is to regularize Θ, and show that (t) and (u) remain true for the regularizations Θ^α, provided that the regularization parameter α is small enough. This has to be done with care because Θ is not known to be continuous, so the regularizations Θ^α will not necessarily converge to Θ uniformly on compact sets.

We fix a nonnegative compactly supported function $\psi : \mathbb{R}^{n+1} \mapsto \mathbb{R}$ of class C^∞ such that $\int_{\mathbb{R}^{n+1}} \psi = 1$ and support$(\psi) \subseteq \{q \in Q : \|q\| \leq 1\}$. We then define, for every positive real number α, a function $\Theta^\alpha : Q \mapsto \mathbb{R}$ by letting

$$\Theta^\alpha(q) = \int_{\mathbb{R}^{n+1}} \Theta(q+\alpha y)\psi(y)\,dy$$

for $q \in Q$. Then Θ^α is continuous on Q and of class C^∞ on $\overset{\circ}{Q}$ whenever $\alpha > 0$. We also define the regularizations W^α of W by letting

$$W^\alpha(q) = \int_{\mathbb{R}^{n+1}} W(q+\alpha y)\psi(y)\,dy\,.$$

Furthermore, for each α, we let

$$\begin{aligned}
\omega_1(\alpha) &= \max\{\Theta(q)-W(q) : q \in Q,\, d(q, Q^*\setminus \overset{\circ}{Q}{}^*) \leq \alpha\}\\
\omega_2(\alpha) &= \max\{W^\alpha(q) - W(q) : q \in Q^*\setminus \overset{\circ}{Q}{}^*\}
\end{aligned}$$

(The maxima exist because $\Theta - W$ is upper semicontinuous and $\Theta^\alpha - W^\alpha$ is continuous.) It is clear that

(v) $W(q) - \Theta^\alpha(q) \geq -\omega_1(\alpha) - \omega_2(\alpha)$ *whenever* $\alpha > 0$ *and* $q \in Q^*\setminus \overset{\circ}{Q}{}^*$.

(Indeed,

$$\begin{aligned}
&W^\alpha(q) - \Theta^\alpha(q)\\
&\quad= \int_{\mathbb{R}^{n+1}} (W(q+\alpha y) - \Theta(q+\alpha y))\psi(y)\,dy\\
&\quad\geq -\omega_1(\alpha) \int_{\mathbb{R}^{n+1}} \psi(y)\,dy\\
&\quad= -\omega_1(\alpha)\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
W(q) - \Theta^\alpha(q) &\geq W(q) - W^\alpha(q) + W^\alpha(q) - \Theta^\alpha(q)\\
&\geq -\omega_2(\alpha) - \omega_1(\alpha)\,,
\end{aligned}$$

as desired.) The numbers $\omega_2(\alpha)$ go to 0 as $\alpha \downarrow 0$, because $W^\alpha \to W$ uniformly on compact sets. On the other hand $\limsup_{\alpha\downarrow 0} \omega_1(\alpha) \leq 0$. (Otherwise there would exist α_j, q_j, β, such that $\beta > 0$, $\alpha_j \downarrow 0$, $q_j \in Q$, $d(q_j, Q^*\setminus \overset{\circ}{Q}{}^*) \leq \alpha_j\}$, and $\Theta(q_j) - W(q_j) \geq \beta$. By passing to a subsequence, we may assume that the q_j converge to a limit $q \in Q^*\setminus \overset{\circ}{Q}{}^*$, and then the upper semicontinuity of $\Theta - W$ implies that $\Theta(q) - W(q) \geq \beta$, contradicting (s).) Therefore

$$\liminf_{\alpha\downarrow 0} \Big(\min\Big\{W(q)-\Theta^\alpha(q) : q \in Q^*\setminus \overset{\circ}{Q}{}^* \Big\}\Big) \geq 0. \quad (32)$$

The value of Θ^α at the point $sq^\#$ can be estimated from below using the facts that ψ is supported in Q and $\Theta(sq^\# + \alpha y) \geq \eta_{0,\varepsilon,\rho}(\zeta.\langle p^*, sq^\# + \alpha y\rangle$ whenever $y \in Q$, because in that case $sq^\# + \alpha y$ belongs to Q_s. Since $\zeta.\langle p^*, sq^\# + \alpha y\rangle \geq 0$, we have

$$\eta_{0,\varepsilon,\rho}(\zeta.\langle p^*, sq^\# + \alpha y\rangle \geq \eta_{0,\varepsilon,\rho}(0) = \varepsilon\,.$$

Therefore

$$\Theta^\alpha(sq^\#) \geq \varepsilon = W(sq^\#) + s \quad \text{whenever} \quad \alpha > 0\,, \quad (33)$$

so that

$$W(sq^\#) - \Theta^\alpha(sq^\#) \leq -s \quad \text{whenever} \quad \alpha > 0\,. \quad (34)$$

It follows from (32) that we can choose α such that

$$W(q) - \Theta^\alpha(q) \geq -\frac{s}{2} \quad \text{whenever} \quad q \in Q^*\setminus \overset{\circ}{Q}{}^*\,. \quad (35)$$

Then (34) and (35) imply that

(w) *The function* $Q^* \ni q \mapsto \mathcal{W}^\alpha(q) \stackrel{\rm def}{=} W(q) - \Theta^\alpha(q)$ *attains a global minimum at some point* $q \in \overset{\circ}{Q}{}^*$.

Furthermore, we claim that

(x) $\Theta^\alpha(q + uw) \geq \Theta^\alpha(q) + u\tilde{\gamma}$ *whenever* $q \in Q$, $w \in \Psi_{\bar{\gamma}}(0)$, *and* $u \geq 0$.

To prove this, we write

$$\begin{aligned}
\Theta^\alpha(q+uw) &= \int_{\mathbb{R}^{n+1}} \Theta(q+uw+\alpha y)\psi(y)\,dy\\
&\geq \int_{\mathbb{R}^{n+1}} \Big(\Theta(q+\alpha y) + u\tilde{\gamma}\Big)\psi(y)\,dy\\
&= \Theta^\alpha(q) + u\tilde{\gamma} \int_{\mathbb{R}^{n+1}} \psi(y)\,dy\\
&= \Theta^\alpha(q) + u\tilde{\gamma}\,,
\end{aligned}$$

where we have used (t) and the fact that $q + \alpha y \in Q$ whenever $\psi(y) \neq 0$ to conclude that the inequality $\Theta(q + uw + \alpha y) \geq \Theta(q + \alpha y) + u\tilde{\gamma}$ holds for all such y.

Now let $\hat{q} = (\hat{x}, \hat{x}_0)$ be a point of Q^* where $\mathcal{W}^\alpha$ has a global minimum. Since Θ^α is everywhere differentiable, we can let (π, π_0) be the differential of Θ^α at $\hat{q}$. Then, if we write $q = (x, x_0)$, we have

$$\lim_{q \to \hat{q}} \frac{\Theta^\alpha(q) - \Theta^\alpha(\hat{q}) - \pi \cdot (x - \hat{x}) - \pi_0(x_0 - \hat{x}_0)}{\|x - \hat{x}\| + |x_0 - \hat{x}_0|} = 0\,. \tag{36}$$

Taking $q = (x, \hat{x}_0)$, this implies

$$\lim_{x \to \hat{x}} \frac{\Theta^\alpha(x, \hat{x}_0) - \Theta^\alpha(\hat{x}, \hat{x}_0) - \pi \cdot (x - \hat{x})}{\|x - \hat{x}\|} = 0\,.$$

Since

$$\begin{array}{rcl} \Theta^\alpha(x, \hat{x}_0) - \Theta^\alpha(\hat{x}, \hat{x}_0) & \geq & W(x, \hat{x}_0) - W(\hat{x}, \hat{x}_0) \\ & = & V(x) - V(\hat{x})\,, \end{array}$$

we conclude that

$$\liminf_{x \to \hat{x}} \frac{V(x) - V(\hat{x}) - \pi \cdot (x - \hat{x})}{\|x - \hat{x}\|} \geq 0\,,$$

so π is a subdifferential of V at $\hat{x}$.

Taking $q = (\hat{x}, x_0)$, (36) implies

$$\lim_{x_0 \to \hat{x}_0} \frac{\Theta^\alpha(\hat{x}, x_0) - \Theta^\alpha(\hat{x}, \hat{x}_0) - \pi_0(x_0 - \hat{x}_0)}{|x_0 - \hat{x}_0|} = 0\,.$$

Since

$$\begin{array}{rcl} \Theta^\alpha(\hat{x}, x_0) - \Theta^\alpha(\hat{x}, \hat{x}_0) & \geq & W(\hat{x}, x_0) - W(\hat{x}, \hat{x}_0) \\ & = & x_0 - \hat{x}_0\,, \end{array}$$

we conclude that

$$\liminf_{x \to \hat{x}} \frac{x_0 - \hat{x}_0 - \pi_0(x_0 - \hat{x}_0)}{|x_0 - \hat{x}_0|} \geq 0\,,$$

so $\pi_0 = 1$.

Now, if $u \in U$, then $w = (f(\hat{x}, u), L(\hat{x}, u)) \in \Phi_{\tilde{\gamma}, \hat{\Sigma}}(0)$, so $w \in \Psi_{\tilde{\gamma}}(0)$, and then $\Theta^\alpha(\hat{q} + \alpha w) \geq \Theta(\hat{q}) + \alpha\tilde{\gamma}$ for all positive α. It follows that

$$\begin{array}{rcl} \pi \cdot f(\hat{x}, u) + L(\hat{x}, u) & = & \displaystyle\lim_{\alpha \downarrow 0} \frac{\Theta^\alpha(\hat{q} + \alpha w) - \Theta^\alpha(\hat{q})}{\alpha} \\ & \geq & \displaystyle\limsup_{\alpha \downarrow 0} \frac{\alpha\tilde{\gamma}}{\alpha} \\ & = & \tilde{\gamma}\,. \end{array}$$

Since this is true for every $u \in U$, we see that

$$\sup\{-\pi \cdot f(\hat{x}, u) - L(\hat{x}, u) : u \in U\} \leq -\tilde{\gamma}\,.$$

Since π is a subdifferential of V at $\hat{x}$, this contradicts our hypothesis that V satisfies (13) in the viscosity sense, and completes our proof. $\diamondsuit$

5. The compactness theorem.

If $\hat{\Sigma} = (\Omega, U, f, L)$ is an augmented control system, and ε is a positive number, an *ε-approximate augmented trajectory of* $\hat{\Sigma}$ is a locally absolutely continuous curve $I \ni t \mapsto \Xi(t) = (\xi(t), \xi_0(t)) \in \mathbb{R}^{n+1}$ having the property that there exists a measurable function $I \ni t \mapsto v(t) \in \mathbb{R}^n$ such that

(1) $\|v(t)\| \leq \varepsilon$ for almost all $t \in I$,

(2) $\dot{\Xi}(t) - (v(t), 0) \in F_{\hat{\Sigma}}(\xi(t), U)$ for almost all $t \in I$.

