Path-integral generalized differentials*

Dedicated to Jack Warga on his 80th birthday

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1. Introduction

This is the second of a series of two papers on generalized differentiation theories (abbr. GDTs). The first paper discussed the definition of the GDT concept, presented several GDTs (the Warga derivate containers, weak multidifferentials, and generalized differential quotients) and compared them showing, in particular, that none of these theories contains all the others.

In this paper, we introduce another concept of generalized differential—the "path-integral generalized differential," abbreviated PIGD—that achieves the desired unification.

2. Preliminaries

If $n, m \in \mathbb{Z}_+$, $\alpha : [0,1] \to \mathbb{R}^n$ is a Lipschitz function, and $h : [0,1] \to \mathbb{R}^{m\times n}$ is integrable, we use $h * \alpha$ to denote the "chronological product" of h and α, that is, the absolutely continous function $\beta : [0,1] \to \mathbb{R}^m$ given by $\beta(t) = \int_0^t h(s)\cdot\dot\alpha(s)\,ds$.

The following lemma says that the chronological product operation $C^0([0,1];\mathbb{R}^n) \times L^1([0,1],\mathbb{R}^{m\times n}) \ni (\alpha,h) \mapsto h*\alpha \in C^0([0,1];\mathbb{R}^m)$ is jointly continuous, as long as the function α varies in a uniformly Lipschitz subset of $C^0([0,1];\mathbb{R}^n)$. The proof is very simple and will be omitted.

Lemma 2.1 *Let $n, m \in \mathbb{Z}_+$. Let $\{(\alpha_j, h_j)\}_{j=1}^\infty$ be a sequence of members of the product space $S = C^0([0,1];\mathbb{R}^n) \times L^1([0,1],\mathbb{R}^{m\times n})$ that converges in S to a limit $(\alpha_\infty, h_\infty)$. Assume that the sequence $\{\alpha_j\}$ is uniformly Lipschitz (that is, there exists a constant $r \in \mathbb{R}$ such that $\|\alpha_j(t) - \alpha_j(s)\| \le r|t-s|$ for all $j \in \mathbb{N}$ and all $t, s \in [0,1]$). Then*

$$h_j * \alpha_j \to h_\infty * \alpha_\infty \quad \text{in} \quad C^0([0,1];\mathbb{R}^m) \quad \text{as} \quad j \to \infty\,.\diamondsuit$$

Let $n \in \mathbb{Z}_+$, and let S be a subset of $\mathbb{R}^n$. We write $\mathcal{A}(S)$ to denote the subset of $C^0([0,1];\mathbb{R}^n)$ consisting of all absolutely continuous curves $\alpha : [0,1] \to \mathbb{R}^n$ such that $\alpha(0) = 0$ and $\dot\alpha(t) \in S$ for almost all $t \in [0,1]$.

If $\mathcal{S} \subseteq C^0([0,1];\mathbb{R}^n)$, we write $\tau(\mathcal{S})$ to denote the set $\tau(\mathcal{S}) \stackrel{\text{def}}{=} \{\alpha(1) : \alpha \in \mathcal{S}\}$, so $\tau(\mathcal{S})$ is the set of all terminal points of curves in $\mathcal{S}$.

The following is then an immediate consequence of our definitions.

Proposition 2.2 *If K is a compact convex subset of $\mathbb{R}^n$, then $\mathcal{A}(K)$ is a compact convex subset of $C^0([0,1];\mathbb{R}^n)$, and $\tau\Big(\mathcal{A}(K)\Big) = K$.* $\diamondsuit$

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If $m \in \mathbb{Z}_+$ and $v \in \mathbb{R}^m$, we use ξ_v to denote the curve

$$[0,1] \ni t \to tv \stackrel{\text{def}}{=} \xi_v(t) \in \mathbb{R}^m\,. \tag{1}$$

If

$$S \subseteq \mathbb{R}^n \text{ and } G : \mathcal{A}(S) \to\!\!\!\to C^0([0,1];\mathbb{R}^{m\times n}) \times \mathbb{R}^m\,, \tag{2}$$

then we can define set-valued maps

$$\mathcal{I}_G : \mathcal{A}(S) \to\!\!\!\to C^0([0,1];\mathbb{R}^m)\,, \quad \Phi_G : S \to\!\!\!\to \mathbb{R}^m\,,$$

by letting

$$\begin{aligned}\mathcal{I}_G(\alpha) &= \Big\{ h*\alpha + \xi_v : (h,v) \in G(\alpha) \Big\}\,,\\ \Phi_G(x) &= \Big\{ y \in \mathbb{R}^m : (\exists (\alpha,h,v) \in \mathrm{Gr}(G))\\ &\qquad (\alpha(1) = x \vee (h*\alpha)(1) + v = y) \Big\}\,.\end{aligned}$$

The following fact is then trivial.

Proposition 2.3 *Let $n, m \in \mathbb{Z}_+$, and let S, G be such that (2) holds. Then:*

1. $\mathrm{Do}(\mathcal{I}_G) = \mathrm{Do}(G)$, *so in particular $\mathcal{I}_G$ is everywhere defined if and only if G is.*
2. *If G is everywhere defined then Φ_G is everywhere defined.*
3. *If G is single-valued at a particular $\alpha \in \mathcal{A}(S)$, then $\mathcal{I}_G$ is single-valued at α; in particular, if G is single-valued and everywhere defined, then $\mathcal{I}_G$ is single-valued and everywhere defined.* $\diamondsuit$

Lemma 2.4 *Let $n, m \in \mathbb{Z}_+$, and let S, G be such that (2) holds. Assume that S is compact and convex. Then:*

1. *If $\mathrm{Gr}(G)$ is compact, then $\mathrm{Gr}(\mathcal{I}_G)$ and $\mathrm{Gr}(\Phi_G)$ are compact.*
2. *If G is single-valued, everywhere defined, and continuous, then $\mathcal{I}_G$ is single-valued, everywhere defined, and continuous.*
3. *If G is regular, then $\mathcal{I}_G$ and Φ_G are regular.*

Proof. To prove the first statement, assume that G has a compact graph. We want to show that $\mathrm{Gr}(\mathcal{I}_G)$ and $\mathrm{Gr}(\Phi_G)$ are compact.

Let $\{(\alpha_j, \beta_j)\}_{j=1}^\infty$ be a sequence in $\mathrm{Gr}(\mathcal{I}_G)$. We want to extract a subsequence that converges to a limit $(\alpha_\infty, \beta_\infty) \in \mathrm{Gr}(\mathcal{I}_G)$. Since $\beta_j \in \mathcal{I}_G(\alpha_j)$, there exist $(h_j, v_j) \in G(\alpha_j)$ such that $\beta_j = h_j * \alpha_j + \xi_{v_j}$ for $j \in \mathbb{N}$. Then $(\alpha_j, h_j, v_j) \in \mathrm{Gr}(G)$. Since $\mathrm{Gr}(G)$ is compact, we may assume, after passing to a subsequence, that (i) the sequences $\{\alpha_j\}_{j=1}^\infty$, $\{h_j\}_{j=1}^\infty$, converge uniformly to limits α_∞, h_∞, (ii) $\{v_j\}_{j=1}^\infty$ converges in $\mathbb{R}^m$ to a limit v_∞, (iii) $\alpha_\infty \in \mathcal{A}(S)$, and (iv) $(h_\infty, v_\infty) \in G(\alpha_\infty)$.

Let $\beta_\infty = h_\infty * \alpha_\infty + \xi_{v_\infty}$. Then $\beta_\infty \in \mathcal{I}_G(\alpha_\infty)$. Lemma 2.4 implies that $\beta_j \to \beta_\infty$ uniformly as $j \to \infty$. So $\mathrm{Gr}(\mathcal{I}_G)$ is compact.

