Path-integral generalized differentials *

Dedicated to Jack Warga on his 80th birthday

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1. Introduction

This is the second of a series of two papers on generalized differentiation theories (abbr. GDTs). The first paper discussed the definition of the GDT concept, presented several GDTs (the Warga derivate containers, weak multidifferentials, and generalized differential quotients) and compared them showing, in particular, that none of these theories contains all the others.

In this paper, we introduce another concept of generalized differential—the "path-integral generalized differential," abbreviated PIGD—that achieves the desired unification.

2. Preliminaries

If $n, m \in \mathbb{Z}_+$, $\alpha : [0,1] \to \mathbb{R}^n$ is a Lipschitz function, and $h : [0,1] \to \mathbb{R}^{m\times n}$ is integrable, we use $h * \alpha$ to denote the "chronological product" of h and α, that is, the absolutely continous function $\beta : [0,1] \to \mathbb{R}^m$ given by $\beta(t) = \int_0^t h(s) \cdot \dot\alpha(s)\, ds$.

The following lemma says that the chronological product operation $C^0([0,1];\mathbb{R}^n) \times L^1([0,1],\mathbb{R}^{m\times n}) \ni (\alpha, h) \mapsto h * \alpha \in C^0([0,1];\mathbb{R}^m)$ is jointly continuous, as long as the function α varies in a uniformly Lipschitz subset of $C^0([0,1];\mathbb{R}^n)$. The proof is very simple and will be omitted.

Lemma 2.1 *Let $n, m \in \mathbb{Z}_+$. Let $\{(\alpha_j, h_j)\}_{j=1}^\infty$ be a sequence of members of the product space $S = C^0([0,1];\mathbb{R}^n) \times L^1([0,1],\mathbb{R}^{m\times n})$ that converges in S to a limit $(\alpha_\infty, h_\infty)$. Assume that the sequence $\{\alpha_j\}$ is uniformly Lipschitz (that is, there exists a constant $r \in \mathbb{R}$ such that $\|\alpha_j(t) - \alpha_j(s)\| \le r|t-s|$ for all $j \in \mathbb{N}$ and all $t, s \in [0,1]$). Then*

$$h_j * \alpha_j \to h_\infty * \alpha_\infty \quad \text{in} \quad C^0([0,1];\mathbb{R}^m) \quad \text{as} \quad j \to \infty\,.\diamondsuit$$

Let $n \in \mathbb{Z}_+$, and let S be a subset of $\mathbb{R}^n$. We write $\mathcal{A}(S)$ to denote the subset of $C^0([0,1];\mathbb{R}^n)$ consisting of all absolutely continuous curves $\alpha : [0,1] \to \mathbb{R}^n$ such that $\alpha(0) = 0$ and $\dot\alpha(t) \in S$ for almost all $t \in [0,1]$.

If $\mathcal{S} \subseteq C^0([0,1];\mathbb{R}^n)$, we write $\tau(\mathcal{S})$ to denote the set $\tau(\mathcal{S}) \stackrel{\text{def}}{=} \{\alpha(1) : \alpha \in \mathcal{S}\}$, so $\tau(\mathcal{S})$ is the set of all terminal points of curves in $\mathcal{S}$.

The following is then an immediate consequence of our definitions.

Proposition 2.2 *If K is a compact convex subset of $\mathbb{R}^n$, then $\mathcal{A}(K)$ is a compact convex subset of $C^0([0,1];\mathbb{R}^n)$, and $\tau\Big(\mathcal{A}(K)\Big) = K$.* $\diamondsuit$

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If $m \in \mathbb{Z}_+$ and $v \in \mathbb{R}^m$, we use ξ_v to denote the curve

$$[0,1] \ni t \to tv \stackrel{\text{def}}{=} \xi_v(t) \in \mathbb{R}^m\,. \tag{1}$$

If

$$S \subseteq \mathbb{R}^n \text{ and } G : \mathcal{A}(S) \to\!\!\!\to C^0([0,1];\mathbb{R}^{m\times n}) \times \mathbb{R}^m\,, \tag{2}$$

then we can define set-valued maps

$$\mathcal{I}_G : \mathcal{A}(S) \to\!\!\!\to C^0([0,1];\mathbb{R}^m)\,, \quad \Phi_G : S \to\!\!\!\to \mathbb{R}^m\,,$$

by letting

$$\begin{aligned} \mathcal{I}_G(\alpha) &= \Big\{ h * \alpha + \xi_v : (h,v) \in G(\alpha) \Big\}, \\ \Phi_G(x) &= \Big\{ y \in \mathbb{R}^m : (\exists (\alpha,h,v) \in \mathrm{Gr}(G)) \\ &\qquad (\alpha(1) = x \vee (h*\alpha)(1) + v = y) \Big\}. \end{aligned}$$

The following fact is then trivial.

