Uniqueness results for the value function via direct trajectory-construction methods

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***Abstract*— We present some new results, together with a number of particularly simple and user-friendly versions of results obtained in recent years by the author and M. Malisoff, on the uniqueness of solutions of the Hamilton-Jacobi-Bellman equation (HJBE) for deterministic finite-dimensional optimal control problems under non-standard hypotheses. Our approach is completely control-theoretic and totally self-contained, using the systematic construction of special trajectories of various kinds, and not involving any PDE methods. We donot assume that the Lagrangian is positive, or that the dynamics is Lipschitz-continuous.**

I. INTRODUCTION

We consider autonomous *Lagrangian optimization problems* involving a *state variable* x with values in an open subset Ω of $\mathbb{R}^n$, a *control variable* u taking values in a *control space* U, and a *target set* $\mathcal{T}$, which is a closed subset of the closure of Ω disjoint from Ω itself. The dynamics is given by an O.D.E.

$$\dot{x} = f(x,u)\,, \tag{1}$$

the cost functional to be minimized is

$$J = \int_{\tau_-(\xi)}^{\tau_+(\xi)} L(\xi(t),\eta(t))\,dt\,, \tag{2}$$

(where $\tau_-(\xi)$, $\tau_+(\xi)$ are the initial and terminal times of the trajectory ξ), and for each $x \in \Omega$ the minimization is over the set $\mathcal{A}^{\hat{\Sigma}}_{x,\mathcal{T}}$ of all pairs $\Xi = (\xi,\xi_0)$ such that

- (i) Ξ consists of a trajectory ξ of (1) (i.e., a locally absolutely continuous (LAC) function ξ that satisfies $\dot{\xi}(t) = f(\xi(t),\eta(t))$ for almost all t) corresponding to some U-valued control η, and a "running cost" function ξ_0 corresponding to ξ and η (i.e., a LAC function ξ_0 such that $\dot{\xi}_0(t) = L(\xi(t),\eta(t))$ for almost all t);
- (ii) ξ starts at x, and "ends at the target" in a sense to be defined precisely later.

We will call a pair $\Xi = (\xi,\xi_0)$ for which (i) above holds an "augmented trajectory" of our system, because it really is a trajectory of the "augmented control system"

$$\dot{x} = f(x,u)\,, \qquad \dot{x}_0 = L(x,u) \tag{3}$$

obtained from (1) by "adding the cost as an extra variable" in a well known way. We will write $J(\Xi)$, rather than just J, for the left-hand side of (2), because it is easy to see that the natural arguments for our cost functional J are really augmented trajectories, since (i) one cannot just regard J as a functional of ξ only, because the integral of (2) involves the control η as well as ξ, but on the other hand (ii) the cost is completely determined once we know ξ and a running cost ξ_0, because in that case $L(\xi(t),\eta(t)) = \dot{\xi}_0(t)$.

The infimum $V(x)$ of the costs $J(\Xi)$ of all augmented trajectories $\Xi \in \mathcal{A}^{\hat{\Sigma}}_{x,\mathcal{T}}$ is the *value* of our problem at x. (If $\mathcal{A}^{\hat{\Sigma}}_{x,\mathcal{T}}$ is empty, then of course $V(x) = +\infty$.) The function $V : \Omega \mapsto \mathbb{R} \cup \{-\infty,+\infty\}$ is the *value function*.

Our goal is to present uniqueness theorems, showing that a viscosity solution of the HJBE that satisfies an appropriate boundary condition is necessarily the value function. "Uniqueness" is to be understood as "uniqueness within a class defined by some additional properties," such as that of all functions that are continuous and bounded below.

We will work with a class of systems which is sufficiently general to capture some new and interesting phenomena, and restricted enough to make it possible to prove strong theorems. In particular, we will asume that the sets $F_{\hat{\Sigma}}(x,U) = \{(f(x,u),L(x,u)) : u \in U\}$ are closed and convex, but will not require them to be compact, and will instead impose the "local coerciveness" condition that locally, an inequality of the form $\|f(x,u)\|^r \le L(x,u) + C$, with $C > 0$ and $r > 1$, hold uniformly with respect to u. We also require $f(x,u)$ and $L(x,u)$ to be continuous with respect to x, uniformly with respect to u locally in x.

On the other hand, we will most definitely *not* require that $f(x,u)$ be Lipschitz-continuous with respect to x, since one of the main purposes of this work is to clarify the exact role of the Lipschitz-continuity assumptions often made in the viscosity literature. The answer we propose is as follows:

- (a) Without any Lipschitz-continuity hypotheses, one can prove, for continuous viscosity solutions V of the HJBE, an *existence theorem for trajectories*, asserting that, starting at every point of x of Ω, there is a maximally defined "augmented trajectory of steepest descent," that is, a maximally defined pair $\Xi = (\xi,\xi_0)$ defined on a interval I such that $0 = \min I$, for which the inequality
$$V(\xi(t)) + \xi_0(t) \ge V(\xi(s)) + \xi_0(s) \tag{4}$$
holds whenever $s,t \in I$ and $s \le t$.
- (b) As a trivial corollary of the existence of steepest descent trajectories (applied to $-V$ and $-L$), we get

the existence of "DPI trajectories" (where "DPI" is an acronym for "Dynamic Programming Inequality"), i.e., augmented trajectories $\Xi = (\xi, \xi_0)$ along which the inequality

$$V(\xi(t)) + \xi_0(t) \leq V(\xi(s)) + \xi_0(s) \qquad (5)$$

(that is, the exact opposite of (4)) is satisfied.