Now $(x,y) \in \mathrm{Gr}(\Phi_G)$ if and only if there exists a pair $(\alpha,\beta) \in \mathrm{Gr}(\mathcal{I}_G)$ such that $\alpha(1) = x$ and $\beta(1) = y$. So $\mathrm{Gr}(\Phi_G)$ is the image of $\mathrm{Gr}(\mathcal{I}_G)$ under the projection

$$C^0(\,[0,1]\,;\mathbb{R}^n\,) \times C^0(\,[0,1]\,;\mathbb{R}^m\,) \ni \\ (\alpha,\beta) \to (\alpha(1),\beta(1)) \in \mathbb{R}^n \times \mathbb{R}^m\,.$$

Since this projection is continuous, $\mathrm{Gr}(\Phi_G)$ is compact, and the proof of the first statement is complete.

If G is single-valued, everywhere defined, and continuous, then $\mathcal{I}_G$ is single-valued and everywhere defined and, moreover, the graph $\mathrm{Gr}(\mathcal{I}_G)$ is compact, because $\mathrm{Gr}(G)$ is compact. This implies that $\mathcal{I}_G$ is continuous, and the second statement is proved.

Finally, let us prove the third statement. Assume that G is regular. We want to show that $\mathcal{I}_G$ and Φ_G are regular. This requires that we prove that

(a) the graphs $\mathrm{Gr}(\mathcal{I}_G)$ and $\mathrm{Gr}(\Phi_G)$ are compact,

(b) $\mathcal{I}_G$ and Φ_G can be approximated in the sense of inward graph convergence by sequences of single-valued continuous maps.

Part (a) follows from the fact that $\mathrm{Gr}(G)$ is compact. We now prove part (b). Using the regularity of G, let $\{G_j\}_{j=1}^\infty$ be a sequence of single-valued, everywhere defined continuous maps from $\mathcal{A}(S)$ to $C^0(\,[0,1]\,;\mathbb{R}^{m\times n}\,) \times \mathbb{R}^m$ such that $G_j \xrightarrow{\mathrm{igr}} G$ as $j \to \infty$. Then the $\mathcal{I}_{G_j}$ are single-valued, everywhere defined, and continuous.

We show that $\mathcal{I}_{G_j} \xrightarrow{\mathrm{igr}} \mathcal{I}_G$ as $j \to \infty$. Let

$$\delta_j = \sup\Big\{\,\mathrm{dist}\Big((\alpha,\beta),\mathrm{Gr}(\mathcal{I}_G)\Big) : (\alpha,\beta) \in \mathrm{Gr}(\mathcal{I}_{G_j}\,)\,\Big\}\,.$$

We want to show that $\delta_j \to 0$ as $j \to \infty$. Assume this is not true. Then there exists an infinite subset J of $\mathbb{N}$ and a strictly positive number θ such that $\delta_j \geq 2\theta$ for all $j \in J$. We can therefore pick members (α_j,β_j) of $\mathrm{Gr}(\mathcal{I}_{G_j})$ for $j \in J$ such that

$$\mathrm{dist}\Big((\alpha_j,\beta_j),\mathrm{Gr}(\mathcal{I}_G)\Big) \geq \theta \ \text{ whenever } \ j \in J\,. \qquad (3)$$

If $j \in J$, then $\beta_j \in \mathcal{I}_{G_j}(\alpha_j)$, so we can pick pairs $(h_j,v_j) \in G_j(\alpha_j)$ such that $\beta_j = h_j * \alpha_j + \xi_{v_j}$. Since $G_j \xrightarrow{\mathrm{igr}} G$, we may assume, after making J smaller, if necessary, that the limit

$$(\alpha_\infty,h_\infty,v_\infty) = \lim_{j\to\infty,j\in J}(\alpha_j,h_j,v_j)$$

exists and belongs to $\mathrm{Gr}(G)$. Let $\beta_\infty = h_\infty * \alpha_\infty + \xi_{v_\infty}$. Then $\beta_\infty \in \mathcal{I}_G(\alpha_\infty)$. Lemma 2.4 implies that $\beta_j \to \beta_\infty$ uniformly as $j \to \infty$ via values in J. But then

$$(\alpha_\infty,\beta_\infty) = \lim_{j\to\infty,j\in J}(\alpha_j,\beta_j)\,.$$

Since $(\alpha_\infty,\beta_\infty) \in \mathrm{Gr}(\mathcal{I}_G)$, we have shown that

$$\lim_{j\to\infty,j\in J}\,\mathrm{dist}\Big((\alpha_j,\beta_j),\mathrm{Gr}(\mathcal{I}_G)\Big) = 0\,,$$

contradicting (3). Therefore $\delta_j \to 0$ as $j \to \infty$, and we have completed the proof that $\mathcal{I}_G$ is regular.

We must now show that Φ_G is regular. For each $x \in S$, let α_x be the curve given by

$$\alpha_x(t) = tx \quad \text{for} \quad t \in [0,1]\,.$$

Then $\alpha_x \in \mathcal{A}(S)$, and the map $S \ni x \to \alpha_x \in \mathcal{A}(S)$ is continuous. Define

$$\Phi^j(x) = \mathcal{I}_{G_j}(\alpha_x)(1) \quad \text{for} \quad x \in S\,.$$

Then Φ^j is a continuous map from S to $\mathbb{R}^m$. (Continuity follows because the map $\mathcal{I}_{G_j} : \mathcal{A}(S) \to C^0(\,[0,1]\,;\mathbb{R}^m\,)$ is continuous, and the maps $x \to \alpha_x$ and $\beta \to \beta(1)$ are continuous as well. The continuity of $\mathcal{I}_{G_j}$ follows from Lemma 2.1.)

We now show that $\Phi^j \xrightarrow{\mathrm{igr}} \Phi_G$. Let $(x_j,y_j) \in \mathrm{Gr}(\Phi^j)$. We want to extract a subsequence of $\{(x_j,y_j)\}_{j=1}^\infty$ that converge to a limit $(x,y) \in \mathrm{Gr}(\Phi_G)$. Pick $\beta_j \in \mathcal{I}_{G_j}(\alpha_{x_j})$. Then

$$\mathrm{dist}\Big(\,(\alpha_{x_j},\beta_j),\mathrm{Gr}(\mathcal{I}_G)\,\Big) \to 0 \quad \text{as} \quad j \to \infty\,,$$

because $\mathcal{I}_{G_j} \xrightarrow{\mathrm{igr}} \mathcal{I}_G$. Since $\mathrm{Gr}(\mathcal{I}_G)$ is compact we may assume, after passing to a susequence, that there exists a pair $(\alpha,\beta) \in \mathrm{Gr}(\mathcal{I}_G)$ such that $\alpha_{x_j} \to \alpha$ and $\beta_j \to \beta$. If we let $x = \alpha(1)$, then $x_j \to x$. Therefore $\alpha_{x_j} \to \alpha_x$, so $\alpha = \alpha_x$. Let $y = \beta(1)$. Then $y \in \Phi_G(x)$, and $(x_j,y_j) \to (x,y)$. So our proof is complete. $\diamondsuit$

3. The main definition

If $n \in \mathbb{Z}_+$, C is a cone in $\mathbb{R}^n$, and $r \in \,]\,0,\infty\,[$, we write $C(r)$ to denote the set $C \cap r\bar{\mathbb{B}}^n$, that is

$$C(r) \stackrel{\mathrm{def}}{=} \{\, x \in C : \|x\| \leq r\,\}\,.$$

Then $C(r)$ is compact convex if C is a closed convex cone.