Remark 5.1 Roughly speaking, an ε-approximate augmented trajectory of $\hat{\Sigma}$ is an augmented trajectory of the "ε-extended system" $\hat{\Sigma}^\varepsilon = (\Omega, U^\varepsilon, f^\varepsilon, L^\varepsilon)$ whose control space U^ε is the Cartesian product $U \times \{v \in \mathbb{R}^n : \|v\| \leq \varepsilon\}$, and whose dynamics f^ε and Lagrangian L^ε are given by

$$f^\varepsilon(x, u, v) = f(x, u) + v\,, \quad L^\varepsilon(x, u, v) = L(x, u)\,.$$

More precisely, a curve $\Xi = (\xi, \xi_0) : I \mapsto \mathbb{R}^{n+1}$ is an ε-approximate augmented trajectory of $\hat{\Sigma}$ if and only if Ξ is locally absolutely continuous and there exist functions $I \ni t \mapsto \eta(t) \in U$, $I \ni t \mapsto v(t) \in \{v \in \mathbb{R}^n : \|v\| \leq \varepsilon\}$ such that v is measurable and $\dot{\Xi}(t) \in F_{\hat{\Sigma}^\varepsilon}(\xi(t), U^\varepsilon)$ for almost every $t \in I$. The definition of an augmented trajectory of $\hat{\Sigma}^\varepsilon$ is exactly the same, except that in that case the requirement that v be measurable is omitted. $\diamondsuit$

Theorem 5.2 *Let $\hat{\Sigma} = (\Omega, U, f, L)$ be an n-dimensional locally uniformly continuous, locally coercive augmented control system such that $F_{\hat{\Sigma}}(x, U)$ is closed and convex for every $x \in \Omega$. Let K be a compact subset of Ω, let T be a positive time, and let $k \in \mathbb{R}$. Let $\{\varepsilon_j\}_{j=1}^\infty$ be a sequence of positive numbers such that $\varepsilon_j \to 0$ as $j \to \infty$, and let $\Xi^j = (\xi^j, \xi_0^j)$ be, for each j, an ε_j-approximate augmented trajectory of $\hat{\Sigma}$, defined on the interval $[0, T]$, such that $\xi^j(t) \in K$ for all $t \in [0, T]$, $\xi_0^j(0) = 0$, and $\xi_0^j(T) \leq k$. Then there exist a subsequence $\{\Xi^{j(k)}\}_{k=1}^\infty$ of the sequence $\{\Xi^j\}_{j=1}^\infty$ and an augmented trajectory $\Xi^\infty = (\xi^\infty, \xi_0^\infty)$ of $\hat{\Sigma} = (\Omega, U, f, L)$ such that*

(i) $\xi_0^\infty(0) = 0$,

(ii) the sequence $\{\xi^{j(k)}\}_{k=1}^\infty$ converges uniformly to ξ^∞, and

(iii) $\liminf_{k \to \infty} \inf\left\{\Delta^{j(k),\infty}(t, s) : 0 \leq s \leq t \leq T\right\} \geq 0$, *where, if $j, j' \in \mathbb{N} \cup \{+\infty\}$, we define*

$$\Delta^{j,j'}(t, s) \stackrel{\text{def}}{=} (\xi_0^j(t) - \xi_0^j(s)) - (\xi_0^{j'}(t) - \xi_0^{j'}(s))\,. \tag{37}$$

Proof. Pick constants r, C, such that $r > 1$, $C > 0$, and $\|f(x, u)\|^r \leq L(x, u) + C$ whenever $x \in K$, $u \in U$, and a function $\omega :]\,0, +\infty\,[\mapsto [0, +\infty]$ such that $\lim_{s \downarrow 0} \omega(s)$ and $\|F_{\hat{\Sigma}}(x, u) - F_{\hat{\Sigma}}(x', u)\| \leq \omega(s)$ whenever $x, x' \in K$ and $\|x - x'\| \leq s$. Choose, for each j, a measurable function $v^j : [0, T] \mapsto \mathbb{R}^n$ and a function $\eta^j : [0, T] \mapsto U$ such that $\|v(t)\| \leq \varepsilon_j$, $\dot{\xi}^j(t) = f(\xi^j(t), \eta^j(t)) + v(t)$, and $\dot{\xi}_0^j(t) = L(\xi^j(s), \eta^j(s))$ for almost all $t \in [0, T]$.

Let $C' = 2^r(C + \varepsilon^r)$. Then for each j the coerciveness condition implies the inequality

$$\|\dot{\xi}^j(t)\|^r \leq 2^r L(\xi^j(t), \eta^j(t)) + C' \text{ for almost all } t\,,$$

because

$$\begin{aligned}\|\dot{\xi}^j(t)\|^r &= \|f(\xi^j(t),\eta^j(t))+v(t)\|^r\\ &\le \big(\|f(\xi^j(t),\eta^j(t))\|+\|v(t)\|\big)^r\\ &\le 2^r\|f(\xi^j(t),\eta^j(t))\|^r+2^r\|v(t)\|^r\\ &\le 2^r(L(\xi^j(t),\eta^j(t))+C)+2^r\varepsilon^r\\ &= 2^rL(\xi^j(t),\eta^j(t))+C'\,.\end{aligned}$$

from which it follows that $\int_0^T\|\dot{\xi}^j(t)\|^r dt\le 2^rk+C'T$. Then the sequence $\{\dot{\xi}^j(t)\}_{j=1}^\infty$ is uniformly bounded in L^r, so we may assume, after passing to a subsequence if necessary, that the weak L^r-limit $\zeta=$ w-$\lim_{j\to\infty}\dot{\xi}^j$ exists. Then the ξ^j converge uniformly as $j\to\infty$ to a limit ξ^∞ such that $\xi^\infty(t)-\xi^\infty(s)=\int_s^t\zeta(r)\,dr$ for all $s,t\in[0,T]$. After passing to a subsequence once more, if necessary, we assume that

$$\|\xi^j(t)-\xi^\infty(t)\|\le 2^{-j}\ \text{for all}\ j\in\mathbb{N}\,,\ \ t\in[0,T]\,. \tag{38}$$

Let $\theta^j(t)=\dot{\xi}^j_0(t)+C$. Then the θ^j are nonnegative, because $\dot{\xi}^j_0(t)=L(\xi^j(t),\eta^j(t))\ge -C$. Furthermore, the sequence $\{\theta^j\}_{j=1}^\infty$ is bounded in $L^1([0,T],\mathbb{R})$, because

$$\begin{aligned}\|\theta^j\|_{L^1} &= \int_0^T\theta^j(t)\,dt\\ &= \int_0^T(\dot{\xi}^j_0(t)+C)\,dt\\ &\le k+CT\,.\end{aligned}$$

The space $L^1([0,T],\mathbb{R})$ can be embedded in the usual way in $C^0([0,T],\mathbb{R})^\dagger$ (the dual of $C^0([0,T],\mathbb{R})$), which is the space of finite Borel measures on $[0,T]$, by means of the map $\psi\mapsto\mu_\psi$ that assigns to each function $\psi\in L^1([0,T],\mathbb{R})$ the Borel measure μ_ψ such that

$$\mu_\psi(\varphi)=\int_0^T\psi(t)\varphi(t)dt\ \text{for every}\ \varphi\in C^0([0,T],\mathbb{R})\,.$$

Then we may assume, after passing to a subsequence for a third time, if necessary, that the weak* limit μ^∞ of the measures $\mu^j\stackrel{\mathrm{def}}{=}\mu_{\theta^j}$ defined by the θ^j exists as $j\to\infty$. The measure μ^∞ then has a decomposition

$$\mu^\infty=\mu^{\infty,ac}+\mu^{\infty,at}+\mu^{\infty,sing}$$

into the sum of an absolutely continuous part, an atomic part, and a singular part. Since the measure μ^∞ is positive, because it is a limit of positive measures, the three components $\mu^{\infty,ac}$, $\mu^{\infty,at}$, $\mu^{\infty,sing}$ are positive as well. Let θ^∞ be the Radon-Nikodym derivative of μ^∞, so θ^∞ is an integrable function on $[0,T]$ such that

$$\mu^{\infty,ac}(\varphi)=\int_0^T\theta^\infty(t)\varphi(t)\,dt\ \text{for}\ \varphi\in C^0([0,T],\mathbb{R})\,.$$

Define

$$\xi_0^\infty(t)=-Ct+\int_0^t\theta^\infty(s)ds \tag{39}$$

and then set $\Xi^\infty=(\xi^\infty,\xi_0^\infty)$.

We will show that Ξ^∞ is an augmented trajectory of the system $\hat{\Sigma}$. To see this, we observe first of all that by construction the functions $\xi^\infty:[0,T]\mapsto\mathbb{R}^n$ and $\xi_0^\infty:[0,T]\mapsto\mathbb{R}$ are absolutely continuous, and their derivatives at t are equal to $\zeta(t)$ and $\theta^\infty(t)$, respectively, for all t in a subset G of $[0,T]$ such that $[0,T]\backslash G$ has measure zero. Let A be the set of atoms of $\mu^{\infty,at}$, and let B be a subset of $[0,T]$ of Lebesgue measure zero such that $\mu^{\infty,sing}([0,T]\backslash B)=0$. Then the set $G'=G\backslash(A\cup B\cup\{T\})$ has measure T and $\mu^{\infty,at}(G')=\mu^{\infty,sing}(G')=0$. Let G'' be the set of points of density of G' that are Lebesgue points of ζ and θ^∞, so G'' has measure T as well. (Recall that a *Lebesgue point* of a scalar- or vector-valued integrable function σ defined on an interval $[a,b]$ is a point $t\in\,]a,b[$ such that $\lim_{h\downarrow 0}\frac{1}{h}\int_{t-h}^{t+h}\|\sigma(s)-\sigma(t)\|\,ds=0$.) Let $t\in G''$, and fix an h such that $0<h<T-t$. Let $E_{t,h}=[t,t+h]\cap G'$, so the Lebesgue measure $|E_{t,h}|$ of $E_{t,h}$ satisfies $\lim_{h\downarrow 0}h^{-1}|E_{t,h}|=1$. Using the facts that the Borel measure $\mu^{\infty,at}+\mu^{\infty,sing}$ is regular and $(\mu^{\infty,at}+\mu^{\infty,sing})(E_{t,h})=0$, we can find a relatively open subset $\tilde{U}_{t,h}$ of $[0,T]$ such that $E_{t,h}\subseteq\tilde{U}_{t,h}$ and $(\mu^{\infty,at}+\mu^{\infty,sing})(\tilde{U}_{t,h})\le h^2$. We then let $U_{t,h}=\tilde{U}_{t,h}\cap\,]t,t+h[$, so $U_{t,h}$ is an open subset of $\mathbb{R}$, $U_{t,h}\subseteq\,]t,t+h[$, and $(\mu^{\infty,at}+\mu^{\infty,sing})(U_{t,h})\le h^2$. Using the regularity of Lebesgue measure we can find a compact subset $K_{t,h}$ of $E_{t,h}\backslash\{t,t+h\}$ such that $|K_{t,h}|\ge|E_{t,h}|-h^2$. Then of course $\lim_{h\downarrow 0}h^{-1}|K_{t,h}|=1$, and $K_{t,h}\subseteq U_{t,h}$ for each h. Let $\tilde{\varphi}_{t,h}(s)=\mathrm{dist}(s,\mathbb{R}\backslash U_{t,h})$, so $\tilde{\varphi}_{t,h}:\mathbb{R}\mapsto\mathbb{R}$ is continuous, $\tilde{\varphi}_{t,h}(s)=0$ whenever $s\notin U_{t,h}$, and $\tilde{\varphi}_{t,h}(s)>0$ whenever $s\in K_{t,h}$. If we let

$$\begin{aligned}\beta_{t,h} &= \min\{\tilde{\varphi}_{t,h}(s):s\in K_{t,h}\}\,,\\ \hat{\varphi}_{t,h}(s) &= \min(\tilde{\varphi}_{t,h}(s),\beta_{t,h})\,,\\ \varphi_{t,h}(s) &= \beta_{t,h}^{-1}\hat{\varphi}_{t,h}(s)\,,\end{aligned}$$

then $\varphi_{t,h}$ is a continuous real-valued function on $\mathbb{R}$ such that $0\le\varphi_{t,h}(s)\le 1$ for all s, $\varphi_{t,h}(s)=1$ for all $s\in K_{t,h}$, and $\varphi_{t,h}(s)=0$ for all $s\in\mathbb{R}\backslash U_{t,h}$. In particular, $\varphi_{t,h}(s)=0$ whenever $s\notin\,]t,t+h[$.