(c) The existence result of (b) says that for every "sufficiently nice" (e.g., piecewise constant) control η *there exists a trajectory of η with the given initial condition along which the DPI holds*. This is almost, but not quite, what is needed to prove that V is bounded above by the value function.

(d) The gap between the existence result for DPI trajectories and what would actually be needed to prove that V is bounded above by the value function is that to achieve the latter goal one needs the DPI to hold for *all* trajectories, and it is not enough to have just one DPI trajectory for every intial condition and every control.

(e) The gap described in (d) clearly does not exist when there is uniqueness of trajectories for every given control and initial condition, and in particular when, for every admissible control η, the corresponding time-varying vector field $(x,t) \mapsto f(x, \eta(t))$ satisfies a Lipschitz-Carathéodory condition that guarantees uniqueness of trajectories, or a weaker condition such as a local bound

$$\langle f(x, \eta(t)) - f(y, \eta(t)), x - y \rangle \leq k(t) \|x - y\|^2 \,, \quad (6)$$

with k integrable.

(h) Even more generally, the only property that really matters is that, if we pick a sequence $\{\eta_j\}_{j=1}$ of piecewise constant controls such that our augmented trajectory Ξ can be approximated by augmented trajectories Ξ_j corresponding to the η_j—with, say, the same initial condition—then the Ξ_j converge to Ξ uniformly *no matter how the Ξ_j are chosen*. We call such trajectories "uniquely limiting," and use this concept in the statement of our main theorem.

Remark 1.1: Our approach owes a great deal to the book [6] by A.I. Subbotin. Subbotin, however, only studies viscosity solutions of PDEs of the form $F(x, u(x), Du(x)) = 0$, where the Hamiltonian $F(x, u, p)$ is required to be globally Lipschitz with respect to p (cf. Equation (2.2) in [6], p. 9). A somewhat weaker hypothesis is also considered later but, even with the weakened requirement, the results do not cover, for example, coercive problems of the kind discussed here, such as linear quadratic optimal control. ♢

II. The main theorem

If n is a positive integer, an *n-dimensional control system* is a triple $\Sigma = (\Omega, U, f)$ such that Ω (the *state space* of Σ) is an open subset of $\mathbb{R}^n$, U (the *control space* of Σ) is a nonempty set, and f (the *dynamics* of Σ) is a map $\Omega \times U \ni (x, u) \mapsto f(x, u) \in \mathbb{R}^n$.

An *n-dimensional augmented control system* is a 4-tuple $\hat{\Sigma} = (\Omega, U, f, L)$ such that $\Sigma = (\Omega, U, f, L)$ is an n-dimensional control system and L (the *Lagrangian* of $\hat{\Sigma}$) is a map $\Omega \times U \ni (x, u) \mapsto L(x, u) \in \mathbb{R}$. (In that case, the state space, control space, and dynamics of Σ are also called the state space, control space, and dynamics of $\hat{\Sigma}$.)

An augmented control system $\hat{\Sigma} = (\Omega, U, f, L)$ is *continuous* if the maps $\Omega \ni x \mapsto f(x, u) \in \mathbb{R}^n$ and $\Omega \ni x \mapsto L(x, u) \in \mathbb{R}$ are continuous for each fixed u. We call $\hat{\Sigma}$ *uniformly continuous* on a subset S of Ω if there exists a function $\omega :]0, \infty[\mapsto [0, \infty]$ such that $\lim_{s \downarrow 0} \omega(s) = 0$, having the property that

$$\|f(x, u) - f(y, u)\| + |L(x, u) - L(y, u)| \leq \omega(\|x - y\|)$$

whenever $x, y \in S$, $u \in U$. We call $\hat{\Sigma}$ *locally uniformly continuous* if it is uniformly continuous on every compact subset of Ω, and *globally uniformly continuous* if it is uniformly continuous on Ω. We call $\hat{\Sigma}$ *Lipschitz continuous* if the maps $\Omega \ni x \mapsto f(x, u) \in \mathbb{R}^n$ and $\Omega \ni x \mapsto L(x, u) \in \mathbb{R}$ are Lipschitz continuous for each fixed u. We call $\hat{\Sigma}$ *uniformly Lipschitz continuous* on a subset S of Ω if there exists a positive constant C such that

$$\|f(x, u) - f(y, u)\| + |L(x, u) - L(y, u)| \leq C\|x - y\|$$

whenever $x, y \in S$ and $u \in U$. We call $\hat{\Sigma}$ *locally uniformly Lipschitz continuous* if it is uniformly Lipschitz continuous on every compact subset of Ω, and *globally uniformly Lipschitz continuous* if it is uniformly Lipschitz continuous on Ω.