Definition 3.1 Let n,m be nonnegative integers, let F be a set-valued map from $\mathbb{R}^n$ to $\mathbb{R}^m$, and let C be a closed convex cone in $\mathbb{R}^n$. We say that Λ is a *path-integral generalized differential of* F *at* $(0,0)$ *in the direction of* C, and write $\Lambda \in PIGD(F,C)$, if (1) Λ is a nonempty compact subset of $\mathbb{R}^{m\times n}$, and (2) for every positive real number δ there exists a number $R \in \,]\,0,\infty\,[$ with the property that for every $r \in \,]\,0,R]$ there exists a regular set-valued map $G : \mathcal{A}(C(r)) \to\!\!\!\to C^0(\,[0,1]\,;\mathbb{R}^{m\times n}\,) \times \mathbb{R}^m$ such that

(2a) $h(t) \in \Lambda^\delta$ and $\|v\| \leq \delta r$ whenever $\alpha \in \mathcal{A}(C(r))$, $(h,v) \in G(\alpha)$, $t \in [0,1]$,

(2b) $\mathrm{Gr}(\Phi_G) \subseteq \mathrm{Gr}(F)$. $\diamondsuit$

4. The chain rule

Theorem 4.1 *Let n_1, n_2, n_3 be nonnegative integers, and let F_i be, for $i = 1,2$, set-valued maps from $\mathbb{R}^{n_i}$ to $\mathbb{R}^{n_{i+1}}$. Assume that*

1. *C_i is a closed convex cone in $\mathbb{R}^{n_i}$ for $i = 1,2$,*
2. *C_2 is polyhedral,*
3. *$\Lambda_i \in PIGD(F_i,C_i)$ for $i = 1,2$,*
4. *$F_1(C_1) \subseteq C_2$,*
5. *$\Lambda_1 \cdot C_1 \subseteq C_2$ (that is, $L \cdot C_1 \subseteq C_2$ for every $L \in \Lambda_1$).*

Then

$$\Lambda_2 \circ \Lambda_1 \in PIGD(F_2 \circ F_1, C_1)\,.$$

Outline of the proof. The crucial point is that, since the cone C_2 is polyhedral, it is possible to pick a vector $\bar{w} \in \mathrm{Int}_S(C_2)$ and a positive constant $\bar{k}$ such that the following "error correction property" holds

(ECP) *If* $m \in \mathbb{N}$, $u_1, \ldots, u_m \in C_2$, $w \in C_2$, *and* $u_1 + \ldots + u_m = w + \bar{w}$, *then there exist vectors* $c_1, \ldots, c_m$ *such that* $\|c_1\| + \ldots + \|c_m\| \le \bar{k}$, $c_1 + \ldots + c_m = \bar{w}$, *and the conditions* $\|c_i\| \le \bar{k}\|u_i\|$ *and* $u_i - c_i \in C_2$ *hold for* $i = 1, \ldots, m$.

(This is not easy to prove, and we will omit the proof for lack of space.)

We then may—and will—assume, without loss of generality, that $\|\bar{w}\| \le 1$. We then fix a number $\rho \in]\,0, \infty\,[$ such that

$$\bar{w} + \rho \bar{\mathbb{B}}^{n_2} \subseteq C_2 \,. \tag{4}$$

Then the vector $\bar{w}$ and the number ρ satisfy

$$\Big(s > 0 \wedge u \in S \wedge \|u\| \le s\Big) \Rightarrow u + \rho^{-1} s \bar{w} \in C_2 \,. \tag{5}$$

Write $F = F_2 \circ F_1$, $\Lambda = \Lambda_2 \circ \Lambda_1$, $n = n_1$, $m = n_3$, $C = C_1$, and let

$$\begin{array}{lcl} S & = & \text{linear span of } C_2 \text{ in } \mathbb{R}^{n_2}\,, \\ \Pi & = & \text{the orthogonal projection from } \mathbb{R}^{n_2} \text{ to } S\,, \\ \kappa_i & = & \sup\{\, \|L\| : L \in \Lambda_i \,\} \quad \text{for} \quad i = 1, 2\,. \end{array}$$

Then $F : \mathbb{R}^n \rightarrow\!\!\!\!\rightarrow \mathbb{R}^m$, and Λ is a nonempty compact subset of $\mathbb{R}^{m \times n}$.

Fix a positive real number δ. We want to find a number $R \in]\,0, \infty\,[$ satisfying Property 2# of Definition 3.1.

Define a function $\Psi : [0, \infty\,[\times [0, \infty\,[\to [0, \infty\,[$ by

$$\begin{aligned} \Psi(\delta_1, \delta_2) \;&\stackrel{\mathrm{def}}{=}\; (1 + \bar{k})\Big(\kappa_1 + 2\delta_1(1 + \rho^{-1})\Big)\delta_2 \\ &+ (\kappa_2 + \delta_2)\delta_1 + 2\delta_1\Big(1 + \rho^{-1}(1 + \bar{k})\Big)(\kappa_2 + \delta_2)\,. \end{aligned} \tag{6}$$

Then

$$\lim_{(\delta_1, \delta_2) \to (0,0)} \Psi(\delta_1, \delta_2) = 0\,. \tag{7}$$

We let δ_1, δ_2, be positive numbers such that

$$\Psi(\delta_1, \delta_2) \le \frac{\delta}{2}\,. \tag{8}$$

For $i = 1, 2$, using the fact that $\Lambda_i \in PIGD(F_i, C_i)$, choose $R_i \in]\,0, \infty\,[$ with the property that for every $r_i \in]\,0, R_i]$ there exists G_i such that

(1) $G_i \in \mathrm{REG}(\mathcal{A}(C_i(r_i)); C^0(\,[0,1]\,; \mathbb{R}^{n_{i+1} \times n_i}\,) \times \mathbb{R}^{n_{i+1}})$,

(2) $\mathrm{Gr}(\Phi_{G_i}) \subseteq \mathrm{Gr}(F_i)$,

(3) $(h_i(t), v_i) \in \Lambda_i^{\delta_i} \times (\delta_i r_i \bar{\mathbb{B}}^{n_{i+1}})$ whenever $t \in [0,1]$, $\zeta \in \mathcal{A}(C_i(r_i))$, and $(h_i, v_i) \in G_i(\zeta)$.

Inequality (8) implies in particular that the inclusion

$$\Lambda_2^{\delta_2} \circ \Lambda_1^{\delta_1} \subseteq \Lambda^{\delta} \tag{9}$$

holds.

We let $\theta_0 = \kappa_1 + 2\delta_1\Big(1 + \frac{1}{\rho}\Big)$, $\theta = (1 + \bar{k})\theta_0$, and choose $R = \min\Big(R_1, \frac{R_2}{\theta}\Big)$. We then have to show that, with this choice of R, the property of Definition 3.1 is satisfied. For this purpose, we pick $r \in \mathbb{R}$ such that $0 < r \le R$, and prove the existence of a G satisfying the conditions of Definition 3.1.

Let $r_1 = r$, $r_2 = \theta r$, and observe that $0 < r_1 \le R_1$ and $0 < r_2 \le R_2$. Pick G_1, G_2, such that (1)-(2)-(3) hold. Let $\omega = 2\delta_1 r_1$, $\tilde{r} = \kappa_1 r$, $\hat{r} = \theta_0 r_1$, so that

$$r_2 = (1 + \bar{k})\hat{r}\,. \tag{10}$$

Let K be the ω-neighborhood of $C_2(\tilde{r})$, that is,

$$K = \{\, x \in \mathbb{R}^{n_2} : \mathrm{dist}(x, C_2(\tilde{r})) \le \omega \,\}\,.$$

It is easy to see tat

$$\mathcal{I}_{G_1}\Big(\mathcal{A}(C_1(r_1))\Big) \subseteq \mathcal{A}(K)\,. \tag{11}$$

We are now ready to begin the long process of defining the set-valued map

$$G : \mathcal{A}(C(r)) \rightarrow\!\!\!\!\rightarrow C^0(\,[0,1]\,; \mathbb{R}^{m \times n}\,) \times \mathbb{R}^m\,.$$

The first step will be to assign, to each triple (α, h_1, v_1) such that $\alpha \in \mathcal{A}(C(r))$ and $(h_1, v_1) \in G_1(\alpha)$, a curve $\beta_{\alpha,h_1,v_1} : [0,1] \to \mathbb{R}^{n_2}$.

Pick $\alpha \in \mathcal{A}(C(r)) = \mathcal{A}(C_1(r_1))$, and $(h_1, v_1) \in G_1(\alpha)$. Then $\beta_{\alpha,h_1,v_1} \in \mathcal{I}_{G_1}(\alpha)$, so (11) implies that $\beta_{\alpha,h_1,v_1} \in \mathcal{A}(K)$. Moreover,

$$\beta_{\alpha,h_1,v_1}(1) \in C_2\,, \tag{12}$$

because $\beta_{\alpha,h_1,v_1}(1) \in \Phi_{G_1}(\alpha(1)) \subseteq F_1(\alpha(1)) \subseteq C_2$.