Let $a_{t,h}=\int_t^{t+h}\varphi_{t,h}(s)\,ds=\int_{-\infty}^{+\infty}\varphi_{t,h}(s)\,ds$. Then

$$|E_{t,h}|-h^2\le a_{t,h}\le h\,,$$

from which it follows that

$$\lim_{h\downarrow 0}\frac{a_{t,h}}{h}=1\,.$$

Let $\psi_{t,h}=a_{t,h}^{-1}\varphi_{t,h}$. Then

$$\int_t^{t+h}\psi_{t,h}(s)\,ds=1\,.$$

If $h>0$, write

$$\delta_j(h)=2^{-j}+h^{1/\rho}(2^r|k|+C'T)^{1/r}\,.$$

Then, if $s\in[t,t+h]$, we have

$$\begin{aligned}&\|\xi^j(s)-\xi^\infty(t)\|\\ &\quad\le \|\xi^j(t)-\xi^\infty(t)\|+\|\xi^j(s)-\xi^j(t)\|\\ &\quad\le 2^{-j}+\int_t^s\|\dot{\xi}^j(\tau)\|\,dt\\ &\quad\le 2^{-j}+(s-t)^{1/\rho}\Big(\int_t^s\|\dot{\xi}^j(\tau)\|^r\,dt\Big)^{1/r}\\ &\quad\le 2^{-j}+h^{1/\rho}\Big(\int_0^T\|\dot{\xi}^j(\tau)\|^r\,dt\Big)^{1/r}\end{aligned}$$

$$\begin{aligned} &\le 2^{-j} + h^{1/\rho}\Big(\int_0^T (2^r L(\xi^j(\tau),\eta^j(\tau)) + C')\,d\tau\Big)^{1/r} \\ &\le 2^{-j} + h^{1/\rho}(2^r|k| + C'T)^{1/r} \\ &= \delta_j(h)\,. \end{aligned}$$

For almost all $s \in [t, t+h]$, the derivative $\dot\Xi^j(s)$ exists and is equal to $F_{\hat\Sigma}(\xi^j(s),\eta^j(s)) + (v^j(s),0)$. Hence

$$\|\,\dot\Xi^j(s) - F_{\hat\Sigma}(\xi^\infty(t),\eta^j(s))\| \le \omega(\delta_j(h)) + \varepsilon_j\,,$$

from which it follows that

$$\mathrm{dist}(\dot\Xi^j(s), F_{\hat\Sigma}(\xi^\infty(t),U) \le \omega(\delta_j(h)) + \varepsilon_j\,.$$

Therefore the average

$$A^j_{t,h} = \int_t^{t+h} \psi_{t,h}(s)\dot\Xi^j(s)\,ds = \int_0^T \psi_{t,h}(s)\dot\Xi^j(s)\,ds$$

also satisfies

$$\mathrm{dist}(A^j_{t,h}, F_{\hat\Sigma}(\xi^\infty(t),U) \le \omega(\delta_j(h)) + \varepsilon_j\,, \qquad (40)$$

because $F_{\hat\Sigma}(\xi^\infty(t),U)$ is closed and convex. As $j\to\infty$, the vector functions $\dot\xi^j$ converge weakly in L^r to ζ, so

$$\begin{aligned} \int_t^{t+h} \psi_{t,h}(s)\dot\xi^j(s)\,ds &\to \int_t^{t+h} \psi_{t,h}(s)\zeta(s)\,ds \\ &= \int_t^{t+h} \psi_{t,h}(s)\dot\xi^\infty(s)\,ds \end{aligned}$$

as $j\to\infty$, $j\in J$. The integral $\int_t^{t+h}\psi_{t,h}(s)\dot\xi^\infty(s)\,ds$ satisfies

$$\begin{aligned} \int_t^{t+h} \psi_{t,h}(s)\dot\xi^\infty(s)\,ds &= \int_t^{t+h} \psi_{t,h}(s)\dot\xi^\infty(t)\,ds + \mathcal{E}_{t,h} \\ &= \dot\xi^\infty(t) + \mathcal{E}_{t,h} \end{aligned}$$

where

$$\mathcal{E}_{t,h} = \int_t^{t+h} \psi_{t,h}(s)(\dot\xi^\infty(s) - \dot\xi^\infty(t))\,ds\,,$$

and we have used the fact that $\int_t^{t+h}\psi_{t,h}(s)\,ds = 1$.

The error term $\mathcal{E}_{t,h}$ satisfies

$$\begin{aligned} |\mathcal{E}_{t,h}| &\le \max\big\{\psi_{t,h}(s) : s\in[0,T]\big\} \\ &\quad\times \int_t^{t+h} \|\dot\xi^\infty(s) - \dot\xi^\infty(t)\|\,ds \\ &= h a_{t,h}^{-1}\hat{\mathcal{E}}_{t,h} \\ &= \alpha_{t,h}\hat{\mathcal{E}}_{t,h}\,, \end{aligned}$$

where $\alpha_{t,h} = h a_{t,h}^{-1}$, so $\alpha_{t,h}\to 1$ as $h\downarrow 0$, and

$$\hat{\mathcal{E}}_{t,h} = \frac{1}{h}\int_t^{t+h} \|\dot\xi^\infty(s) - \dot\xi^\infty(t)\|\,ds\,.$$

Then

$$\begin{aligned} &\limsup_{j\to\infty, j\in J} \Big\|\int_t^{t+h} \psi_{t,h}(s)\dot\xi^j(s)\,ds - \dot\xi^\infty(t)\Big\| \\ &\qquad \le \alpha_{t,h}\hat{\mathcal{E}}_{t,h}\,. \end{aligned} \qquad (41)$$

To analyze the behavior of the integrals

$$I^j_{t,h} \stackrel{\mathrm{def}}{=} \int_t^{t+h} \psi_{t,h}(s)\dot\xi^j_0(s)\,ds\,,$$

we write $\dot\xi^j_0(s) = \theta^j(s) - C$, so

$$\begin{aligned} I^j_{t,h} &= -Ch + \int_t^{t+h} \psi_{t,h}(s)\theta^j(s)\,ds \\ &= -Ch + \int_0^T \psi_{t,h}(s)\theta^j(s)\,ds \\ &= -Ch + \int_{[0,T]} \psi_{t,h}\,d\mu^j\,. \end{aligned}$$

Hence

$$I^j_{t,h} \to -Ch + \int_{[0,T]} \psi_{t,h}(s)\,d\mu^\infty(s) \text{ as } j\to\infty\,. \qquad (42)$$

Write

$$\begin{aligned} \int_{[0,T]} \psi_{t,h}(s)\,d\mu^\infty(s) &= \int_0^T \psi_{t,h}(s)\theta^\infty(s)\,ds \\ &\quad + \int_{[0,T]} \psi_{t,h}(s)\,d\hat\mu^\infty(s)\,, \end{aligned}$$

where $\hat\mu^\infty = \mu^{\infty,at} + \mu^{\infty,sing}$. Since $\dot\xi^\infty_0(s) = \theta^j(s) - C$, (42) implies

$$\begin{aligned} I^j_{t,h} &\to \int_{[0,T]} \psi_{t,h}(s)\dot\xi^\infty_0(s)\,ds \\ &\quad + \int_{[0,T]} \psi_{t,h}(s)\,d\hat\mu^\infty(s) \text{ as } j\to\infty\,. \end{aligned} \qquad (43)$$

The integral $\int_{[0,T]}\psi_{t,h}(s)\,d\hat\mu^\infty(s)$ that occcurs in (43) is a nonnegative number, and is bounded above by $\max\{\psi_{t,h}(s) : s\in[0,T]\}$ times $\hat\mu^\infty(U_{t,h})$, since $\psi_{t,h}$ vanishes outside $U_{t,h}$. Therefore

$$0 \le \int_{[0,T]} \psi_{t,h}(s)\,d\hat\mu^\infty(s) \le h^2 a_{t,h}^{-1} = h\alpha_{t,h}\,, \qquad (44)$$

On the other hand, the integral $\int_{[0,T]}\psi_{t,h}(s)\dot\xi^\infty_0(s)\,ds$ satisfies

$$\begin{aligned} \int_{[0,T]} \psi_{t,h}(s)\dot\xi^\infty_0(s)\,ds &= \int_t^{t+h} \psi_{t,h}(s)\dot\xi^\infty_0(s)\,ds \\ &= \dot\xi^\infty_0(t))\,ds + E_{t,h} \\ &= \dot\xi^\infty_0(t) + E_{t,h}\,, \end{aligned} \qquad (45)$$

where

$$E_{t,h} = \int_t^{t+h} \psi_{t,h}(s)(\dot\xi^\infty_0(s) - \dot\xi^\infty_0(t))\,ds\,,$$

and we have used the fact that $\int_t^{t+h}\psi_{t,h}(s)\,ds = 1$.

The error term $E_{t,h}$ satisfies

$$\begin{aligned} |E_{t,h}| &\le \max\big\{\psi_{t,h}(s) : s\in[0,T]\big\} \\ &\quad\times \int_t^{t+h} |\dot\xi^\infty_0(s) - \dot\xi^\infty_0(t)|\,ds \\ &= h a_{t,h}^{-1}\hat E_{t,h} \\ &= \alpha_{t,h}\hat E_{t,h}\,, \end{aligned}$$

where

$$\hat E_{t,h} = \frac{1}{h}\int_t^{t+h} |\dot\xi^\infty_0(s) - \dot\xi^\infty_0(t)|\,ds\,.$$

It follows from (43), (44), (45), and the bound $|E_{t,h}| \le \alpha_{t,h}\hat E_{t,h}$, that

$$\limsup_{j\to\infty, j\in J} \Big| I^j_{t,h} - \dot\xi^\infty_0(t)\Big| \le \alpha_{t,h}(h + \hat E_{t,h})\,. \qquad (46)$$

If we now combine (41) and (46), we find that

$$\limsup_{j\to\infty, j\in J} \Big\|A^j_{t,h} - \dot\Xi^\infty(t)\Big\| \le \alpha_{t,h}(h+\hat E_{t,h}+\mathcal{E}_{t,h}). \qquad (47)$$

Then (40) implies that

$$\begin{aligned}&\mathrm{dist}(\dot\Xi^\infty(t), F_{\hat\Sigma}(\xi^\infty(t)))\\ &\qquad \le \alpha_{t,h}(h+\hat E_{t,h}+\mathcal{E}_{t,h}) + \limsup_{j\to\infty}(\omega(\delta_j(h))+\varepsilon_j)\,.\end{aligned}$$

Hence, given any j_*, we have

$$\begin{aligned}&\mathrm{dist}(\dot\Xi^\infty(t), F_{\hat\Sigma}(\xi^\infty(t)))\\ &\qquad \le \alpha_{t,h}(h+\hat E_{t,h}+\mathcal{E}_{t,h}) + \sup\{\omega(\delta_j(h))+\varepsilon_j : j\ge j_*\}\,.\end{aligned}$$

Given any positive number β, we can find a positive γ such that $\omega(s) < \beta$ whenever $0 < s \le \gamma$, and then find j_*, h_* such that $\delta_j(h) \le \gamma$ whenever $j \ge j_*$ and $0 < h \le h_*$, and $\varepsilon_j < \beta$ whenever $j \ge j_*$. Then we can pick h such that $0 < h \le h_*$, and $\alpha_{t,h}(h+\hat E_{t,h}+\mathcal{E}_{t,h}) < \beta$. Then

$$\mathrm{dist}(\dot\Xi^\infty(t), F_{\hat\Sigma}(\xi^\infty(t))) < 3\beta\,.$$

Since β was arbitrary, we conclude that $\mathrm{dist}(\dot\Xi^\infty(t), F_{\hat\Sigma}(\xi^\infty(t))) = 0$, so $\dot\Xi^\infty(t) \in F_{\hat\Sigma}(\xi^\infty(t))$, because $F_{\hat\Sigma}(\xi^\infty(t))$ is closed. Since this is true for almost all $t \in [0,T]$, and Ξ^∞ is absolutely continuous, we have shown that Ξ^∞ is an augmented trajectory of $\hat\Sigma$.