The augmented control system $\hat{\Sigma} = (\Omega, U, f, L)$ is *coercive* on a subset S of Ω if there exist real constants r, A, C, such that $A > 0$, $C > 0$, $r > 1$, and

$$\|f(x, u)\|^r \leq AL(x, u) + C \text{ for all } x \in S \,,\ u \in U \,. \quad (7)$$

We call $\hat{\Sigma}$ *locally coercive* if it is coercive on every compact subset of Ω, and *globally coercive* if it is coercive on Ω.

Remark 2.1: If $\hat{\Sigma}$ is coercive on a set S, then *it is always possible to choose C, r, such that $C > 0$, $r > 1$, and (7) holds with $A = 1$.* ♢

For an augmented control system $\hat{\Sigma} = (\Omega, U, f, L)$, we define a map $F_{\hat{\Sigma}} : \Omega \times U \mapsto \mathbb{R}^{n+1}$ (called the *augmented dynamics* of $\hat{\Sigma}$) by letting $F_{\hat{\Sigma}}(x, u) = (f(x, u), L(x, u))$ for $x \in \Omega$, $u \in U$. We say that $\hat{\Sigma}$ *satisfies the convexity and upper semicontinuity condition* if, for each $x \in \Omega$,

$$F_{\hat{\Sigma}}(x, U) = \bigcap_{\varepsilon > 0} \overline{\mathrm{co}} \Big(\bigcup \{ F_{\hat{\Sigma}}(x', U) : \|x' - x\| \leq \varepsilon \} \Big) \,, \quad (8)$$

where "$\overline{\mathrm{co}}$" stands for "closed convex hull," and $F_{\hat{\Sigma}}(x, U) \stackrel{\mathrm{def}}{=} \{ (f(x, u), L(x, u)) : u \in U \}$.

Remark 2.2: It is easy to see that *if $\hat{\Sigma} = (\Omega, U, f, L)$ is locally uniformly continuous, then $\hat{\Sigma}$ satisfies the convexity and upper semicontinuity condition iff the set $F_{\hat{\Sigma}}(x, U)$ is closed and convex for every $x \in \Omega$.* ♢

A *target* for an augmented control system $\hat{\Sigma} = (\Omega, U, f, L)$ is a closed subset $\mathcal{T}$ of $\mathbb{R}^n$ such that $\mathcal{T} \subseteq \text{Closure}\,\Omega$ and $\mathcal{T} \cap \Omega = \emptyset$.

A *trajectory* of $\hat{\Sigma} = (\Omega, U, f, L)$ is a LAC curve

$$I \ni t \mapsto \xi(t) \in \Omega\,, \tag{9}$$

defined on a nonempty subinterval I of $\mathbb{R}$, such that $\dot{\xi}(t) \in f(\xi(t), U)$ for almost every $t \in I$. An *augmented trajectory* of $\hat{\Sigma}$ is a LAC curve

$$I \ni t \mapsto \Xi(t) = (\xi(t), \xi_0(t)) \in \Omega \times \mathbb{R}\,, \tag{10}$$

defined on a subinterval I of $\mathbb{R}$, having the property that $\dot{\Xi}(t) \in F_{\hat{\Sigma}}(\xi(t), U)$ for almost every $t \in I$.

The *initial time*, or *starting time* of a trajectory ξ (resp. an augmented trajectory $\Xi = (\xi, \xi_0)$) with domain I is the time $\tau_-(\xi) \stackrel{\text{def}}{=} \min I$ (resp. $\tau_-(\Xi) \stackrel{\text{def}}{=} \min I$)), if the minimum exists, i.e., if I is bounded below and its infimum belongs to I. If the initial time of ξ (resp. Ξ) exists, then (i) the point $x_-(\xi) \stackrel{\text{def}}{=} \xi(\tau_-(\xi))$ (resp. $x_-(\Xi) \stackrel{\text{def}}{=} \xi(\tau_-(\Xi))$) is the *starting point*, or *initial point*, of ξ (resp. Ξ), and (ii) the ordered pair $\partial_-(\xi) \stackrel{\text{def}}{=} (\tau_-(\xi), x_-(\xi))$ (resp. $\partial_-(\Xi) \stackrel{\text{def}}{=} (\tau_-(\Xi), x_-(\Xi))$) is the *initial condition* of ξ (resp. Ξ). If $\partial_-(\xi) = (t, x)$ (resp. $\partial_-(\Xi) = (t, x)$), we say that ξ (resp. Ξ) *starts at x at time t*.

If $\mathcal{T}$ is a target for $\hat{\Sigma} = (\Omega, U, f, L)$, then a trajectory ξ or augmented trajectory $\Xi = (\xi, \xi_0)$ with domain I *ends at $\mathcal{T}$* if the limit $\xi(\uparrow) \stackrel{\text{def}}{=} \lim_{t \uparrow \sup I} \xi(t)$ exists and belongs to $\mathcal{T}$.

For each $x \in \Omega$, we let $\mathcal{A}^{\hat{\Sigma}}_{x,\mathcal{T}}$ be the set of all augmented trajectories $\Xi = (\xi, \xi_0)$ of $\hat{\Sigma}$ such that (i) $\partial_-(\Xi) = (0, x)$, (ii) $\xi_0(0) = 0$, (iii) Ξ ends at the target, and (iv) the limit $\xi_0(\uparrow) \stackrel{\text{def}}{=} \lim_{t \uparrow \sup \text{domain}\,\Xi} \xi_0(t)$ exists.