The second step is to correct the error arising from the fact that $\beta \in \mathcal{A}(K)$ rather than in $\mathcal{A}(C_2(\tilde{r}))$. For this purpose we define, whenever α belongs to $\mathcal{A}(C(r))$ and $(h_1, v_1) \in G_1(\alpha)$, a new curve $\gamma_{\alpha,h_1,v_1} \in C^0(\,[0,1]\,; \mathbb{R}^{n_2}\,)$ by letting $\gamma_{\alpha,h_1,v_1}(t) = \Pi(\beta_{\alpha,h_1,v_1}(t)) + \frac{\omega t}{\rho}\bar{w}$ for $t \in [0,1]$.

We claim that

$$\gamma_{\alpha,h_1,v_1} \in \mathcal{A}(C_2(\hat{r}))\,. \tag{13}$$

To see this, write $\beta = \beta_{\alpha,h_1,v_1}$, $\gamma = \gamma_{\alpha,h_1,v_1}$, and observe that γ is an absolutely continuous curve, and $\gamma(0) = 0$. In addition, for almost all $t \in [0,1]$, $\dot{\gamma}(t)$ exists and is equal to $\Pi(\dot{\beta}(t)) + \frac{\omega}{\rho}\bar{w}$, and $\dot{\beta}(t) \in K$. Let E be the set of all $t \in [0,1]$ for which this is true. Then $\mathrm{meas}(E) = 1$. Moreover,

$$\dot{\gamma}(t) \in C_2(\hat{r}) \quad \text{whenever} \quad t \in E\,. \tag{14}$$

(*Proof.* Fix $t \in E$. Since $\dot{\beta}(t) \in K$, we can write $\dot{\beta}(t) = b_1 + b_2$, with $b_1 \in C_2(\tilde{r})$ and $\|b_2\| \le \omega$. Then $\Pi(\dot{\beta}(t)) = \Pi(b_1) + \Pi(b_2) = b_1 + \Pi(b_2)$, since $b_1 \in S$. Moreover, $\|\Pi(b_2)\| \le \omega$, because $\|b_2\| \le \omega$ and Π is an orthogonal projection. Then (5) implies that $\Pi(b_2) + \rho^{-1}\omega\bar{w} \in C_2$. Since $b_1 \in C_2$, and $\dot{\gamma}(t) = b_1 + \Pi(b_2) + \rho^{-1}\omega\bar{w}$, we conclude that $\dot{\gamma}(t) \in C_2$. Furthermore,

$$\begin{array}{lcl} \|\dot{\gamma}(t)\| & \le & \|\Pi(\dot{\beta}(t))\| + \rho^{-1}\omega\|\bar{w}\| \le \|\dot{\beta}(t)\| + \rho^{-1}\omega \\ & \le & \tilde{r} + \omega + \rho^{-1}\omega = (\kappa_1 + 2\delta_1 + 2\rho^{-1}\delta_1)r_1 \\ & = & \theta_0 r_1 = \hat{r}\,. \end{array}$$

So $\dot{\gamma}(t) \in C_2(\hat{r})$, and the proof of (14) is complete.) Therefore (13) holds.

The third step is to make a piecewise linear approximation of the curves γ_{α,h_1,v_1}, by first choosing a large positive integer N as follows. The fact that G_2 is regular implies that the set $G_2(\mathcal{A}(C_2(r_2)))$ is compact in $C^0(\,[0,1]\,; \mathbb{R}^{n_3 \times n_2}\,) \times \mathbb{R}^{n_3}$. Let H be the set of those $h_2 \in C^0(\,[0,1]\,; \mathbb{R}^{n_3 \times n_2}\,)$ such that $(h_2, v_2) \in G_2\Big(\mathcal{A}(C_2(r_2))\Big)$ for some $v_2 \in \mathbb{R}^{n_3}$. Then H is compact as well. Hence H is uniformly equicontinuous, so we can choose $N \in \mathbb{N}$ such that $\|h_2(t) - h_2(s)\| \le \frac{\delta}{4\theta_0}$

whenever $h_2 \in H$, $t,s \in [0,1]$, and $|t-s| \leq \frac{1}{N}$. With this choice of N, we define

$$\begin{aligned}\eta_{\alpha,h_1,v_1}(t) &= (j-Nt)\gamma_{\alpha,h_1,v_1}(N^{-1}(j-1)) \\ &+(Nt+1-j)\gamma_{\alpha,h_1,v_1}(N^{-1}j) \qquad (15)\end{aligned}$$

whenever $\alpha \in \mathcal{A}(C(r))$, $(h_1,v_1) \in G_1(\alpha)$, $\frac{j-1}{N} \leq t \leq \frac{j}{N}$, $j \in \mathbb{N}$, $1 \leq j \leq N$.

Then $\eta_{\alpha,h_1,v_1} \in C^0([0,1];\mathbb{R}^{n_2})$, and

$$\begin{aligned}\eta_{\alpha,h_1,v_1}(0) &= 0\,, & (16)\\ \eta_{\alpha,h_1,v_1}(1) &= \gamma_{\alpha,h_1,v_1}(1) \\ &= \beta_{\alpha,h_1,v_1}(1)+\rho^{-1}\omega\bar{w} \\ &\in C_2+\rho^{-1}\omega\,. & (17)\end{aligned}$$

Moreover, the map η_{α,h_1,v_1} is linear on each interval $I_j = [N^{-1}(j-1), N^{-1}j\,]$. The derivative $\dot{\eta}_{\alpha,h_1,v_1}(t)$ of η_{α,h_1,v_1} is equal, for $t \in I_j$, to $u^N_{\alpha,h_1,v_1,j}$, where

$$\begin{aligned}u^N_{\alpha,h_1,v_1,j} &= N\gamma_{\alpha,h_1,v_1}(N^{-1}j)-N\gamma_{\alpha,h_1,v_1}(N^{-1}(j-1)) \\ &= N\int_{\frac{j-1}{N}}^{\frac{j}{N}} \dot{\gamma}_{\alpha,h_1,v_1}(t)\,dt\,. \qquad (18)\end{aligned}$$

Since $\dot{\gamma}_{\alpha,h_1,v_1}(t) \in C_2(\hat{r})$ for almost all t, the vectors $u^N_{\alpha,h_1,v_1,j}$ belong to $C_2(\hat{r})$ as well. So $\dot{\eta}_{\alpha,h_1,v_1}(t) \in C_2(\hat{r})$ for almost all $t \in [0,1]$, and then (16) implies that

$$\eta_{\alpha,h_1,v_1} \in \mathcal{A}(C_2(\hat{r}))\,. \qquad (19)$$

The fourth step is to take care of the undesirable fact that η_{α,h_1,v_1} satisfies (17), and produce a curve whose terminal point is $\beta_{\alpha,h_1,v_1}(1)$ rather than $\beta_{\alpha,h_1,v_1}(1)+\rho^{-1}\omega\bar{w}$. For this purpose, we define $\tilde{u}^N_{\alpha,h_1,v_1,j} = \frac{\rho}{N\omega}u^N_{\alpha,h_1,v_1,j}$. Then

$$\sum_{j=1}^{N} \tilde{u}^N_{\alpha,h_1,v_1,j} = \frac{\rho}{\omega}\beta_{\alpha,h_1,v_1}(1)+\bar{w}\,.$$

It then follows from (ECP) that there exists an N-tuple $\tilde{\mathbf{c}} = (\tilde{c}_1,\ldots,\tilde{c}_N)$ of vectors that satisfies

$$\begin{aligned}\tilde{u}^N_{\alpha,h_1,v_1,j}-\tilde{c}_j &\in C_2 & (20)\\ \tilde{c}_1+\cdots+\tilde{c}_N &= \bar{w}\,, & (21)\\ \|\tilde{c}_1\|+\cdots+\|\tilde{c}_N\| &\leq \bar{k}\,, & (22)\\ \|\tilde{c}_j\| &\leq \bar{k}\|\tilde{u}^N_{\alpha,h_1,v_1,j}\| \quad , & (23)\end{aligned}$$

for $j = 1,\ldots,N$. Define $c_j = \rho^{-1}N\omega\tilde{c}_j$ and $\mathbf{c} = (c_1,\ldots,c_N)$. Then, for $j = 1,\ldots,N$,

$$\begin{aligned}u^N_{\alpha,h_1,v_1,j}-c_j &\in C_2 \quad , & (24)\\ c_1+\cdots+c_N &= \rho^{-1}N\omega\bar{w}\,, & (25)\\ \|c_1\|+\cdots+\|c_N\| &\leq \rho^{-1}N\omega\bar{k}\,, & (26)\\ \|c_j\| &\leq \bar{k}\|u^N_{\alpha,h_1,v_1,j}\| \quad . & (27)\end{aligned}$$