By construction, the ξ^j converge uniformly to ξ^∞. Also, it is clear from (39) that $\xi^\infty_0(0) = 0$. To conclude our proof, we have to show that

$$\liminf{}_{j\to\infty} \inf\big\{\Delta^{j,\infty}(t,s) : 0\le s\le t\le T\big\} \ge 0\,, \qquad (48)$$

where $\Delta^{j,\infty}(t,s)$ is the quantity defined in (37). Suppose that $\liminf_{j\to\infty}\inf\big\{\Delta^{j,\infty}(t,s) : 0\le s\le t\le T\big\} < 0$. Pick a number β such that $\beta > 0$ and

$$\liminf_{j\to\infty}\inf\big\{\Delta^{j,\infty}(t,s) : 0\le s\le t\le T\big\} \le -3\beta\,.$$

Then there exists a subsequence $\{\Xi^{j(k)}\}_{k=1}^\infty$ of $\{\Xi^j\}_{j=1}^\infty$ such that $\inf\big\{\Delta^{j,\infty}(t,s) : 0\le s\le t\le T\big\} \le -2\beta$ for all k. We can then choose, for each k, members s_k and t_k of $[0,T]$ such that $s_k \le t_k$ and

$$\big(\xi^{j(k)}_0(t_k) - \xi^{j(k)}_0(s_k)\big) - \big(\xi^\infty_0(t_k) - \xi^\infty_0(s_k)\big) \le -\beta\,. \quad (49)$$

By passing to a subsequence, if necessary, we may assume that the s_k and the t_k converge to limits s, t. Clearly, then, $s \le t$.

If $s = t$, then

$$\begin{aligned}\xi^{j(k)}_0(t_k) - \xi^{j(k)}_0(s_k) &= \int_{s_k}^{t_k} \dot\xi^{j(k)}_0(v)\,dv\\ &= \int_{s_k}^{t_k} \big(\theta^{j(k)}(v) - C\big)\,dv\\ &\ge -C(t_k - s_k)\,,\end{aligned}$$

so the fact that $\lim_{k\to\infty}(\xi^\infty_0(t_k) - \xi^\infty_0(s_k)) = 0$ implies the inequalities

$$\begin{aligned}&\liminf_{k\to\infty}\Big(\big(\xi^{j(k)}_0(t_k) - \xi^{j(k)}_0(s_k)\big) - \big(\xi^\infty_0(t_k) - \xi^\infty_0(s_k)\big)\Big)\\ &\ = \liminf_{k\to\infty}\big(\xi^{j(k)}_0(t_k) - \xi^{j(k)}_0(s_k)\big) - \lim_{k\to\infty}\big(\xi^\infty_0(t_k) - \xi^\infty_0(s_k)\big)\\ &\ = \liminf_{k\to\infty}\big(\xi^{j(k)}_0(t_k) - \xi^{j(k)}_0(s_k)\big)\\ &\ \ge \liminf_{k\to\infty}(-C(t_k - s_k))\\ &\ = \lim_{k\to\infty}(-C(t_k - s_k))\\ &\ = 0\,,\end{aligned}$$

which clearly contradict (49).

Now assume that $s < t$. Fix a positive number γ such that $2\gamma < t-s$, and let $\Phi_{s,t,\gamma}$ be the set of all continuous nonnegative functions $\varphi : \mathbb{R} \mapsto \mathbb{R}$ that vanish outside the interval $[s+\gamma, t-\gamma]$ and are such that $\varphi(v) \le 1$ for all $v \in \mathbb{R}$. Then

$$\begin{aligned}&\liminf_{k\to\infty}\Big(\xi^{j(k)}_0(t_k) - \xi^{j(k)}_0(s_k)\Big)\\ &\qquad = \liminf_{k\to\infty}\int_{s_k}^{t_k} \dot\xi^{j(k)}_0(v)\,dv\\ &\qquad = \liminf_{k\to\infty}\int_{s_k}^{t_k} (\theta^{j(k)}(v) - C)\,dv\\ &\qquad = \liminf_{k\to\infty}\Big(- C(t_k - s_k) + \int_{s_k}^{t_k} \theta^{j(k)}(v)\,dv\Big)\\ &\qquad = -C(t-s) + \liminf_{k\to\infty}\int_{s_k}^{t_k} \theta^{j(k)}(v)\,dv\\ &\qquad \ge -C(t-s) + \liminf_{k\to\infty}\int_{s_k}^{t_k} \varphi(v)\theta^{j(k)}(v)\,dv\\ &\qquad = -C(t-s) + \liminf_{k\to\infty}\int_{[0,T]} \varphi(v)d\mu^{j(k)}(v)\\ &\qquad = -C(t-s) + \lim_{k\to\infty}\int_{[0,T]} \varphi(v)d\mu^{j(k)}(v)\\ &\qquad = -C(t-s) + \int_{[0,T]} \varphi(v)d\mu^{\infty}(v)\\ &\qquad \ge -C(t-s) + \int_{[0,T]} \varphi(v)d\mu^{\infty,ac}(v)\\ &\qquad = -C(t-s) + \int_s^t \varphi(v)\theta^\infty(v)\,dv\\ &\qquad = \int_s^t \varphi(v)(\theta^\infty(v) - C)\,dv\\ &\qquad = \int_s^t \varphi(v)\dot\xi^\infty_0(v)\,dv\,,\end{aligned}$$

where, for the first inequality, we have used the fact that $[s+\gamma, t-\gamma] \subseteq [s_k, t_k]$ when k is large enough. Therefore

$$\liminf_{k\to\infty}\Big(\xi^{j(k)}_0(t_k) - \xi^{j(k)}_0(s_k)\Big) \ge \int_s^t \varphi(v)\dot\xi^\infty_0(v)\,dv$$

for every $\varphi \in \Phi_{s,t,\gamma}$. Hence

$$\begin{aligned}&\liminf_{k\to\infty}\Big(\xi^{j(k)}_0(t_k) - \xi^{j(k)}_0(s_k)\Big)\\ &\qquad \ge \sup\Big\{\int_s^t \varphi(v)\dot\xi^\infty_0(v)\,dv : \varphi \in \Phi_{s,t,\gamma}\Big\}\\ &\qquad = \int_{s+\gamma}^{t-\gamma} \dot\xi^\infty_0(v)\,dv\\ &\qquad = \xi^\infty_0(t-\gamma) - \xi^\infty_0(s+\gamma)\,.\end{aligned}$$

Since γ is arbitrary, we can let $\gamma \downarrow 0$, and conclude that

$$\begin{aligned}\liminf_{k\to\infty} \Big(\xi_0^{j(k)}(t_k) - \xi_0^{j(k)}(s_k)\Big) & \\ \geq\ & \xi_0^{\infty}(t) - \xi_0^{\infty}(s) \\ =\ & \lim_{k\to\infty} \Big(\xi_0^{\infty}(t_k) - \xi_0^{\infty}(s_k)\Big)\,,\end{aligned}$$

so

$$\liminf_{k\to\infty} \Big(\big(\xi_0^{j(k)}(t_k) - \xi_0^{j(k)}(s_k)\big) - \big(\xi_0^{\infty}(t_k) - \xi_0^{\infty}(s_k)\big)\Big) \geq 0\,,$$

contradicting (49). This completes the proof of (48). $\diamondsuit$

6. Trajectories of steepest descent.

We recall from §2 that $\mathcal{MSD}_{\hat\Sigma,V,x}$ denotes the set of all maximal augmented trajectories of $\hat\Sigma$ from x of steepest descent with respect to V and that, as explained in Proposition 2.6 and the remarks preceding its statement, $\mathcal{MSD}_{\hat\Sigma,V,x}$ is always nonempty for trivial reasons, because the trivial trajectory Ξ_x^{triv} always belongs to $\mathcal{SD}_{\hat\Sigma,V,x}$, and once we know that $\mathcal{SD}_{\hat\Sigma,V,x} \neq \emptyset$ it follows immediately from Zorn's Lemma that $\mathcal{SD}_{\hat\Sigma,V,x}$ must have a maximal element.

The truly nontrivial and useful result is the statement that maximal steepest descent trajectories not only exist but are "large," in the sense that they are "right-unbounded." Precisely, if $\xi : I \mapsto \Omega$ is a curve, we say that ξ is *right-unbounded* if (i) the interval I is open on the right (that is, if $\tau = \sup I$, then either (a) $\tau = +\infty$ or (b) τ is finite and does not belong to I), and (ii) if τ is finite, then for every compact subset K of Ω there exists a τ_K such that $0 \leq \tau_K < \tau$ and $\xi(t) \notin K$ whenever $\tau_K < t < \tau$. (Equivalently, condition (ii) asserts that $\lim_{t\uparrow\tau} \xi(t) = \infty_\Omega$, where ∞_Ω is the point at infinity of the one-point compactification of Ω.)

Theorem 6.1 *Let $\hat\Sigma = (\Omega, U, f, L)$ be a locally coercive, locally uniformly continuous augmented control system such that $F_{\hat\Sigma}(x, U)$ is closed and convex for every $x \in \Omega$. Let $V : \Omega \mapsto \mathbb{R}$ be a continuous function that satisfies (13) on Ω in the viscosity sense. Let $x_* \in \Omega$, and let $\Xi = (\xi, \xi_0)$ be a maximal augmented trajectory of $\hat\Sigma$ from x_* of steepest descent with respect to V, defined on an interval I. Then ξ is right-unbounded.*

Proof. We assume that the conclusion is not true and derive a contradiction. Pick a $\Xi = (\xi, \xi_0) \in \mathcal{MSD}_{\hat\Sigma,V,x_*}$ that violates the conclusion. This means, to begin with, that Ξ is defined on a bounded interval I, and, in addition, this interval is either of the form $[0, \tau]$, with $0 \leq \tau < +\infty$ (the "right-closed case"), or of the form $[0, \tau[$, with $0 < \tau < +\infty$ (the "right-open case"). Furthermore, in the right-open case there exist a compact subset K of Ω and a sequence $\{t_j\}_{j=1}^\infty$ such that $t_j \in I$ and $\xi(t_j) \in K$ for all j, and $\lim_{j\to\infty} t_j = \tau$. In order to treat the right-open and right-closed cases together, we also choose a compact subset K of Ω and a sequence $\{t_j\}_{j=1}^\infty$ in I in the right-closed case, subject to the only requirements that $\xi(t) \in K$ for all $t \in [0, \tau]$ and $t_j \to \tau$ as $j \to \infty$. (For example, we could just choose $K = \{\xi(t) : 0 \leq t \leq \tau\}$, $t_j = \tau$ for all j.) Then in both cases the t_j belong to I and converge to τ, and the $\xi(t_j)$ belong to K. In addition, we pick a function $\eta : I \mapsto U$ such that $\dot\Xi(t) = F(\xi(t), \eta(t))$ for almost all $t \in I$.