If $\Xi = (\xi, \xi_0) \in \mathcal{A}^{\hat{\Sigma}}_{x,\mathcal{T}}$ then the *cost* of Ξ is the number

$$J(\Xi) \stackrel{\text{def}}{=} \xi_0(\uparrow)\,. \tag{11}$$

The *value function* of the optimal control problem defined by $\hat{\Sigma}$ and the target $\mathcal{T}$ is the function $\mathcal{V}^{\hat{\Sigma}}_{\mathcal{T}} : \Omega \cup \mathcal{T} \mapsto \mathbb{R} \cup \{-\infty, +\infty\}$ given by

$$\mathcal{V}^{\hat{\Sigma}}_{\mathcal{T}}(x) = \begin{cases} \inf\{J(\Xi) : \Xi \in \mathcal{A}^{\hat{\Sigma}}_{x,\mathcal{T}}\} & \text{if} \quad x \in \Omega \\ 0 & \text{if} \quad x \in \mathcal{T}\,. \end{cases}$$

If $V : \Omega \mapsto \mathbb{R}$ is a function, then an augmented trajectory $\Xi = (\xi, \xi_0)$ of $\hat{\Sigma}$ is said to be *of steepest descent with respect to V* if $\xi_0(s) + V(\xi(s)) \geq \xi_0(t) + V(\xi(t))$ whenever $s \leq t$ and $s, t \in \text{domain}\,\Xi$. We use $\mathcal{SD}_{\hat{\Sigma},V}$ to denote the set of all augmented trajectories of $\hat{\Sigma}$ of steepest descent with respect to V, and $\mathcal{SD}_{\hat{\Sigma},V,x}$ to denote the set of all $\Xi \in \mathcal{SD}_{\hat{\Sigma},V}$ such that $\partial_-(\Xi) = (0, x)$.

If $x \in \Omega$, a *maximal augmented trajectory of $\hat{\Sigma}$ from x of steepest descent with respect to V* is a $\Xi \in \mathcal{SD}_{\hat{\Sigma},V,x}$ that cannot be extended to a $\tilde{\Xi} \in \mathcal{SD}_{\hat{\Sigma},V,x}$ defined on an interval which is strictly larger that the domain of Ξ. We use $\mathcal{MSD}_{\hat{\Sigma},V,x}$ to denote the set of all maximal augmented trajectories of $\hat{\Sigma}$ from x of steepest descent with respect to V.

Definition 2.3: . If Ω is an open subset of $\mathbb{R}^n$, and $\xi : I \mapsto \Omega$ is a curve, we say that ξ is *right-unbounded* if

(i) the interval I is open on the right (that is, if $\tau = \sup I$, then either (a) $\tau = +\infty$ or (b) τ is finite and does not belong to I),

(ii) if τ is finite, then for every compact subset K of Ω there exists a τ_K such that $0 \leq \tau_K < \tau$ and $\xi(t) \notin K$ whenever $\tau_K < t < \tau$. $\diamondsuit$

The following observation is a completely trivial consequence of Zorn's lemma, given our definitions, but we state it explicitly as a separate result for future reference.

Proposition 2.4: If $\hat{\Sigma} = (\Omega, U, f, L)$ is an augmented control system, $V : \Omega \mapsto \mathbb{R}$ is a function, and $x \in \Omega$, then the set $\mathcal{MSD}_{\hat{\Sigma},V,x}$ of maximal augmented trajectories of $\hat{\Sigma}$ from x of steepest descent with respect to V is nonempty. $\diamondsuit$

An *augmented arc* is an augmented trajectory whose domain is a compact interval. If $\Xi = (\xi, \xi_0)$ is an augmented arc with domain $[a, b]$, then an *improvement* of Ξ is an augmented arc $\Xi' = (\xi', \xi'_0)$, with domain $[a', b']$, that satisfies $\xi'(a') = \xi(a)$, $\xi'(b') = \xi(b)$, and $\xi'_0(b') - \xi'_0(a') \leq \xi_0(b) - \xi_0(a)$.

If $\hat{\Sigma} = (\Omega, U, f, L)$ is an augmented control system, then an augmented arc $\Xi = (\xi, \xi_0)$ of $\hat{\Sigma}$ with domain $[a, b]$ is *uniquely limiting* if there exists a sequence $\{\eta_j\}_{j=1}^{\infty}$ of piecewise constant functions $\eta_j : [a, b] \mapsto U$ such that

(*) *if $\{\Xi_j\}_{j=1}^{\infty}$ is an arbitrary sequence of maximally defined augmented trajectories of $\hat{\Sigma}$ such that $a \in \text{domain}(\Xi_j)$ and $\Xi_j(a) = \Xi(a)$ for every j, then $[a, b] \subseteq \text{domain}(\Xi_j)$ if j is large enough, and $\Xi_j \to \Xi$ uniformly on $[a, b]$ as $j \to \infty$.*

Example 2.5: Suppose $\Xi = (\xi, \xi_0)$ is an augmented arc of $\hat{\Sigma}$ with domain $[a, b]$ such that