Let $\mu_{\mathbf{c}}$ be the function such that

$$\begin{aligned}\mu_{\mathbf{c}} &\in C^0([0,1];\mathbb{R}^{n_2})\,, & (28)\\ \mu_{\mathbf{c}}(0) &= 0\,, & (29)\\ \dot{\mu}_{\mathbf{c}} &\equiv c_j \quad \text{on} \quad I_j \quad \text{for} \quad j = 1,\ldots,N\,. & (30)\end{aligned}$$

Then

$$\int_0^1 \|\dot{\mu}_{\mathbf{c}}(t)\|\,dt = \frac{1}{N}(\,\|c_1\|+\cdots+\|c_N\|\,) \leq \frac{\omega\bar{k}}{\rho} \qquad (31)$$

and

$$\mu_{\mathbf{c}}(1) = \frac{1}{N}(\,c_1+\cdots+c_N\,) = \frac{\omega}{\rho}\bar{w}\,. \qquad (32)$$

Define a curve $\zeta_{\alpha,h_1,v_1,\mathbf{c}}$ by letting

$$\zeta_{\alpha,h_1,v_1,\mathbf{c}}(t) = \eta_{\alpha,h_1,v_1}(t)-\mu_{\mathbf{c}}(t) \quad \text{for} \quad t \in [0,1]\,. \qquad (33)$$

Then $\zeta_{\alpha,h_1,v_1,\mathbf{c}}$ satisfies

$$\begin{aligned}\zeta_{\alpha,h_1,v_1,\mathbf{c}} &\in C^0([0,1];\mathbb{R}^{n_2})\,, & (34)\\ \zeta_{\alpha,h_1,v_1,\mathbf{c}}(0) &= 0\,, & (35)\\ \zeta_{\alpha,h_1,v_1,\mathbf{c}}(1) &= \eta_{\alpha,h_1,v_1}(1)-\mu_{\mathbf{c}}(1) \\ &= \beta_{\alpha,h_1,v_1}(1)\,. & (36)\end{aligned}$$

Moreover, $\zeta_{\alpha,h_1,v_1,\mathbf{c}}$ is linear on each interval I_j, and the derivative $\dot{\zeta}_{\alpha,h_1,v_1,\mathbf{c}}(t)$ of $\zeta_{\alpha,h_1,v_1,\mathbf{c}}$ is equal, for $t \in I_j$, to $u^N_{\alpha,h_1,v_1,j}-c_j$. It follows from (24) that the vectors $v_j = u^N_{\alpha,h_1,v_1,j}-c_j$ belong to C_2. Moreover, the bound (27) implies that

$$\|v_j\| \leq (1+\bar{k})\|u^N_{\alpha,h_1,v_1,j}\| \leq (1+\bar{k})\hat{r} = r_2\,.$$

Therefore

$$\dot{\zeta}_{\alpha,h_1,v_1,\mathbf{c}} \in C_2(r_2) \quad \text{for a.e. } t \in [0,1]\,, \qquad (37)$$

and then (35) implies that

$$\zeta_{\alpha,h_1,v_1,\mathbf{c}} \in \mathcal{A}(C_2(r_2))\,. \qquad (38)$$

We have now finally succeeded in producing, for each curve $\alpha \in \mathcal{A}(C(r))$ and each pair $(h_1,v_1) \in G_1(\alpha)$, a curve $\zeta_{\alpha,h_1,v_1,\mathbf{c}} \in \mathcal{A}(C_2(r_2))$ whose terminal point is exactly $\beta_{\alpha,h_1,v_1}(1)$. Moreover, this curve is "close" to β_{α,h_1,v_1}, in the sense that it is close to η_{α,h_1,v_1}, which is close to γ_{α,h_1,v_1}, which is close to β_{α,h_1,v_1}.

This curve need not be unique, because $\mathbf{c}$ may fail to be unique, so this nonuniqueness will have to be taken care of. As a first step in that direction, we introduce the notation $\mathbf{C}_{\alpha,h_1,v_1}$ to refer to the set of all N-tuples $\mathbf{c} = (c_1,\ldots,c_N)$ that belong to $(\mathbb{R}^{n_2})^N$ and satisfy (24), (25), (26), and (27).

Given a curve $\alpha \in \mathcal{A}(C(r))$, a pair $(h_1,v_1) \in G_1(\alpha)$, and a $\mathbf{c} \in \mathbf{C}_{\alpha,h_1,v_1}$, pick $(h_2,v_2) \in G_2(\zeta_{\alpha,h_1,v_1,\mathbf{c}})$. Define

$$\begin{aligned}h_{\alpha,h_1,v_1,\mathbf{c},h_2,v_2}(t) &= h_2(t)\cdot h_1(t) \text{ for } t \in [0,1]\,, \\ v_{\alpha,h_1,v_1,\mathbf{c},h_2,v_2} &= v_2+\Big(\int_0^1 h_2(t)\,dt\Big)\cdot v_1 \\ &\quad -\int_0^1 h_2(t)\cdot(\dot{\beta}(t)-\dot{\zeta}(t))\,dt\,.\end{aligned}$$

We let $G(\alpha)$ be the set of all pairs $(h_{\alpha,h_1,v_1,\mathbf{c},h_2,v_2}, v_{\alpha,h_1,v_1,\mathbf{c},h_2,v_2})$, as α varies over $\mathcal{A}(C(r))$, (h_1,v_1) varies over all members of $G_1(\alpha)$, $\mathbf{c}$ varies over all members of $\mathbf{C}_{\alpha,h_1,v_1}$, and the pair (h_2,v_2) varies over all members of $G_2(\zeta_{\alpha,h_1,v_1,\mathbf{c}})$.

With this definition, it is clear that G is a set-valued map from $\mathcal{A}(C(r))$ to $C^0([0,1];\mathbb{R}^{m\times n})\times\mathbb{R}^m$. Moreover, if $(h,v) \in G(\alpha)$ for an $\alpha \in \mathcal{A}(C(r))$, then the following can be verified

(F1) *The matrix-valued function h takes values in Λ^δ.*

(F2) *If $x = \alpha(1)$, $\sigma = h * \alpha + \xi_v$, and $z = \sigma(1)$, then $z \in F(x)$.*

(F3) $\|v\| \leq \delta r$.

In view of (F1), (F2) and (F3), our conclusion will follow if we prove that G is regular. To prove the regularity of G, we express G as a composite of regular maps.