Fix a positive number δ such that the compact set
$$K_\delta = \{x \in \mathbb{R}^n : \mathrm{dist}(x, K)\| \leq \delta\}$$
is contained in Ω. Then use the fact that $\hat\Sigma$ is locally coercive to choose r, C such that $r > 1$, $C > 0$, and $\|f(x, u)\|^r \leq L(x, u) + \tilde C$ for all $x \in K_\delta$, $u \in U$. Let

$$\bar V = \max\{|V(x)| : x \in K_\delta\}\,.$$

If $\rho = \frac{r}{r-1}$, so that $\frac{1}{r} + \frac{1}{\rho} = 1$, and we define
$$\bar\delta(\sigma) \overset{\mathrm{def}}{=} 2\Big(\sigma^{1/\rho}(2|\bar V| + (C+1)\sigma)^{1/r} + \sigma\Big)\,,$$
then $\bar\delta(\sigma)$ goes to zero as $\sigma \downarrow 0$. Therefore we can pick σ such that $\bar\delta(\sigma) < \delta$.

Now suppose that we are in the right-open case. Given any j, the point $x_j = \xi(t_j)$ belongs to K. Let

$$S_j = \Big\{t \in I : t \geq t_j \wedge (\forall s)\big((t_j \leq s \leq t) \Rightarrow \xi(s) \in K_\delta\big)\Big\}\,.$$

Then S_j is a subinterval of I, whose left endpoint is t_j.

If $t, t' \in S_j$, and $t' \leq t$, then the steepest descent property of Ξ implies that

$$\xi_0(t) + V(\xi(t)) \leq \xi_0(t') + V(\xi(t'))\,,$$

so
$$\xi_0(t) - \xi_0(t') \leq V(\xi(t)) - V(\xi(t')) \leq 2|\bar V|\,,$$
since both $\xi(t)$ and $\xi(t')$ belong to K_δ. Furthermore, the inequality

$$\|f(\xi(s), \eta(s))\|^r \leq L(\xi(s), \eta(s)) + C$$

is true for all $s \in S_j$, since $\xi(s) \in K_\delta$ for all such s. Therefore

$$\begin{aligned}
\|\xi(t) - \xi(t')\| & \\
=\ & \Big\| \int_{t'}^{t} \dot\xi(s)\, ds \Big\| \\
\leq\ & \int_{t'}^{t} \|\dot\xi(s)\|\, ds \\
=\ & \int_{t'}^{t} \|f(\xi(s), \eta(s))\|\, ds \\
\leq\ & (t - t')^{1/\rho} \Big(\int_{t'}^{t} \|f(\xi(s), \eta(s))\|^r\, ds\Big)^{1/r} \\
\leq\ & (t - t')^{1/\rho} \Big(\int_{t'}^{t} (L(\xi(s), \eta(s)) + C)\, ds\Big)^{1/r} \\
=\ & (t - t')^{1/\rho} (C(t - t') + \xi_0(t) - \xi_0(t'))^{1/r} \\
=\ & (t - t')^{1/\rho} (C(t - t') + 2|\bar V|))^{1/r} \\
\leq\ & \frac{1}{2} \bar\delta(t - t')\,.
\end{aligned}$$

Now pick j so large that $\tau - t_j < \sigma$, and apply the above inequality with $t' = t_j$. Then

$$\|\xi(t) - \xi(t_j)\| \leq \frac{1}{2}\bar\delta(\sigma) \leq \frac{1}{2}\delta \quad \text{for all} \quad t \in S_j\,. \tag{50}$$

Hence, if we let $\bar t = \sup S_j$, it is impossible that $\bar t < \tau$, because if $\bar t < \tau$ then $\bar t \in I$, and (50) implies that

$\|\xi(\bar{t}) - \xi(t_j)\| \leq \frac{1}{2}\delta$, from which it follows, by continuity, that there exists a positive α such that $\|\xi(t)-\xi(t_j)\| < \delta$ for $\bar{t} \leq t \leq \bar{t}+\alpha$; but then $\xi(t) \in K_\delta$ if $\bar{t} \leq t \leq \bar{t}+\alpha$, since $\xi(t) \in K$; therefore $\bar{t}+\alpha \in S_j$, contradicting the definition of $\bar{t}$.

Therefore $\bar{t} = \tau$, and this implies that $S_j = [t_j, \tau[$, so $\xi(t) \in K_\delta$ for all $t \in [t_j, \tau[$. Then the bound $L(\xi(s), \eta(s)) + C \geq \|f(\xi(s), \eta(s))\|^r$ holds for almost all $s \in [t_j, \tau[$, showing in particular that the function

$$[t_j, \tau[\, \ni s \mapsto \psi(s) \stackrel{\text{def}}{=} L(\xi(s), \eta(s)) + C$$

which is measurable because $L(\xi(s), \eta(s)) = \dot{\xi}_0(s)$, is nonnegative. Hence, to prove that ψ is Lebesgue-integrable on $[t_j, \tau[$, it suffices to show that the integrals $\int_{t_j}^t (L(\xi(s), \eta(s)) + C)\, ds$, for $t_j \leq t < \tau$, are bounded above by a fixed constant. But, if $t_j \leq t < \tau$, then

$$\begin{aligned}\int_{t_j}^t (L(\xi(s), \eta(s)) + C)\, ds &= \xi_0(t) - \xi_0(t_j) + C(t - t_j) \\ &\leq 2|V| + C\sigma\,.\end{aligned}$$

Therefore ψ is Lebesgue-integrable on $[t_j, \tau[$, and then $\dot{\xi}_0$ is also Lebesgue-integrable on $[t_j, \tau[$, because $\dot{\xi}_0 = \psi - C$. Since $\dot{\xi}_0$ is Lebesgue-integrable on $[0, t_j]$, we conclude that $\dot{\xi}_0$ is Lebesgue-integrable on $[0, \tau[$. Hence the limit $x_0^\# = \lim_{t\uparrow\tau} \xi_0(t)$ exists, and the extended function $\xi_0^\# : [0, \tau] \mapsto \mathbb{R}$ defined by $\xi_0^\#(t) = \xi_0(t)$ if $0 \leq t < \tau$, $\xi_0^\#(\tau) = x_0^\#$, is absolutely continuous.

In addition, the bound

$$\|\dot{\xi}(s)\|^r = \|f(\xi(s), \eta(s)\|^r \leq L(\xi(s), \eta(s)) + C = \psi(s)\,,$$

valid on $[t_j, \tau[$, shows that $\dot{\xi}$ in L^r on $[s_j, \tau[$, so *a fortiori* $\dot{\xi}$ is Lebesgue-integrable on $[s_j, \tau[$, and then the limit $x^\# = \lim_{t\uparrow\tau} \xi(t)$ exists and belongs to K_δ (because $x^\# = \lim_{\ell\to\infty} \xi(t_\ell)$ and the $\xi(t_\ell)$ belong to K_δ), and the extended function $\xi^\# : [0, \tau] \mapsto \Omega$ defined by $\xi^\#(t) = \xi(t)$ if $0 \leq t < \tau$, $\xi^\#(\tau) = x_0^\#$, is absolutely continuous.

Hence we have constructed an absolutely continuous extension $\Xi^\# = (\xi^\#, \xi_0^\#)$ of Ξ to the closed interval $[0, \tau]$. Clearly, $\Xi^\#$ is also an augmented trajectory of $\hat{\Sigma}$ starting at x_* at time 0, and the fact that Ξ is of steepest descent with respect to V and V is continuous implies that $\Xi^\#$ is of steepest descent with respect to V as well. Therefore Ξ is not maximal, because we have constructed an extension to a strictly larger interval. It follows that the "right-open case" cannot arise at all.

We now analyze the right-closed case, and show that it cannot arise either. We do this by constructing an extension $\Xi^\#$ of Ξ to a $\Xi^\# \in \mathcal{MSD}_{\hat{\Sigma},V,x_*}$ defined on the interval $[0, \tau+\sigma]$. This will, of course, contradict the assumed maximality of Ξ, and conclude our proof.

To construct $\Xi^\#$, we construct ε-approximate augmented trajectories $\Xi^\varepsilon : [\tau, \tau+\sigma] \mapsto \Omega \times \mathbb{R}$ of $\hat{\Sigma}$ that are "ε-approximately of steepest descent," and such that $\Xi^\varepsilon(\tau) = \Xi(\tau)$. We then pass to the limit as $\varepsilon \downarrow 0$, using the compactness theorem (5.2), and get an exact augmented trajectory $\tilde{\Xi} : [\tau, \tau+\sigma] \mapsto \Omega \times \mathbb{R}$ of $\hat{\Sigma}$ such that $\tilde{\Xi}(\tau) = \Xi(\tau)$, which is exactly of steepest descent.

Write $x^\# = \xi(\tau)$, $x_0^\# = \xi_0(\tau)$. Fix an ε such that $0 < \varepsilon < 1$. Let $\mathcal{Z}_\varepsilon$ be the set of all triples (I', Ξ', S') such that

(1) I' is a subinterval of $[\tau, \tau+\sigma]$ such that $\tau \in I'$,

(2) $\Xi' = (\xi', \xi_0') : I' \mapsto K_\delta \times \mathbb{R}$ is an ε-approximate augmented trajectory of $\hat{\Sigma}$,

(3) $\Xi'(\tau) = (x^\#, x_0^\#)$,

(4) S' is a strongly ε-dense subset of I' such that $\xi_0'(s) + V(\xi'(s)) + \varepsilon(t-s) \geq \xi_0'(t) + V(\xi'(t))$ for all $s, t \in S'$ such that $s \leq t$.

(We say that a subset S of an interval J is *strongly ε-dense* if for every $t \in J$ there exist $s, s' \in S$ such that $s \leq t \leq s'$ and $s' - s \leq \varepsilon$.)

We partially order $\mathcal{Z}_\varepsilon$ by stipulating that, if $(I_i, \Xi_i, S_i) \in \mathcal{Z}_\varepsilon$ for $i = 1, 2$, then $(I_1, \Xi_1, S_1) \preceq (I_2, \Xi_2, S_2)$ iff $I_1 \subseteq I_2$, Ξ_1 is the restriction of Ξ_2 to I_1, and $S_1 = S_2 \cap I_1$.