(#) *there exist a positive $\delta \in \mathbb{R}$, a function $\eta : [a, b] \mapsto U$, and an integrable function $\varphi : [a, b] \mapsto [0, \infty]$, such that*

(i) $\dot{\Xi}(t) = \Big(f(x, \eta(t)), L(x, \eta(t))\Big)$ *for a. e.* $t \in [a, b]$,

(ii) *the map* $t \mapsto \Big(f(x, \eta(t)), L(x, \eta(t))\Big)$, *on the compact set* $I_x \stackrel{\text{def}}{=} \{t : a \leq t \leq b \wedge \|x - \xi(t)\| \leq \delta\}$, *is measurable for each* $x \in \Omega$,

(iii) *the map* $x \mapsto \Big(f(x, \eta(t)), L(x, \eta(t))\Big)$, *on the compact set* $I^t \stackrel{\text{def}}{=} \{x \in \Omega : \|x - \xi(t)\| \leq \delta\}$, *is continuouse for each* $t \in [a, b]$,

(iv) $\Big\langle f(x, \eta(t)) - f(x', \eta(t)), x - x' \Big\rangle \leq \varphi(t)\|x - x'\|^2$ *whenever* $t \in [a, b]$, $\|x - \xi(t)\| \leq \delta$, *and* $\|x' - \xi(t)\| \leq \delta$,

(v) *the inequality* $|L(x, \eta(t))| \leq \varphi(t)$ *holds whenever* $t \in [a, b]$ *and* $\|x - \xi(t)\| \leq \delta$.

Then Ξ is uniquely limiting. $\diamondsuit$

An augmented trajectory $\Xi = (\xi, \xi_0)$ with domain I is *locally uniquely limiting* if for every compact subinterval I' of I the restriction of Ξ to I' is uniquely limiting.

An augmented trajectory $\Xi = (\xi, \xi_0)$ with domain I is *almost uniquely limiting* if there exists a finite subset B of I such that the restriction of Ξ to every subinterval of $I \backslash B$ is locally uniquely limiting.

If Ω is an open subset of $\mathbb{R}^n$, we say that a function $V : \Omega \mapsto \mathbb{R}$ *satisfies the inequality*

$$\sup\{-\langle \nabla V(x), f(x,u)\rangle - L(x,u) : u \in U\} \geq 0 \quad (12)$$

on Ω *in the viscosity sense* if

(V_+) *whenever $x \in \Omega$ and $p \in \mathbb{R}^n$ is a subdifferential of V at x, it follows that*

$$\sup\{-\langle p, f(x,u)\rangle - L(x,u) : u \in U\} \geq 0\,.$$

(We recall that, if Ω is open in $\mathbb{R}^n$, then a *subdifferential* of a function $V : \Omega \mapsto \mathbb{R}$ at a point $\bar{x} \in \Omega$ is a vector $p \in \mathbb{R}^n$ such that

$$\liminf_{x \to \bar{x}} \frac{V(x) - V(\bar{x}) - p \cdot (x - \bar{x})}{\|x - \bar{x}\|} \geq 0\,.)$$

Similarly, we say that V *satisfies the inequality*

$$\sup\{-\langle \nabla V(x), f(x,u)\rangle - L(x,u) : u \in U\} \leq 0 \quad (13)$$

on Ω *in the viscosity sense* if

(V_-) *whenever $x \in \Omega$ and $p \in \mathbb{R}^n$ is a superdifferential of V at x, it follows that*
$$\sup\{-\langle p, f(x,u)\rangle - L(x,u) : u \in U\} \leq 0\,.$$

(A *superdifferential* of V at x_* is a vector p such that $-p$ is a subdifferential of $-V$ at x_*.)

We say that V *satisfies the equation*

$$\sup\{-\langle \nabla V(x), f(x,u)\rangle - L(x,u) : u \in U\} = 0 \quad (14)$$

on Ω *in the viscosity sense* if it satisfies (12) and (13) in the viscosity sense.

Remark 2.6: The definition of "viscosity solution" given here is known to be equivalent to the more common one involving test functions, cf. [1]. $\diamondsuit$

Our main result is the following theorem:

Theorem 2.7: Let $\hat{\Sigma} = (\Omega, U, f, L)$ be an augmented control system, let $\mathcal{T}$ be a target for $\hat{\Sigma}$, and let $V : \Omega \cup \mathcal{T} \mapsto \mathbb{R}$ be a function. Assume that

(1) *$\hat{\Sigma}$ is locally uniformly continuous, locally coercive, and such that $F_{\hat{\Sigma}}(x, U)$ is closed and convex for every $x \in \Omega$.*
(2) *V is continuous.*
(3) *V satisfies (14) on Ω in the viscosity sense.*
(4) *V vanishes on $\mathcal{T}$.*
(5) *Every augmented arc has an almost locally uniquely limiting improvement.*
(6) *Whenever $x \in \Omega$, $\Xi = (\xi, \xi_0) \in \mathcal{MSD}_{\hat{\Sigma},V,x}$, and ξ is right-unbounded, it follows that $\Xi \in \mathcal{A}^{\hat{\Sigma}}_{x,\mathcal{T}}$.*