We define

$$\begin{aligned}
\mathcal{U} &= C^0([0,1];\mathbb{R}^{n_2\times n}),\\
\hat{\mathcal{U}} &= \Big\{ h_1\in\mathcal{U} : \|h_1(t)\|\le\kappa_1+\delta_1 \text{ for all } t\in[0,1]\Big\},\\
\mathcal{Y} &= C^0([0,1];\mathbb{R}^{n_2}),\\
\mathcal{B} &= \Big\{ \beta\in\mathcal{Y} : (\kappa_1+2\delta_1)\|\beta(t)-\beta(s)\|\le|t-s| \\
&\qquad \text{whenever } t,s\in[0,1]\Big\},\\
\tilde{\mathcal{Y}} &= (\mathbb{R}^{n_2})^N,\\
\mathcal{Z} &= \Big\{ \eta\in\mathcal{Y} : \eta \text{ is linear on } I_j \text{ for } j=1,\dots,N\Big\},\\
\tilde{\mathcal{Z}} &= \mathcal{Z}\cap\mathcal{A}(C_2(r_2)),\\
\mathcal{W} &= C^0([0,1];\mathbb{R}^{m\times n_2}),\\
\hat{\mathcal{W}} &= \Big\{ h_2\in\mathcal{W} : \|h_2(t)\|\le\kappa_2+\delta_2 \text{ for all } t\in[0,1]\Big\},\\
\mathcal{Q} &= C^0([0,1];\mathbb{R}^{m\times n}),\\
\mathcal{X}_1 &= \mathcal{A}(C(r))\times\hat{\mathcal{U}}\times\mathbb{R}^{n_2},\\
\mathcal{X}_2 &= \mathcal{A}(C(r))\times\hat{\mathcal{U}}\times\mathbb{R}^{n_2}\times\mathcal{B}\times\mathcal{Y}\times\mathcal{Z},\\
\mathcal{X}_3 &= \mathcal{A}(C(r))\times\hat{\mathcal{U}}\times\mathbb{R}^{n_2}\times\mathcal{B}\times\mathcal{Y}\times\mathcal{Z}\times\tilde{\mathcal{Y}},
\end{aligned}$$

We then let $\Gamma_1:\mathcal{A}(C(r))\to\!\!\!\to\mathcal{X}_1$ be the set-valued map that sends $\alpha\in\mathcal{A}(C(r))$ to the set $\{\alpha\}\times G_1(\alpha)$, so that

$$\Gamma_1(\alpha)=\Big\{(\alpha,h_1,v_1):(h_1,v_1)\in G_1(\alpha)\Big\} \text{ if } \alpha\in\mathcal{A}(C(r)).$$

Then

$$\Gamma_1\in\mathrm{REG}(\mathcal{A}(C(r));\mathcal{X}_1), \tag{39}$$

because of the identity $\Gamma_1=(\mathbb{1}_{\mathcal{A}(C(r))}\times G_1)\circ\Delta_1$, where $\Delta_1:\mathcal{A}(C(r))\to\mathcal{A}(C(r))\times\mathcal{A}(C(r))$ is the diagonal map (i.e., the map that sends $\alpha\in\mathcal{A}(C(r))$ to the pair (α,α)).

We then let $\Gamma_2:\mathcal{X}_1\to\mathcal{X}_2$ be the ordinary map that sends each triple $(\alpha,h_1,v_1)\in\mathcal{X}_1$ to the 6-tuple $(\alpha,h_1,v_1,\beta,\gamma,\eta)\in\mathcal{X}_2$, where $\beta=h_1*\alpha+\xi_{v_1}$, $\gamma(t)=\Pi(\beta(t))+\frac{\omega t}{\rho}\bar{w}$ for $t\in[0,1]$, and

$$\eta(t)=(j-Nt)\gamma\Big(\frac{j-1}{N}\Big)+(Nt+1-j)\gamma\Big(\frac{j}{N}\Big) \tag{40}$$

whenever $\frac{j-1}{N}\le t\le\frac{j}{N}$, $j\in\mathbb{N}$, and $1\le j\le N$.

We then let $\Gamma_3^0:\mathcal{Z}\to\tilde{\mathcal{Y}}$ be the map that sends each $\eta\in\mathcal{Z}$ to the N-tuple

$$\Gamma_3^0(\eta)=(u_1,\dots,u_N)\in\tilde{\mathcal{Y}}, \tag{41}$$

where

$$u_j=N\left(\eta\Big(\frac{j}{N}\Big)-\eta\Big(\frac{j-1}{N}\Big)\right) \text{ for } j=1,\dots,N. \tag{42}$$

Next, we let $\Gamma_3:\mathcal{X}_2\to\mathcal{X}_3$ be the map that sends each 6-tuple $(\alpha,h_1,v_1,\beta,\gamma,\eta)$ to the 7-tuple

$$\Gamma_3(\alpha,h_1,v_1,\beta,\gamma,\eta)=(\alpha,h_1,v_1,\beta,\gamma,\eta,\Gamma_3^0(\eta)). \tag{43}$$

Next, we define $\tilde{\mathcal{K}}_0=C_2(\hat{r})^N$, and let $\tilde{\mathcal{K}}$ be the set of all $(u_1,\dots,u_N)\in\tilde{\mathcal{K}}_0$ such that $N(u_1+\ldots+u_N)=w_0+\frac{\omega}{\rho}\bar{w}$ for some $w_0\in C_2$. Then $\tilde{\mathcal{K}}_0$ and $\tilde{\mathcal{K}}$ are compact convex subsets of $\tilde{\mathcal{Y}}$.

We let Γ_4^0 be a continuous retraction from $\tilde{\mathcal{Y}}$ onto $\tilde{\mathcal{K}}$, and define $\mathcal{X}_4=\mathcal{A}(C(r))\times\hat{\mathcal{U}}\times\mathbb{R}^{n_2}\times\mathcal{B}\times\mathcal{Y}\times\mathcal{Z}\times\tilde{\mathcal{K}}$. We then let $\Gamma_4:\mathcal{X}_3\to\mathcal{X}_4$ be the map that sends each 7-tuple $(\alpha,h_1,v_1,\beta,\gamma,\eta,\mathbf{u})$ to the 7-tuple

$$\Gamma_4(\alpha,h_1,v_1,\beta,\gamma,\eta,\mathbf{u})=(\alpha,h_1,v_1,\beta,\gamma,\eta,\Gamma_4^0(\mathbf{u})). \tag{44}$$

We now define a set-valued map $\Gamma_5^0:\tilde{\mathcal{K}}\to\!\!\!\to\tilde{\mathcal{Y}}$, by letting $\Gamma_5^0(u_1,\dots,u_N)$ be, if $(u_1,\dots,u_N)\in\tilde{\mathcal{K}}$, he set of all N-tuples $(c_1,\dots,c_N)$ that satisfy, for $j=1,\dots,N$, the conditions

$$\begin{aligned}
u_j-c_j &\in C_2\,, && (45)\\
c_1+\cdots+c_N &= \frac{N\omega}{\rho}\bar{w}, && (46)\\
\|c_1\|+\cdots+\|c_N\| &\le \frac{N\omega\bar{k}}{\rho}, && (47)\\
\|c_j\| &\le \bar{k}\|u_j\|. && (48)
\end{aligned}$$

Then Γ_5^0 has convex values and a compact graph. Moreover, (ECP) implies that the values of Γ_5^0 are nonempty. It then follows from Theorem 5.2 of [1] that

$$\Gamma_5^0\in\mathrm{REG}(\tilde{\mathcal{K}};\tilde{\mathcal{Y}}). \tag{49}$$

We then define

$$\mathcal{X}_5=\mathcal{A}(C(r))\times\hat{\mathcal{U}}\times\mathbb{R}^{n_2}\times\mathcal{B}\times\mathcal{Y}\times\mathcal{Z}\times\tilde{\mathcal{K}}\times\tilde{\mathcal{Y}},$$

and let $\Gamma_5:\mathcal{X}_4\to\!\!\!\to\mathcal{X}_5$ be the set-valued map that sends each 7-tuple $(\alpha,h_1,v_1,\beta,\gamma,\eta,\mathbf{u})\in\mathcal{X}_4$ to the set

$$\begin{aligned}&\Gamma_5(\alpha,h_1,v_1,\beta,\gamma,\eta,\mathbf{u})\\ &=\Big\{(\alpha,h_1,v_1,\beta,\gamma,\eta,\mathbf{u},\mathbf{c}):\mathbf{c}\in\Gamma_5^0(\mathbf{u})\Big\}\subseteq\mathcal{X}_5.\end{aligned} \tag{50}$$

Then

$$\Gamma_5=\Big(\mathbb{1}_{\mathcal{X}_4}\times\Gamma_5\Big)\circ\Delta_2, \tag{51}$$

where $\Delta_2:\mathcal{X}_4\to\mathcal{X}_4\times\tilde{\mathcal{K}}$ is the map that sends a 7-tuple $(\alpha,h_1,v_1,\beta,\gamma,\eta,\mathbf{u})$ to the 8-tuple $(\alpha,h_1,v_1,\beta,\gamma,\eta,\mathbf{u},\mathbf{u})$. It follows from (49) and (51) that $\Gamma_5\in\mathrm{REG}(\mathcal{X}_4;\mathcal{X}_5)$.