It is clear that $\mathcal{Z}_\varepsilon \neq \emptyset$, because the triple $(\{\tau\}, \Xi_{x^\#,\tau}, \{\tau\})$—where $\Xi_{x^\#,\tau}$ is the map $\Xi : \{\tau\} \mapsto \Omega \times \mathbb{R}$ such that $\Xi(\tau) = (x^\#, \xi_0(\tau))$—belongs to $\mathcal{Z}_\varepsilon$. If Z is a totally ordered subset of $\mathcal{Z}_\varepsilon$, we show that Z has an upper bound (I'_*, Ξ'_*, S'_*) in $\mathcal{Z}_\varepsilon$. This conclusion is trivial if $Z = \emptyset$, for in that case we can take $(I'_*, \Xi'_*, S'_*) = (\{\tau\}, \Xi_{x^\#,\tau}, \{\tau\})$. Assume that $Z \neq \emptyset$. Let I'_* be the union of the intervals I' for all the members (I', Ξ', S') of Z. Then I'_* is clearly a subinterval of $[\tau, \tau+\sigma]$, and $\tau \in I'$. If $t \in I'_*$, we define $\Xi'_*(t) = \Xi'(t)$, where (I', Ξ', S') is any member of Z such that $t \in I$. Write $\Xi'_* = (\xi'_*, \xi'_{0,*})$. Then Ξ'_* is obviously well defined, and is an ε-approximate augmented trajectory of $\hat{\Sigma}$ such that $\xi'_*(\tau) = x^\#$ and $\xi'_{0,*}(\tau) = x_0^\#$. We then let

$$S'_* = \bigcup_{(I', \Xi', S') \in Z} S'\,. \tag{51}$$

We want to prove that the triple (I'_*, Ξ'_*, S'_*) is an upper bound for Z in $\mathcal{Z}_\varepsilon$. If we show that

$$(I'_*, \Xi'_*, S'_*) \in \mathcal{Z}_\varepsilon\,, \tag{52}$$

then the fact that (I'_*, Ξ'_*, S'_*) is an upper bound for Z is immediate, so all we really need is to prove (52).

It is evident that the triple (I'_*, Ξ'_*, S'_*) satisfies the first three of the four conditions in the definition of $\mathcal{Z}_\varepsilon$. Let us show that it satisfies the fourth one as well. If $t \in I'_*$, we can find $(I'', \Xi'', S'') \in Z$ such that $t \in I''$. Since S'' is a strongly ε-dense subset of I'', there exist $s_1, s_2 \in S''$ such that $s_1 \leq t \leq s_2$ and $s_2 - s_1 \leq \varepsilon$. Then s_1 and s_2 belong to S'_*, and this establishes that S'_* is a strongly ε-dense subset of I'_*. Now, if s_1, s_2 are members of S'_* such that $s_1 \leq s_2$, then we can find (using the fact that Z is totally ordered) a member (I', Ξ', S') of Z such that s_1 and s_2 belong to I'. It then follows easily that s_1 and s_2 must belong to S'. Then, if we write $\Xi' = (\xi', \xi_0')$, the fact that $s_1 \in S'$, $s_2 \in S'$, $s_1 \leq s_2$, and $(I', \Xi', S') \in \mathcal{Z}_\varepsilon$, imply that

$$\xi_0'(s_1)+V(\xi'(s_1))+\varepsilon(s_2-s_1)\geq \xi_0'(s_2)+V(\xi'(s_2))\,.$$

Hence

$$\xi_{0,*}'(s_1)+V(\xi_*'(s_1))+\varepsilon(s_2-s_1)\geq \xi_{0,*}'(s_2)+V(\xi_*'(s_2))\,.$$

Since this is true for any two members s_1, s_2 of S_*' such that $s_1\leq s_2$, we conclude that the fourth condition holds as well, and the proof of (52) is complete.

We have shown that every totally ordered subset of $\mathcal{Z}_\varepsilon$ has an upper bound in $\mathcal{Z}_\varepsilon$. Therefore Zorn's Lemma implies that $\mathcal{Z}_\varepsilon$ has a maximal element (I',Ξ',S'). We claim that $I'=[\tau,\tau+\sigma]$. Suppose this was not true. Then either

(A) $I'=[\tau,\zeta\,[$ for some ζ such that $\tau<\zeta\leq\tau+\sigma$,

or

(B) $I'=[\tau,\zeta]$ for some ζ such that $\tau\leq\zeta<\tau+\sigma$.

We shall exclude both possibilities.

Write $\Xi'=(\xi',\xi_0')$. Let $\mathcal{B}_\varepsilon=\{v\in\mathbb{R}^n:\|v\|\leq\varepsilon\}$. Let $I'\ni t\mapsto(\eta'(t),v(t))\in U\times\mathcal{B}_\varepsilon$ be a function such that the function $v(\cdot)$ is measurable, and the equalities $\dot\xi'(t)=f(\xi'(t),\eta'(t))+v(t)$ and $\dot\xi_0'(t)=L(\xi'(t),\eta'(t))$ hold for almost all $t\in I'$.

Let $t\in I'$. Then there must exist a $\bar t\in S'$ such that $\bar t\geq t$. Also, τ must belong to S', because τ is the leftmost point of I'. Then

$$\xi_0'(\tau)+V(\xi'(\tau))+\varepsilon(t-\tau)\geq \xi_0'(\bar t)+V(\xi'(\bar t))\,,$$

so (since $\varepsilon<1$)

$$\begin{aligned}\xi_0'(\bar t)-\xi_0'(\tau)&\leq \varepsilon(t-\tau)+V(\xi'(\tau))-V(\xi'(\bar t))\\ &\leq 2|\bar V|+\sigma\,.\end{aligned}$$

We then have (using the fact that $\|v(s)\|<1$, which is true because $\varepsilon<1$),

$$\begin{aligned}&\int_\tau^t\|\dot\xi'(s)\|ds\\
&=\int_\tau^t\|f(\xi'(s),\eta'(s))+v(s)\|ds\\
&\leq\int_\tau^{\bar t}\|f(\xi'(s),\eta'(s))+v(s)\|ds\\
&\leq\int_\tau^{\bar t}\|f(\xi'(s),\eta'(s))\|ds+\int_\tau^{\bar t}\|v(s))\|ds\\
&\leq(\bar t-\tau)^{1/\rho}\Big(\int_\tau^{\bar t}\|f(\xi'(s),\eta'(s))\|^r ds\Big)^{1/r}+\bar t-\tau\\
&\leq\sigma^{1/\rho}\Big(\int_\tau^{\bar t}\big(L(\xi'(s),\eta'(s))+C\big)ds\Big)^{1/r}+\sigma\\
&\leq\sigma^{1/\rho}\Big(\xi_0'(\bar t)-\xi_0'(\tau)+C(\bar t-\tau)\Big)^{1/r}+\sigma\\
&\leq\sigma^{1/\rho}\big(2|\bar V|+\sigma+C\sigma\big)^{1/r}+\sigma\\
&=\frac12\bar\delta(\sigma)\\
&\leq\frac12\delta\,.\end{aligned}$$

Also, if we let $\theta(s)=\dot\xi_0'(s)+C=L(\xi'(s),\eta'(s))+C$, then θ is nonnegative and

$$\begin{aligned}\int_\tau^t\theta(s)ds&\leq\int_\tau^{\bar t}\theta(s)ds\\
&=\xi_0'(\bar t)-\xi_0'(\tau)+C(\bar t-\tau)\\
&\leq 2|\bar V|+(C+1)\sigma\,.\end{aligned}$$

Since t is an arbitrary member of I', the above inequalities imply that the functions $\dot\xi'$ and θ are integrable on I'. Since I' is bounded, $\dot\xi'$ is integrable as well. This implies, in particular, that the limits $\lim_{t\uparrow\tau+\zeta}\xi'(t)$ and $\lim_{t\uparrow\tau+\zeta}\xi_0'(t)$ exist. Hence, if (A) holds, we can extend Ξ' to the closed interval $\tilde I'\stackrel{\mathrm{def}}{=}[\tau,\tau+\zeta]$, and the result is a curve $\tilde\Xi'$ in $\mathbb{R}^{n+1}$ which is obviously an ε-approximate augmented trajectory of $\hat\Sigma$. If we then define $\tilde S'=S'\cup\{\tau+\zeta\}$, then $\tilde S'$ is a strongly ε-dense subset of $\tilde I'$. Now, if s_1,s_2 are members of $\tilde S'$ such that $s_1\leq s_2$ we have to prove that $\tilde\xi_0'(s_1)+V(\tilde\xi'(s_1))+\varepsilon(s_2-s_1)\geq\tilde\xi_0'(s_2)+V(\tilde\xi'(s_2))$. This is clearly true if $s_2<\tau+\zeta$ or if $s_1=s_2$. So the only remaining case for us to consider is when $s_1<s_2=\tau+\zeta$. But in that case we can take a sequence $\{v_\ell\}_{\ell=1}^\infty$ of points of I' such that $v_\ell\uparrow\tau+\zeta$ as $\ell\to\infty$, and $v_\ell>s_1$ for all ℓ. Then we can pick, for each ℓ, a $w_\ell\in S'$ such that $v_\ell\leq w_\ell$. On the other hand, $s_1\in S'$, since $s_1<\tau+\zeta$. Therefore $\tilde\xi_0'(s_1)+V(\tilde\xi'(s_1))+\varepsilon(w_\ell-s_1)\geq\tilde\xi_0'(w_\ell)+V(\tilde\xi'(w_\ell))$ for all ℓ. If we let $\ell\to\infty$ and use the continuity of $\tilde\xi_0'$ and V, we find that

$$\tilde\xi_0'(s_1)+V(\tilde\xi'(s_1))+\varepsilon(s_2-s_1)\geq\tilde\xi_0'(s_2)+V(\tilde\xi'(s_2))\,,$$

as desired. This completes the proof that the extension $(\tilde I',\tilde\Xi',\tilde S')$ of (I',Ξ',S') is also in $\mathcal{Z}_\varepsilon$, a fact that of course contradicts the maximality of (I',Ξ',S') if (A) holds. We have thus derived a contradiction from the assumption that (A) is true. Hence (A) is excluded.

We are thus left with Case (B), that is, the possibility that $I'=[\tau,\tau+\zeta]$ and $\zeta<\sigma$. We now proceed to exclude this case. The integral calculation done above shows that

$$\int_\tau^{\tau+\zeta}\|\dot\xi'(t)\|dt\leq\frac{\delta}{2}\,,$$

so

$$\|\xi'(\tau+\zeta)-\xi'(\tau)\|\leq\frac{\delta}{2}\,. \qquad (53)$$

Since $\xi'(\tau)\in K$, (53) implies that, if $q=\xi'(\tau+\zeta)$, then $\mathrm{dist}(q,K)\leq\frac{\delta}{2}$. In particular, q is an interior point of K_δ. Using Theorem 4.1, we construct sequences $\{x_j\}_{j=1}^\infty$, $\{v_j\}_{j=1}^\infty$, $\{\lambda_j\}_{j=1}^\infty$, $\{h_j\}_{j=1}^\infty$, $\{\gamma_j\}_{j=1}^\infty$ in Ω, $\mathbb{R}^n$, $\mathbb{R}$, $\mathbb{R}$, and $\mathbb{R}$, respectively, such that

(1) the inequalities

$$\begin{aligned}h_j&>0\,,\\ \gamma_j&>0\,,\\ \|x_j-q-h_jv_j\|&\leq h_j\gamma_j\,,\\ V(x_j)&\leq V(q)-h_j\lambda_j+h_j\gamma_j\,,\end{aligned}$$

hold for all j,

(2) $(v_j, \lambda_j) \in F_{\hat{\Sigma}}(q, U)$ for all j,

(3) $h_j \downarrow 0$, $\gamma_j \downarrow 0$, and $x_j \to q$ as $j \to \infty$.

We let $(v_j, \lambda_j) = F_{\hat{\Sigma}}(q, u_j)$, $u_j \in U$, and write $\tilde{v}_j = h_j^{-1}(x_j - q)$, so that

$$\|\tilde{v}_j - v_j\| \leq \gamma_j \,. \tag{54}$$

Choose j so large that the following conditions are fulfilled:

$$h_j < \varepsilon \,, \tag{55}$$

$$\gamma_j < \frac{\varepsilon}{2} \,, \tag{56}$$

$$\|x_j - q\| < \beta \,, \tag{57}$$

where β is a positive number such that $\beta < \frac{\delta}{2}$ and $\omega(\beta) < \frac{\varepsilon}{2}$.