Then $V \equiv \mathcal{V}^{\hat{\Sigma}}_{\mathcal{T}}$. $\diamondsuit$

Remark 2.8: Condition (6) was essentially introduced by M. Malisoff, cf. especially [4]. $\diamondsuit$

III. Examples

Example 3.1: (Linear-quadratic optimal control.) Consider the standard linear-quadratic optimal control problem, in which x, u take values in $\mathbb{R}^n$, $\mathbb{R}^m$, respectively, the dynamical law is

$$\dot{x} = Ax + Bu\,, \quad (15)$$

the Lagrangian is given by

$$L(x,u) = x^\dagger Rx + u^\dagger Su\,,$$

the square matrices R, S are strictly positive definite, and the pair (A, B) is stabilizable. We take the target set $\mathcal{T}$ to consist of the origin of $\mathbb{R}^n$. (In order to satisfy the condition that $F_{\hat{\Sigma}}(x, U)$ is convex for every $x \in \Omega$, we add a new scalar nonnegative control variable v, in such a way that the dynamical law (15) remains unchanged but the Lagrangian L is replaced by $\tilde{L}$, where $\tilde{L}(x,u,v) \stackrel{\text{def}}{=} L(x,u) + v$.) The crucial technical issue here is the fact that the Lagrangian is not bounded away from zero. The hypotheses of our main theorem (including the coerciveness, which follows from the positive definiteness of S) are easily verified as long as V is bounded below. The only nontrivial point is the verification of condition (6). To prove that this holds, let $\Xi : [0, \tau[\, \mapsto \mathbb{R}^n \times \mathbb{R}$ be a right-unbounded maximal trajectory of steepest descent with respect to V that does not end at the target, and write $\Xi = (\xi, \xi_0)$ in the usual way. Then τ has to be infinite, because if τ was finite then the boundedness of the cost (arising from the fact that $\Xi = (\xi, \xi_0)$ is of steepest descent and V is bounded below) would trivially imply an L^2 bound on the control, from which it would follow that Ξ can be extended to the closed interval $[0, \tau]$, and then the assumption that Ξ does not end at the target would enable us to extend Ξ even further, contradicting maximality. So τ is infinite. On the other hand, the fact that V is bounded below and Ξ is of steepest descent implies that the integral

$$\int_0^\infty \Big(\xi(t)^\dagger R\xi(t) + \eta(t)^\dagger S\eta(t)\Big)dt$$

is finite, if η is an open-loop control that generates Ξ. But then ξ and η are square-integrable, so the condition that $\dot{\xi} = A\xi + B\eta$ implies that ξ is square-integrable and has a square-integrable derivative, and then Barbalat's lemma implies that ξ ends at the target, as desired. $\diamondsuit$

Example 3.2: (Fuller's problem, cf., e.g., [8].) This is the optimal control problem for the dynamical law

$$\dot{x} = y\,, \qquad \dot{y} = u\,,$$

with control constraint $-1 \leq u \leq 1$. The target set $\mathcal{T}$ consists of the origin of $\mathbb{R}^2$. The Lagragian is $L(x,y,u) = x^2$. The crucial technical issue here is the fact that the Lagrangian is not bounded away from zero, and in fact has a whole line of zeros. The hypotheses of our main theorem are easily verified as long as V is bounded below. The only nontrivial point is the verification of condition (6). To prove that this holds, let $\Xi : [0,\tau[\mapsto \mathbb{R}^3$ be a right-unbounded maximal trajectory of steepest descent with respect to V that does not end at the target, and write $\Xi = (\xi, \xi_0)$ in the usual way. Then τ has to be infinite, because if τ was finite then the boundedness of the control would trivially imply that Ξ can be extended to the closed interval $[0,\tau]$, and then the assumption that Ξ does not end at the target enables us to extend Ξ even further, contradicting maximality. So τ is infinite. On the other hand, the fact that V is bounded below and Ξ is of steepest descent implies that the integral $\int_0^\infty x(t)^2 dt$ is finite, if we write $\xi(t) = (x(t), y(t))$. But then $x(\cdot)$ is a square-integrable function on $[0,\infty[$ whose second derivative is bounded. By a straightforward generalization of Barbalat's lemma, this implies that both $x(\cdot)$ and $y(\cdot)$ go to zero, i.e., that ξ ends at the target, as desired. $\diamondsuit$

Example 3.3: (The reflected brachistochrone problem.) This is the minimum time problem for the dynamical law

$$\begin{aligned} \dot{x} &= u\sqrt{|y|}\,, \\ \dot{y} &= v\sqrt{|y|}\,, \end{aligned}$$