Next, define

$$\mathcal{X}_6=\mathcal{A}(C(r))\times\hat{\mathcal{U}}\times\mathbb{R}^{n_2}\times\mathcal{B}\times\mathcal{Y}\times\mathcal{Z}\times\tilde{\mathcal{K}}\times\tilde{\mathcal{Y}}\times\mathcal{Z}\times\mathcal{Z},$$

and let $\Gamma_6:\mathcal{X}_5\to\mathcal{X}_6$ be the ordinary map that sends each 8-tuple $(\alpha,h_1,v_1,\beta,\gamma,\eta,\mathbf{u},\mathbf{c})\in\mathcal{X}_5$ to the 10-tuple

$$\begin{aligned}&\Gamma_6(\alpha,h_1,v_1,\beta,\gamma,\eta,\mathbf{u},\mathbf{c})\\ &\qquad=(\alpha,h_1,v_1,\beta,\gamma,\eta,\mathbf{u},\mathbf{c},\mu_{\mathbf{c}},\eta-\mu_{\mathbf{c}})\in\mathcal{X}_6.\end{aligned}$$

We now observe that the real linear space $\mathcal{Z}$ is finite-dimensional, and $\tilde{\mathcal{Z}}$ is a nonempty compact convex subset of $\mathcal{Z}$. Let Γ_7^0 be a continuous retraction from $\mathcal{Z}$ onto $\tilde{\mathcal{Z}}$. Define

$$\mathcal{X}_7=\mathcal{A}(C(r))\times\hat{\mathcal{U}}\times\mathbb{R}^{n_2}\times\mathcal{B}\times\mathcal{Y}\times\mathcal{Z}\times\tilde{\mathcal{K}}\times\tilde{\mathcal{Y}}\times\mathcal{Z}\times\tilde{\mathcal{Z}},$$

and let $\Gamma_7:\mathcal{X}_6\to\mathcal{X}_7$ be the map that sends each 10-tuple $(\alpha,h_1,v_1,\beta,\gamma,\eta,\mathbf{u},\mathbf{c},\mu,\zeta)\in\mathcal{X}_6$ to the 10-tuple

$$\begin{aligned}&\Gamma_7(\alpha,h,v,\beta,\gamma,\eta,\mathbf{u},\mathbf{c},\mu,\zeta)\\ &\qquad=(\alpha,h_1,v_1,\beta,\gamma,\eta,\mathbf{u},\mathbf{c},\mu,\Gamma_7^0(\zeta))\in\mathcal{X}_7.\end{aligned} \tag{52}$$

Next, we let Γ_8^0 be the set-valued map $\tilde{\mathcal{Z}}\to\!\!\!\to\hat{\mathcal{W}}\times\mathbb{R}^m$ that sends $\zeta\in\tilde{\mathcal{Z}}$ to the set $G_2(\zeta)\subseteq\hat{\mathcal{W}}\times\mathbb{R}^m$. Then $\Gamma_8^0=G_2\circ\iota_{\mathcal{A}(C_2(r_2)),\tilde{\mathcal{Z}}}$, where $\iota_{\mathcal{A}(C_2(r_2)),\tilde{\mathcal{Z}}}$ is the inclusion map from $\tilde{\mathcal{Z}}$ to $\mathcal{A}(C_2(r_2))$. So $\Gamma_8^0\in\mathrm{REG}(\tilde{\mathcal{Z}};\hat{\mathcal{W}}\times\mathbb{R}^m)$.

Define $\mathcal{X}_8$ to be the product $\mathcal{A}(C(r))\times\hat{\mathcal{U}}\times\mathbb{R}^{n_2}\times\mathcal{B}\times\mathcal{Y}\times\mathcal{Z}\times\tilde{\mathcal{K}}\times\tilde{\mathcal{Y}}\times\mathcal{Z}\times\tilde{\mathcal{Z}}\times\hat{\mathcal{W}}\times\mathbb{R}^m$, and let $\Gamma_8:\mathcal{X}_7\to\!\!\!\to\mathcal{X}_8$ be the set-valued map that sends each 10-tuple $(\alpha,h_1,v_1,\beta,\gamma,\eta,\mathbf{u},\mathbf{c},\mu,\zeta)\in\mathcal{X}_7$ to the set $\Gamma_8(\alpha,h_1,v_1,\beta,\gamma,\eta,\mathbf{u},\mathbf{c},\mu,\zeta)$

$$=\Big\{(\alpha,h_1,v_1,\beta,\gamma,\eta,\mathbf{u},\mathbf{c},\mu,\zeta)\Big\}\times G_2(\zeta)\subseteq\mathcal{X}_8$$

$$=\Big\{(\alpha,h_1,v_1,\beta,\gamma,\eta,\mathbf{u},\mathbf{c},\mu,\zeta,h_2,v_2):(h_2,v_2)\in G_2(\zeta)\Big\}.$$

It is then clear that $\Gamma_8=\Big(\mathbb{1}_{\mathcal{X}_7}\times G_2\Big)\circ\Delta_3$, where $\Delta_3:\mathcal{X}_7\to\mathcal{X}_7\times\tilde{\mathcal{Z}}$ is the map that sends a 10-tuple $(\alpha,h_1,v_1,\beta,\gamma,\eta,\mathbf{u},\mathbf{c},\mu,\zeta)\in\mathcal{X}_7$ to the 11-tuple

$$(\alpha,h_1,v_1,\beta,\gamma,\eta,\mathbf{u},\mathbf{c},\mu,\zeta,\zeta)\in\mathcal{X}_7\times\tilde{\mathcal{Z}}.$$

Therefore $\Gamma_8\in\mathrm{REG}(\mathcal{X}_7;\mathcal{X}_8)$.

Finally, we define an ordinary map $\Gamma_9:\mathcal{X}_8\to\mathcal{Q}\times\mathbb{R}^m$ by letting

$$\Gamma_9(\alpha,h_1,v_1,\beta,\gamma,\eta,\mathbf{u},\mathbf{c},\mu,\zeta,h_2,v_2)$$
$$=v_2+\Big(\int_0^1 h_2(t)\,dt\Big)\cdot v_1-\Big(h_2*(\beta-\zeta)\Big)(1).$$

We have thus defined nine set-valued maps $\Gamma_1\in\mathrm{REG}(\mathcal{A}(C(r));\mathcal{X}_1)$, $\Gamma_2\in C^0(\mathcal{X}_1;\mathcal{X}_2)$, $\Gamma_3\in C^0(\mathcal{X}_2;\mathcal{X}_3)$, $\Gamma_4\in C^0(\mathcal{X}_3;\mathcal{X}_4)$, $\Gamma_5\in\mathrm{REG}(\mathcal{X}_4;\mathcal{X}_5)$, $\Gamma_6\in C^0(\mathcal{X}_5;\mathcal{X}_6)$, $\Gamma_7\in C^0(\mathcal{X}_6;\mathcal{X}_7)$, $\Gamma_8\in\mathrm{REG}(\mathcal{X}_7;\mathcal{X}_8)$, $\Gamma_9\in C^0(\mathcal{X}_8;\mathcal{Q}\times\mathbb{R}^m)$.

Since $G=\Gamma_9\circ\Gamma_8\circ\Gamma_7\circ\Gamma_6\circ\Gamma_5\circ\Gamma_4\circ\Gamma_3\circ\Gamma_2\circ\Gamma_1$, it follows that G is regular, and our proof is complete. $\diamondsuit$

5. The open mapping theorem

We now show that path-integral generalized differentials have the directional open mapping property.