Then

$$h_j\|\tilde{v}_j\| \leq \|x_j - q\| \leq \beta \leq \frac{\delta}{2} \,.$$

Define a new trajectory $\xi^{\#} : I^{\#} \mapsto \mathbb{R}^n$, where

$$I^{\#} \stackrel{\text{def}}{=} [\tau, \tau + \zeta + h_j] \,,$$

by letting

$$\xi^{\#}(s) = \begin{cases} \xi'(s) & \text{if} \quad \tau \leq s \leq \tau + \zeta \,, \\ q + (s - \tau - \zeta)\tilde{v}_j & \text{if} \quad \tau + \zeta \leq s \leq \tau + \zeta + h_j \,, \end{cases}$$

and augment it by defining

$$\xi_0^{\#}(s) = \begin{cases} \xi_0'(s) & \text{if} \quad \tau \leq s \leq \tau + \zeta \,, \\ q_0 + \int_{\tau+\zeta}^{s} L(\xi^{\#}(v), u_j)\, dv & \text{if} \quad \tau + \zeta \leq s \leq \tau + \zeta + h_j \,, \end{cases}$$

where $q_0 = \xi_0'(\tau + \zeta)$. Let $\Xi^{\#} = (\xi^{\#}, \xi_0^{\#})$. Then $\Xi^{\#}$ is clearly absolutely continuous, and its restriction to $[\tau, \tau + \zeta]$ is an ε-approximate augmented trajectory of $\hat{\Sigma}$. If s belongs to $[\tau + \zeta, \tau + \zeta + h_j]$, then

$$\|\xi^{\#}(s) - q\| \leq h_j\|\tilde{v}_j\| \leq \frac{\delta}{2} \,.$$

Therefore $\xi^{\#}(s) \in K_\delta$. Then

$$\begin{aligned} \dot{\Xi}^{\#}(s) &= (v_j, L(\xi^{\#}(s), u_j)) \\ &= (f(q, u_j), L(\xi^{\#}(s), u_j)) \\ &= F_{\hat{\Sigma}}(\xi^{\#}(s), u_j) + (w_j(s), 0) \,, \end{aligned}$$

where

$$w_j(s) = f(q, u_j) - f(\xi^{\#}(s), u_j) \,.$$

Since

$$\|\xi^{\#}(s) - q\| \leq h_j\|v_j\| \leq \beta \,,$$

we have

$$\|w_j(s)\| \leq \omega(\beta) \leq \varepsilon \,.$$

Therefore $\Xi^{\#}$ is an ε-approximate augmented trajectory of $\hat{\Sigma}$.

Clearly, $\xi^{\#}(\tau + \zeta + h_j) = x_j$. Therefore,

$$\begin{aligned} &V(\xi^{\#}(\tau + \zeta + h_j)) \\ &\quad = V(x_j) \\ &\quad \leq V(q) - h_j\lambda_j + h_j\gamma_j \\ &\quad = V(\xi^{\#}(\tau + \zeta)) - h_j L(q, u_j) + h_j\gamma_j \\ &\quad = V(\xi^{\#}(\tau + \zeta)) - \int_{\tau+\zeta}^{\tau+\zeta+h_j} L(\xi^{\#}(v), u_j)\, dv \\ &\qquad - \int_{\tau+\zeta}^{\tau+\zeta+h_j} \Big(L(q, u_j) - L(\xi^{\#}(v), u_j)\Big)\, dv + h_j\gamma_j \\ &\quad = V(\xi^{\#}(\tau + \zeta)) + \xi_0^{\#}(\tau + \zeta) - \xi_0^{\#}(\tau + \zeta + h_j) + E \,, \end{aligned}$$

where

$$E = -\int_{\tau+\zeta}^{\tau+\zeta+h_j} \Big(L(q, u_j) - L(\xi^{\#}(v), u_j)\Big)\, dv + h_j\gamma_j \,.$$

Then

$$E \leq h_j\omega(\beta) + h_j\gamma_j \leq \varepsilon h_j \,.$$

Therefore

$$\begin{aligned} &V(\xi^{\#}(\tau + \zeta + h_j)) \leq \\ &\quad V(\xi^{\#}(\tau + \zeta)) + \xi_0^{\#}(\tau + \zeta) - \xi_0^{\#}(\tau + \zeta + h_j) + \varepsilon h_j E \,, \end{aligned}$$

that is,

$$\begin{aligned} &V(\xi^{\#}(\tau + \zeta)) + \xi_0^{\#}(\tau + \zeta) + \varepsilon h_j \\ &\quad \geq V(\xi^{\#}(\tau + \zeta + h_j)) + \xi_0^{\#}(\tau + \zeta + h_j) \,. \end{aligned} \tag{58}$$

This last inequality implies that, if we define

$$S^{\#} = S' \cup \{\tau + \zeta + h_j\} \,,$$

then we can easily show that $S^{\#}$ is a strongly ε-dense subset of $I^{\#}$ such that

$$\begin{aligned} &V(\xi^{\#}(s_1)) + \xi_0^{\#}(s_1) + \varepsilon(s_2 - s_1) \\ &\quad \geq V(\xi^{\#}(s_2)) + \xi_0^{\#}(s_2) \end{aligned} \tag{59}$$

whenever $s_1, s_2 \in S^{\#}$ and $s_2 \geq s_1$. The strong ε-density follows because S' is a strongly ε-dense subset of I' and $h_j < \varepsilon$, since $\tau + \zeta$ necessarily belongs to S'. Inequality (59) is clearly true if $s_1 = s_2$ or both s_1 and s_2 belong to S'. To verify that it holds in the remaining case, that is, when $s_1 \in S'$ and $s_2 = \tau + \zeta + h_j$, if suffices to use once again the fact that $\tau + \zeta \in S'$, so

$$\begin{aligned} &V(\xi^{\#}(s_1)) + \xi_0^{\#}(s_1) + \varepsilon(\tau + \zeta - s_1) \\ &\quad \geq V(\xi^{\#}(\tau + \zeta) + \xi_0^{\#}(\tau + \zeta) \,. \end{aligned} \tag{60}$$

If we add (58) and (60), and cancel the sum $V(\xi^{\#}(\tau + \zeta)) + \xi_0^{\#}(\tau + \zeta)$ that appears on both sides of the result, we get

$$\begin{aligned} &V(\xi^{\#}(s_1)) + \xi_0^{\#}(s_1) + \varepsilon(\tau + \zeta + h_j - s_1) \\ &\quad \geq V(\xi^{\#}(\tau + \zeta + h_j)) + \xi_0^{\#}(\tau + \zeta + h_j) \,, \end{aligned} \tag{61}$$

that is,

$$\begin{aligned} &V(\xi^{\#}(s_1)) + \xi_0^{\#}(s_1) + \varepsilon(s_2 - s_1) \\ &\quad \geq V(\xi^{\#}(s_2)) + \xi_0^{\#}(s_2) \,. \end{aligned} \tag{62}$$

It then follows that $(I^\#, \Xi^\#, S^\#)$ belongs to $\mathcal{Z}_\varepsilon$. Since $(I', \Xi', S') \preceq (I^\#, \Xi^\#, S^\#)$ but $(I', \Xi', S') \neq (I^\#, \Xi^\#, S^\#)$, we have arrived at a contradiction, which this time has arisen from the assumption that (B) holds. Hence (B) is excluded as well.

It now follows that $I' = [\tau, \tau+\sigma]$. In other words, we have shown that there exists an ε-approximate augmented trajectory $\Xi_\varepsilon = (\xi_\varepsilon, \xi_{0,\varepsilon})$ of $\hat{\Sigma}$ which is defined on $[\tau, \tau+\sigma]$, and is such that there is a strongly ε-dense subset S_ε of $[\tau, \tau+\sigma]$ having the property that

$$V(\xi_\varepsilon(s_1)) + \xi_{0,\varepsilon}(s_1) + \varepsilon(s_2 - s_1) \geq V(\xi_\varepsilon(s_2)) + \xi_{0,\varepsilon}(s_2) \quad (63)$$

whenever $s_1, s_2 \in S_\varepsilon$ and $s_1 \leq s_2$.

Clearly, the points τ, $\tau+\sigma$ must belong to S_ε. If we aplly (63) with $s_1 = \tau$, $s_2 = \tau+\sigma$, we find the bound

$$\xi_{0,\varepsilon}(\tau+\sigma) - \xi_{0,\varepsilon}(\tau) \leq 2|\bar{V}| + \sigma\,.$$

Therefore, if $\{\varepsilon_j\}_{j=1}^\infty$ is a sequence of positive numbers that converges to 0, we can apply Theorem (5.2) with $T = \sigma$ and $\Xi^j = (\xi^j, \xi_0^j) = (\check{\xi}_{\varepsilon_j}, \check{\xi}_{0,\varepsilon_j})$, where $\check{\xi}_{\varepsilon_j}(s) = \xi_{\varepsilon_j}(s+\tau)$ and $\check{\xi}_{0,\varepsilon_j}(s) = \xi_{0,\varepsilon_j}(s+\tau) - \xi_{0,\varepsilon_j}(\tau)$ for $s \in [0, \sigma$. Then, after passing to a subsequence, we may assume that there exists an augmented trajectory $\Xi^\infty = (\xi^\infty, \xi_0^\infty)$ of $\hat{\Sigma}$ such that

(i) $\xi_0^\infty(\tau) = 0$,

(ii) the sequence $\{\xi^j\}_{j=1}^\infty$ converges uniformly to ξ^∞, and

(iii) $\liminf_{j\to\infty} \inf\left\{\Delta^{j,\infty}(t,s) : 0 \leq s \leq t \leq T\right\} \geq 0$, where, if $j, j' \in \mathbb{N} \cup \{+\infty\}$, we define $\Delta^{j,j'}$ as in (37).

We now show that Ξ^∞ is an augmented trajectory of steepest descent of $\hat{\Sigma}$ from $x^\#$ with respect to V. For this purpose, we pick $s_1, s_2 \in [0, \sigma]$ such that $s_1 \leq s_2$, and prove that

$$V(\xi^\infty(s_1)) + \xi_0^\infty(s_1) \geq V(\xi^\infty(s_2)) + \xi_0^\infty(s_2)\,. \quad (64)$$

For this purpose we pick, for each sufficiently large j, points s_1^j, s_2^j in S_{ε_j} such that

$$s_1^j \leq s_1 + \tau \leq s_1 + \varepsilon_j < s_2^j - \varepsilon_j \leq s_2 + \tau \leq s_2^j\,.$$

(We assume that $s_1 < s_2$, because (64) is trivially true if $s_1 = s_2$.)