with control constraint $u^2+v^2 \leq 1$. The target set $\mathcal{T}$ consists of a single point $B \in \mathbb{R}^2$. The crucial technical issue here is the fact that the dynamical law is not Lipschitz-continuous with respect to the state. The hypotheses of our main theorem are easily verified. The only nontrivial point is the verification of condition (5). To prove that this holds, we pick an arbitrary integral arc $\xi : [a,b] \mapsto \mathbb{R}^2$, and observe that either (i) $\xi(t)$ never belongs to the x axis $X = \{(x,y) \in \mathbb{R}^2 : y = 0\}$ or (ii) there exist $t_-, t_+ \in [a,b]$ such that $t_- \leq t_+$, $\xi(t) \notin X$ whenever $a \leq t < t_-$ or $t_+ > t \leq b$, $\xi(t_-) \in X$, and $\xi(t_+) \in X$. If (i) holds, then ξ satisfies the conditions of Example 2.5, so ξ is uniquely limiting. If (ii) holds and $t_+ = t_-$, then the restriction of ξ to each of the intervals $[a, t_-[+,,]\, t_+, b]$, is locally uniquely limiting, so ξ is almost locally uniquely limiting. Finally, if (ii) holds and $t_+ < t_-$, then the restriction $\tilde{\xi}$ of ξ to the interval $[t_-, t_+]$ is such that the set $S = \{t \in [a,b] : \xi(t) \notin X\}$ is the union of a finite or countable infinite collection $\mathcal{I}$ of pairwise disjoint relatively open subintervals of $[a,b]$. If $I \in \mathcal{I}$, then the restriction $\tilde{\xi}_I$ of $\tilde{\xi}$ to I is entirely contained in the open upper half-plane or in the open lower half-plane. By reflecting $\tilde{\xi}_I$ with respect to X, if necessary, we may assume that $\tilde{\xi}_I$ is entirely contained in the open upper half-plane for every $I \in \mathcal{I}$. Then $\tilde{\xi}$ is a trajectory of our system entirely contained in the closed upper half-plane. It is well known that the problem in the closed upper half-plane H_+ is the famous "brachistochrone problem," whose time-optimal trajectories $\zeta : [\alpha, \beta] \mapsto H_+$ are cycloids such that $\zeta(t)$ is an interior point of H_+ whenever $\alpha < t < \beta$. It follows that we can always replace $\tilde{\xi}$ by a cycloid ζ, thereby obtaining an almost locally uniquely limiting improvement of ξ. $\diamondsuit$

Example 3.4: (An example with a continuous non-Lipschitz dynamics where uniqueness fails.) Let $\varphi : [0,1] \mapsto \mathbb{R}$ be a nonnegative continuous function such that (a) the set $\{x \in [0,1] : \varphi(x) = 0\}$ is exactly the Cantor set, and (b) $\int_0^1 \frac{dx}{\varphi(x)} < \infty$. (For example, we may take φ to be given by $\varphi(x) = \mathrm{dist}(x,C)^\rho$, where C is the Cantor set and ρ is a positive number such that $\rho < 1 - \log_3 2$. An explicit calculation shows that $\int_0^1 \frac{dx}{\varphi(x)} = (1-\rho)^{-1} 2^\rho \sum_{j=1}^\infty \theta^j$, where $\theta = \frac{2}{3} \times 3^\rho$. Our choice of ρ guarantees that $\theta < 1$, so the integral is finite.) Extend φ to a function defined on $\mathbb{R}$ by making it periodic of period 1. Then consider the optimal control problem on $\mathbb{R}$ whose dynamics is $\dot{x} = u\varphi(x)$, $|u| \leq 1$, and where the goal is to reach the origin in minimum time. It is easy to see that the optimal trajectory from each point x exists and is obtained by "moving towards the target as fast as possible." Precisely, this means the we use the control $u = -1$ as long as we are to the right of the origin, and we use $u = 1$ if we are to the left. This, however, does not suffice to specify the optimal trajectories, because of the lack of uniqueness of solutions. The complete specification of the optimal trajectories is as follows. Suppose $\bar{x} < 0$. Define a function $\tau : [\bar{x}, 0] \mapsto \mathbb{R}$ by letting $\tau(x) = \int_{\bar{x}}^x \frac{dy}{\varphi(y)}$. Then τ is abosolutely continuous, strictly increasing, and such that $\tau(\bar{x}) = 0$. Therefore τ maps the interval $[\bar{x}, 0]$ homeomorphically onto the interval $[0, \tau(0)]$. Let ξ be the inverse function, so ξ maps $[0,\tau(0)]$ homeomorphically onto $[\bar{x}, 0]$. Then ξ is absolutely continuous, and $\dot{\xi}(t) = \varphi(\xi(t))$ for almost all $t \in [0, \tau(0)]$. So ξ is a trajectory of our system which goes from $\bar{x}$ to 0 in time $\tau(0)$, and it is easy to see that ξ is the optimal trajectory from $\bar{x}$ to 0. It follows that optimal time to go from $\bar{x}$ to 0 is $\tau(0)$, that is, $\int_{\bar{x}}^0 \frac{dy}{\varphi(y)}$. A similar contruction applies when $\bar{x} > 0$. Then the value function $\bar{V}$ for our problem is given by