Theorem 5.1 *Assume that $n,m\in\mathbb{Z}_+$, $F\in\mathcal{SVM}(\mathbb{R}^n,\mathbb{R}^m)$, $\bar{w}\in\mathbb{R}^m$, C is a polyhedral convex cone in $\mathbb{R}^n$, Λ belongs to $PIGD(F;0,0;C)$, and $\bar{w}\in\bigcap_{L\in\Lambda}\mathrm{Int}(LC)$. Then there exists a closed convex cone D in $\mathbb{R}^m$ such that $\bar{w}\in\mathrm{Int}(D)$, having the property that for every $\delta\in]0,\infty[$ there exists an $\varepsilon(\delta)\in]0,\infty[$ such that*

$$\boxed{D\cap\{y\in\mathbb{R}^m:\|y\|\le\varepsilon(\delta)\}\subseteq F(C\cap\{x\in\mathbb{R}^n:\|x\|\le\delta\})}.$$

Proof. Let us assume that $\bar{w}\neq 0$. Pick a closed convex cone $\hat{D}$ in $\mathbb{R}^n$ such that $\bar{w}\in\mathrm{Int}(\hat{D})$, a compact neighborhood Λ' of Λ such that $\hat{D}\subseteq LC$ for every $L\in\Lambda'$, and a continuous map $\Lambda'\times\hat{D}\ni(L,y)\mapsto\eta(L,y)\in C$ which is positively homogeneous of degree 1 with respect to y and such that $L\cdot\eta(L,y)=y$ for all $(L,y)\in\Lambda'\times\hat{D}$. Pick θ such that $0<\theta<\|\bar{w}\|$ and the ball $\{y\in\mathbb{R}^m:\|y-\bar{w}\|\le 2\theta\}$ is entirely contained in $\hat{D}$. Let D be the smallest closed convex cone that contains the ball $\{y\in\mathbb{R}^m:\|y-\bar{w}\|\le\theta\}$. Then D satisfies:

$$\Big(y\in D\wedge\|z\|\le\frac{\theta\|y\|}{\|\bar{w}\|+\theta}\Big)\Rightarrow y+z\in\hat{D}. \tag{53}$$

(Indeed, if $y=0$ then $z=0$, so $y+z=0\in\hat{D}$. Assume that $y\neq 0$. Then we can write $y=su$ with $s>0$ and $\|u-\bar{w}\|\le\theta$. Then $\|u\|\le\|\bar{w}\|+\|u-\bar{w}\|\le\|\bar{w}\|+\theta$. Therefore $\|y\|\le s(\|\bar{w}\|+\theta)$, so $s\le\frac{\|y\|}{\|\bar{w}\|+\theta}$. Furthermore, $y+z=s\tilde{u}$, where $\tilde{u}=u+\frac{z}{s}$. Then

$$\|\tilde{u}-\bar{w}\|=\|u-\bar{w}+\frac{z}{s}\|\le\|u-\bar{w}\|+\frac{\|z\|}{s}\le\theta+\frac{\|z\|}{\|y\|}(\|\bar{w}\|+\theta),$$

so $\|\tilde{u}-\bar{w}\|\le 2\theta$, and then $y+z\in\hat{D}$.)

Let $M=\sup\{\|\eta(L,y)\|:L\in\Lambda',y\in\hat{D},\|y\|\le 1\}$. Then $\|\eta(L,y)\|\le M\|y\|$ whenever $L\in\Lambda'$ and $y\in\hat{D}$.

Fix a $\delta\in]0,\infty[$. Let δ' be such that $0<\delta'\le\delta$, $\Lambda^{\delta'}\subseteq\Lambda'$, $2M\delta'\le\frac{\theta}{\|\bar{w}\|+\theta}$. Then choose $R\in\mathbb{R}$ such that $R>0$ and a family $\{G_r:0<r\le R\}$ of regular set-valued maps $G_r:\mathcal{A}(C(r))\to\!\!\!\to C^0([0,1];\mathbb{R}^{m\times n})\times\mathbb{R}^m$ such that

(a) $h(t)\in\Lambda^{\delta'}$ and $\|v\|\le\delta' r$ whenever $\alpha\in\mathcal{A}(C(r))$, $(h,v)\in G_r(\alpha)$, $t\in[0,1]$,

(b) $\mathrm{Gr}(\Phi_{G_r})\subseteq\mathrm{Gr}(F)$.

By making R smaller, if necessary, we may assume that $R\le\delta$.

We choose ε such that $2M\varepsilon<R$, and show that this choice of ε satisfies our requirements.

Fix y such that $0<\|y\|\le\varepsilon$. Let $\rho=\|y\|$, and choose $r=2M\rho$. Then $r<R$. We will show that there is an $x\in C$ such that $\|x\|\le\delta$ and $y\in F(x)$. For this purpose, it suffices to find a triple $(\alpha,h,v)\in\mathrm{Gr}(G_r)$ such that $v+\int_0^1 h(t)\cdot\dot{\alpha}(t)\,dt=y$ and $\|\alpha(1)\|\le\delta$. The first equality, in turn, will follow if α satisfies

$$h(t)\cdot\dot{\alpha}(t)=y-v\quad\text{for a.e.}\quad t, \tag{54}$$

as well as $\|\alpha(1)\|\le\delta$. If $y-v\in D$, then (54) will follow if $\dot{\alpha}(t)=\eta(h(t),y-v)$ for a.e. t, i.e., if

$$\alpha(t)=\int_0^t\eta(h(s),y-v)\,ds\quad\text{for all}\quad t. \tag{55}$$

Let Σ be the set-valued map on $\mathcal{A}(C(r))$ that asigns to each $\alpha\in\mathcal{A}(C(r))$ the set $\Sigma(\alpha)$ of all paths $\beta=\beta_{\alpha,h,v}$ for all $(h,v)\in G_r(\alpha)$, where $\beta_{\alpha,h,v}(t)=\int_0^t\eta(h(s),y-v)\,ds$. Then Σ is well defined and takes values in $\mathcal{A}(C(r))$. (*Proof.* Let $\alpha\in\mathcal{A}(C(r))$ and $(h,v)\in G_r(\alpha)$. Then $\|v\|\le\delta' r=2M\delta'\rho=2M\delta'\|y\|\le\frac{\theta\|y\|}{\|\bar{w}\|+\theta}$. Therefore (53) implies that $y-v\in\hat{D}$. Since $h(t)\in\Lambda^{\delta'}\subseteq\Lambda'$ for each t, $\eta(h(t),y-v)$ is defined for each t. Since the map $t\mapsto\eta(h(t),y-v)$ is continuous, $\beta_{\alpha,h,v}$ is well defined. Moreover, $\dot{\beta}_{\alpha,h,v}(t)=\eta(h(t),y-v)\in C$. On the other hand, $\|\dot{\beta}_{\alpha,h,v}(t)\|=\|\eta(h(t),y-v)\|\le M\|y-v\|$. But $\|y-v\|\le\|y\|+\|v\|\le\rho+\delta' r$, and this implies that $\|\dot{\beta}_{\alpha,h,v}(t)\|\le M\rho+M\delta' r$. But $M\rho=\frac{r}{2}$, and we know that $M\delta' r\le\frac{\theta r}{2(\|\bar{w}\|+\theta)}<\frac{r}{2}$. So $\|\dot{\beta}_{\alpha,h,v}(t)\|\le r$. Therefore $\dot{\beta}_{\alpha,h,v}(t)\in C(r)$. Hence $\beta_{\alpha,h,v}\in\mathcal{A}(C(r))$.)

It is easy to see that Σ is regular. Hence Σ is a regular map from the compact convex set $\mathcal{A}(C(r))$ to itself. By the obvious extension of Schauder's fixed point theorem to regular maps, Σ has a fixed point α_*. Then, for some $(h,v)\in G_r(\alpha_*)$, we have $h(t)\cdot\dot{\alpha}_*(t)=y-v$ for almost all t. Hence if we let $x=\alpha(1)$, we have $y=v+\int_0^1 h(t)\cdot\dot{\alpha}_*(t)\,dt$, so that $y\in\Phi_{G_r}(x)$, and then $y\in F(x)$. Finally the fact that $\alpha\in\mathcal{A}(C(r))$ implies that $\|x\|\le r=2M\rho\le 2M\varepsilon<R\le\delta$. This completes the proof. $\diamondsuit$

References

[1] Sussmann, H.J., "Warga derivate containers and other generalized differentials." Submitted for publication in *Proc. 41st IEEE Conf. Decision and Control, Las Vegas, December 2002.*