Then

$$\begin{aligned} V(\xi_{\varepsilon_j}(s_1^j)) - V(\xi_{\varepsilon_j}(s_2^j)) + \varepsilon_j(s_2^j - s_1^j) & \\ \geq \quad \xi_{0,\varepsilon_j}(s_2^j) - \xi_{0,\varepsilon_j}(s_1^j)\,, & \end{aligned} \quad (65)$$

that is,

$$\begin{aligned} & V(\xi^j(s_1^j - \tau)) - V(\xi^j(s_2^j - \tau)) + \varepsilon_j(s_2^j - s_1^j) \\ & \quad \geq \quad \xi_0^j(s_2^j - \tau) - \xi_0^j(s_1^j - \tau) \\ & \quad = \quad \xi_0^\infty(s_2^j - \tau) - \xi_0^\infty(s_1^j - \tau) + \Delta^{j,\infty}(s_2^j - \tau, s_1^j - \tau) \\ & \quad \geq \quad \xi_0^\infty(s_2^j - \tau) - \xi_0^\infty(s_1^j - \tau) \\ & \qquad\quad + \inf\{\Delta^{j,\infty}(t,s) : 0 \leq s \leq t \leq \sigma\} \end{aligned}$$

Then

$$\begin{aligned} & V(\xi^\infty(s_1)) - V(\xi^\infty(s_2)) \\ & \quad = \lim_{j\to\infty} \Big(V(\xi^j(s_1^j - \tau)) - V(\xi^j(s_2^j - \tau)) + \varepsilon_j(s_2^j - s_1^j) \Big) \\ & \quad \geq \liminf_{j\to\infty} \Big(\xi_0^j(s_2^j - \tau) - \xi_0^j(s_1^j - \tau) \Big) \\ & \quad = \liminf_{j\to\infty} \Big(\xi_0^\infty(s_2^j - \tau) - \xi_0^\infty(s_1^j - \tau) \\ & \qquad\quad + \Delta^{j,\infty}(s_2^j - \tau, s_1^j - \tau) \Big) \\ & \quad \geq \liminf_{j\to\infty} \Big(\xi_0^\infty(s_2^j - \tau) - \xi_0^\infty(s_1^j - \tau) \\ & \qquad\quad + \inf\{\Delta^{j,\infty}(t,s) : 0 \leq s \leq t \leq \sigma\} \Big) \\ & \quad = \lim_{j\to\infty} \Big(\xi_0^\infty(s_2^j - \tau) - \xi_0^\infty(s_1^j - \tau) \Big) \\ & \qquad\quad + \liminf_{j\to\infty} \Big(\inf\{\Delta^{j,\infty}(t,s) : 0 \leq s \leq t \leq \sigma\} \Big) \\ & \quad = \xi_0^\infty(s_2) - \xi_0^\infty(s_1) \\ & \qquad\quad + \liminf_{j\to\infty} \Big(\inf\{\Delta^{j,\infty}(t,s) : 0 \leq s \leq t \leq \sigma\} \Big) \\ & \quad \geq \xi_0^\infty(s_2) - \xi_0^\infty(s_1)\,. \end{aligned}$$

We have thus proved (64), thereby establishing that Ξ^∞ is an augmented trajectory of $\hat{\Sigma}$ from $x^\#$ of steepest descent with respect to V. If we now concatenate Ξ and Ξ^∞ in the obvious way, by defining

$$\xi^\#(s) = \begin{cases} \xi(s) & \text{if} \quad 0 \leq s \leq \tau\,, \\ \xi^\infty(s-\tau) & \text{if} \quad \tau \leq s \leq \tau+\sigma\,, \end{cases}$$

$$\xi_0^\#(s) = \begin{cases} \xi_0(s) & \text{if} \quad 0 \leq s \leq \tau\,, \\ \xi_0^\infty(s-\tau) + \xi_0(\tau) & \text{if} \quad \tau \leq s \leq \tau+\sigma\,, \end{cases}$$

then $\Xi^\#$ is an augmented trajectory of $\hat{\Sigma}$ from x_* of steepest descent with respect to V, defined on $[0, \tau+\sigma]$. This contradicts the maximality of Ξ, and concludes our proof. $\diamondsuit$

7. The dynamic programming inequality.

Theorem 7.1 *Let Ω be an open subset of $\mathbb{R}^n$, and let $f : \Omega \mapsto \mathbb{R}^n$, $L : \Omega \mapsto \mathbb{R}$ be continuous maps. Let $V : \Omega \mapsto \mathbb{R}$ be a continuous function that satisfies*

$$-\nabla V(x) \cdot f(x) - L(x) \leq 0 \tag{66}$$

on Ω in the viscosity sense. Then for every $x_* \in \Omega$ there exists a curve ξ in Ω, defined on an interval I that contains 0, such that

(1) ξ is an integral curve of f (that is, ξ is locally absolutely continuous and $\dot{\xi}(t) = f(\xi(t))$ for almost every $t \in I$, from which it follows that ξ is continuously differentiable and $\dot{\xi}(t) = f(\xi(t))$ for every $t \in I$),

(2) $\xi(0) = x_*$,

(3) $V(\xi(s)) \leq V(\xi(t)) + \int_s^t L(\xi(v))\,dv$ whenever $s, t \in I$ and $s \leq t$,

(4) ξ is right-unbounded.

Proof. Let U be a set consisting of a single point $\bar{u}$. Let $\tilde{\Sigma}$ be the augmented control system $(\Omega, U, \tilde{f}, \tilde{L})$, where $\tilde{f}(x, \bar{u}) = f(x)$, $\tilde{L}(x, \bar{u}) = -L(x)$. Then $\tilde{\Sigma}$ satisfies all the hypotheses of Theorem 6.1.

Let $\mathcal{V} = -V$. We claim that $\mathcal{V}$ satisfies

$$\sup\{-\nabla\mathcal{V}(x) \cdot \tilde{f}(x, u) - \tilde{L}(x, u) : u \in U\} \geq 0$$

in the viscosity sense. To prove this, we have to pick a point $x \in \Omega$ and a subdifferential p of $\mathcal{V}$ at x, and show that

$$\sup\{-p \cdot \tilde{f}(x, u) - \tilde{L}(x, u) : u \in U\} \geq 0\,,$$

i.e., that

$$-p \cdot f(x) + L(x) \geq 0\,. \tag{67}$$

But, if p is a subdifferential of $-V$ at x, and we let $\pi = -p$, then it follows that π is a superdifferential of V at x. Since V satisfies (66) in the viscosity sense, this implies that $-\pi \cdot f(x) - L(x) \leq 0$. But then $p \cdot f(x) - L(x) \leq 0$, so $-p \cdot f(x) + L(x) \geq 0$, and (67) has been proved.

We can therefore apply the trivial Proposition 2.6 to the augmented system $\tilde{\Sigma}$ and the function $\mathcal{V}$, and conclude that there exists a maximal augmented trajectory $\Xi = (\xi, \xi_0)$ of $\tilde{\Sigma}$ from x_* of steepest descent with respect to $\mathcal{V}$, and then use the nontrivial Theorem 6.1 to conclude that ξ is right-unbounded. The fact that ξ is a trajectory of $\tilde{\Sigma}$ means, of course, that ξ is an integral curve of f. The steepest descent condition says that

$$\mathcal{V}(\xi(s)) \geq \mathcal{V}(\xi(t)) + \int_s^t (-L(\xi(v)))\,dv$$

whenever $0 \leq s \leq t < \tau$. But this says precisely that

$$V(\xi(s)) \leq V(\xi(t)) + \int_s^t L(\xi(v))\,dv$$

whenever $0 \leq s \leq t < \tau$. Hence ξ satisfies all the desired properties, and our proof is complete. $\diamondsuit$

The following result is then a trivial corollary of Theorem 7.1.

Theorem 7.2 *Let $\hat{\Sigma} = (\Omega, U, f, L)$ be an n-dimensional augmented system such that the map $\Omega \ni x \mapsto (f(x, u), L(x, u)$ is continuous for each $u \in U$. Let $V : \Omega \mapsto \mathbb{R}$ be a continuous function that satisfies (14) on Ω in the viscosity sense. Then for every $x_* \in \Omega$ and every piecewise constant function $\eta : [0, \infty[\mapsto U$ there exists a curve $\xi : I \mapsto \Omega$, defined on a subinterval I of $[0, \infty[$, such that*

(1) $0 \in I$ and $\xi(0) = x_*$,

(2) ξ is a trajectory for the control η (that is, ξ is locally absolutely continuous and $\dot{\xi}(t) = f(\xi(t), \eta(t))$ for almost every $t \in I$)

(3) $V(\xi(s)) \leq V(\xi(t)) + \int_s^t L(\xi(v), \eta(v))\,dv$ whenever $s, t \in I$ and $s \leq t$,

(4) ξ is right-unbounded. $\diamondsuit$

Theorem 7.2 has the following immediate consequence.

Theorem 7.3 *Let $\hat{\Sigma} = (\Omega, U, f, L)$ be an n-dimensional augmented system such that the map $\Omega \ni x \mapsto (f(x, u), L(x, u)$ is continuous for each $u \in U$. Let $V : \Omega \mapsto \mathbb{R}$ be a continuous function that satisfies (14) on Ω in the viscosity sense. Let $\Xi(\xi, \xi_0) : I \mapsto \Omega \times \mathbb{R}$ be a locally uniquely limiting augmented trajectory of $\hat{\Sigma}$. Then the dynamic programming inequality*

$$V(\xi(s)) \leq V(\xi(t)) + \xi_0(t) - \xi_0(s)$$

holds for all $s, t \in I$ such that $s \leq t$. $\diamondsuit$

8. Proof of Theorem 2.9

According to Theorem 7.3, the dynamic programming inequality holds along every almost locally uniquely limiting augmented trajectory. The hypothesis that every augmented arc has an almost locally uniquely limiting improvement then implies that the dynamic programming inequality holds along every augmented trajectory. If we apply the inequality to an augmented arc Ξ that starts at a point x and ends at the target, and use the fact that $V = 0$ on the target, we find that $V(x) \leq J(\Xi)$. Hence V is bounded above by the value function $\mathcal{V}_{\mathcal{T}}^{\hat{\Sigma}}$.

To prove that $V \geq \mathcal{V}_{\mathcal{T}}^{\hat{\Sigma}}$, we pick $x \in \Omega$ and use Proposition 2.6 to conclude that there exists a maximal augmented trajectory $\Xi = (\xi, \xi_0)$ of $\hat{\Sigma}$ from x of steepest descent with respect to $\mathcal{V}$, and then use Theorem 6.1 to conclude that ξ is right-unbounded. We then invoke our hypotheses to conclude that Ξ ends at the target. Then $V(x) \geq J(\Xi)$, so V is bounded below by the value function, and our proof is complete. $\diamondsuit$

References

[1] Bardi, M., and I. Capuzzo-Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Birkhäuser, Boston, 1997.

[2] Bardi, Martino, and Francesca Da Lio, "On the Bellman equation for some unbounded control problems," *NoDEA Nonlinear Differential Equations Appl.*, **4**(1997), pp. 276-285.

[3] Fleming, W.H., and H.M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer, New York, 1993.

[4] Malisoff, M., "Viscosity Solutions of the Bellman Equation for Optimal Control Problems with Exit Times," Ph.D. Thesis, Rutgers University, New Brunswick, NJ, January 2000.

[5] Malisoff, M., "Viscosity solutions of the Bellman equation for exit time optimal control problems with vanishing Lagrangians," *SIAM J. Control Optim.*, to appear.

[6] Malisoff, M., "Viscosity solutions of the Bellman equation for exit time optimal control problems with non-Lipschitz dynamics," *ESAIM: Control, Optimisation and Calculus of Variations*, **6**(2001), pp. 415-441.

[7] Malisoff, M., "Bounded-from-below solutions of the Hamilton-Jacobi equation for optimal control problems with exit times: Vanishing Lagrangians, eikonal equations, and shape-from-shading," submitted.

[8] Sussmann, H.J., "From the Brachystochrone problem to the maximum principle," in *Proc. 35th IEEE CDC*, IEEE, New York, 1996, pp. 1588-1594.

[9] Zelikin, M.I., and V.F. Borisov, *Theory of Chattering Control with Applications to Astronautics, Robotics, Economics, and Engineering*, Birkhäuser, Boston, 1994.