$$\bar{V}(x) = \int_{\min(x,0)}^{\max(x,0)} \frac{dy}{\varphi(y)}\,.$$

The HJBE for our problem is

$$|V'(x)|\varphi(x) - 1 = 0\,. \tag{16}$$

The function $\bar{V}$ is a solution of this equation on $\mathbb{R}\backslash\{0\}$ in the viscosity sense. (This follows from the fact that, for problems such as this one, the value function is automatically a viscosity solution of the HJBE. In addition, one can also verify this directly. Let $\mathcal{O}$ be the set of points where $\varphi(x) > 0$. Then on $\mathcal{O}$ the function $\bar{V}$ is smooth, and its derivative is $\frac{1}{\varphi}$ when $x < 0$, and $-\frac{1}{\varphi}$ when $x > 0$, so (16) holds. At points x where $\varphi(x) = 0$, the viscosity solution requirements say that $-1 \geq 0$ whenever p is a subdifferential of $\bar{V}$ at x, and $-1 \leq 0$ whenever p is a superdifferential of $\bar{V}$ at x. The second condition is trivially true. To verify the first condition, we need to show that it is satisfied vacuously, i.e., that there are no subdifferentials of $\bar{V}$ at x. But this easy. Suppose, say, that $x < 0$. The difference quotient $\frac{1}{h}(V(x+h) - V(x))$ is equal, if $h > 0$, to $-\frac{1}{h}\int_x^{x+h} \frac{dy}{\varphi(y)}$, which is bounded above by $\zeta(h) = -\frac{1}{\max\{\varphi(y):y\in[x,x+h]\}}$. Since $\varphi(x) = 0$, $\zeta(h)$ goes to $-\infty$ as $h \to 0$. This shows that the right derivative of $\bar{V}$ at x is equal to ∞, from which it follows easily that there exist no subdifferentials of $\bar{V}$ at x. A similar argument shows that if $x > 0$ the left derivative of $\bar{V}$ at x is equal to $+\infty$, from which it follows once again that there are no subdifferentials of $\bar{V}$ at x.

We now show that there exist nonnegative continuous functions $\hat{V} : \mathbb{R} \mapsto \mathbb{R}$ other than $\bar{V}$ that satisfy the HJBE on $\mathbb{R}\backslash\{0\}$ and are such that $\hat{V}(0) = 0$. To see this, we let W be a continuous monotonically nondecreasing real-valued function on $[0, \infty[$ such that (a) $W(0) = 0$, (b) W is constant on each connected component of the set $\{x : x > 0 \wedge \varphi(x) > 0\}$, and (c) $W(x) < W(y)$ whenever $0 \leq x < y$ and the interval $[x, y]$ contains a zero of φ. (Such a function is easily constructed using the well known Cantor function.) We then extend W to all of $\mathbb{R}$ by defining $W(x) = W(-x)$ when $x < 0$. Using W, we define $\hat{V} = \bar{V} + W$. Then $\hat{V}$ is continuous, $\hat{V}(0) = 0$, and $\hat{V}(x) > \bar{V}(x)$ whenever $x \neq 0$. Let us show that $\hat{V}$ is also a solution of the HJBE for our problem on $\mathbb{R}\backslash\{0\}$. Near points x such that $\varphi(x) > 0$, the functions $\bar{V}$ and $\hat{V}$ differ by a constant, so the fact that $\bar{V}$ satisfies the HJBE implies that the same is true for $\hat{V}$. If $x \neq 0$ but $\varphi(x) = 0$, the viscosity solution requirements say that $-1 \geq 0$ whenever p is a subdifferential of $\hat{V}$ at x, and $-1 \leq 0$ whenever p is a superdifferential of $\hat{V}$ at x, and the second one of these conditions is trivially true. As for the first condition, if $x < 0$ then we have already shown that the right derivative of $\bar{V}$ at x is equal to ∞, and this clearly implies that the right derivative of $\hat{V}$ at x is equal to ∞ as well, since $\hat{V} = \bar{V} + W$ and W is monotonically nonincreasing near x. Hence there exist no subdifferentials of $\hat{V}$ at x. A similar argument applies if $x > 0$, and we conclude that first one of the viscosity requirements is satisfied vacuously.

It follows that for our example the value function is not the unique continuous nonnegative function that vanishes at the target and satisfies the HJBE. In the example, the reason for the failure of uniqueness is easy to understand, and clearly related to the non-uniqueness of trajectories. Notice that the spurious value function $\hat{V}$ is bounded below by the true value function, so what goes wrong is the other inequality, which is related to the dynamic programming inequality (DPI). And, indeed, the DPI fails, and this makes it impossible to draw the conclusion that $\hat{V} \leq \bar{V}$. Furthermore, the failure of the DPI happens exactly as described in our general analysis: given any control $u(\cdot)$ and any initial condition x_0, it is easy to construct a maximal trajectory ξ for $u(\cdot)$ starting at x_0 along which the DPI for $\hat{V}$ holds. (It suffices to follow the only possible trajectory for $u(\cdot)$ as long as $\varphi \neq 0$, and stopping at $\bar{x}$ and staying there for ever as soon as we reach the first point $\bar{x}$ where φ vanishes.) This ξ is not, however, the only trajectory for $u(\cdot)$ starting at x_0. And the fact that the DPI holds along ξ does not imply that that the DPI holds for every trajectory for $u(\cdot)$ that starts at x_0. (Indeed, if for example $x_0 < 0$ and $u(t) \equiv 1$, then in addition to the ξ given by our construction we could also consider ξ_{opt}, the optimal trajectory described earlier. The DPI for $\hat{V}$ clearly fails along the curve ξ_{opt}, because if it was true it would follow that $\hat{V}(x_0) \leq \bar{V}(x_0)$, whereas we know that $\hat{V}(x_0) > \bar{V}(x_0)$.) $\diamondsuit$

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