Proposition 2.3 *Let $n, m \in \mathbb{Z}_+$, and let S, G be such that (2) holds. Then:*

1. *$\mathrm{Do}(\mathcal{I}_G) = \mathrm{Do}(G)$, so in particular $\mathcal{I}_G$ is everywhere defined if and only if G is.*
2. *If G is everywhere defined then Φ_G is everywhere defined.*
3. *If G is single-valued at a particular $\alpha \in \mathcal{A}(S)$, then $\mathcal{I}_G$ is single-valued at α; in particular, if G is single-valued and everywhere defined, then $\mathcal{I}_G$ is single-valued and everywhere defined.* $\diamondsuit$

Lemma 2.4 *Let $n, m \in \mathbb{Z}_+$, and let S, G be such that (2) holds. Assume that S is compact and convex. Then:*

1. *If $\mathrm{Gr}(G)$ is compact, then $\mathrm{Gr}(\mathcal{I}_G)$ and $\mathrm{Gr}(\Phi_G)$ are compact.*
2. *If G is single-valued, everywhere defined, and continuous, then $\mathcal{I}_G$ is single-valued, everywhere defined, and continuous.*
3. *If G is regular, then $\mathcal{I}_G$ and Φ_G are regular.*

Proof. To prove the first statement, assume that G has a compact graph. We want to show that $\mathrm{Gr}(\mathcal{I}_G)$ and $\mathrm{Gr}(\Phi_G)$ are compact.

Let $\{(\alpha_j, \beta_j)\}_{j=1}^\infty$ be a sequence in $\mathrm{Gr}(\mathcal{I}_G)$. We want to extract a subsequence that converges to a limit $(\alpha_\infty, \beta_\infty) \in \mathrm{Gr}(\mathcal{I}_G)$. Since $\beta_j \in \mathcal{I}_G(\alpha_j)$, there exist $(h_j, v_j) \in G(\alpha_j)$ such that $\beta_j = h_j * \alpha_j + \xi_{v_j}$ for $j \in \mathbb{N}$. Then $(\alpha_j, h_j, v_j) \in \mathrm{Gr}(G)$. Since $\mathrm{Gr}(G)$ is compact, we may assume, after passing to a subsequence, that (i) the sequences $\{\alpha_j\}_{j=1}^\infty$, $\{h_j\}_{j=1}^\infty$, converge uniformly to limits α_∞, h_∞, (ii) $\{v_j\}_{j=1}^\infty$ converges in $\mathbb{R}^m$ to a limit v_∞, (iii) $\alpha_\infty \in \mathcal{A}(S)$, and (iv) $(h_\infty, v_\infty) \in G(\alpha_\infty)$.

Let $\beta_\infty = h_\infty * \alpha_\infty + \xi_{v_\infty}$. Then $\beta_\infty \in \mathcal{I}_G(\alpha_\infty)$. Lemma 2.4 implies that $\beta_j \to \beta_\infty$ uniformly as $j \to \infty$. So $\mathrm{Gr}(\mathcal{I}_G)$ is compact.

Now $(x, y) \in \mathrm{Gr}(\Phi_G)$ if and only if there exists a pair $(\alpha, \beta) \in \mathrm{Gr}(\mathcal{I}_G)$ such that $\alpha(1) = x$ and $\beta(1) = y$. So $\mathrm{Gr}(\Phi_G)$ is the image of $\mathrm{Gr}(\mathcal{I}_G)$ under the projection

$$C^0([0,1]; \mathbb{R}^n) \times C^0([0,1]; \mathbb{R}^m) \ni (\alpha, \beta) \to (\alpha(1), \beta(1)) \in \mathbb{R}^n \times \mathbb{R}^m .$$

Since this projection is continuous, $\mathrm{Gr}(\Phi_G)$ is compact, and the proof of the first statement is complete.

If G is single-valued, everywhere defined, and continuous, then $\mathcal{I}_G$ is single-valued and everywhere defined and, moreover, the graph $\mathrm{Gr}(\mathcal{I}_G)$ is compact, because $\mathrm{Gr}(G)$ is compact. This implies that $\mathcal{I}_G$ is continuous, and the second statement is proved.

Finally, let us prove the third statement. Assume that G is regular. We want to show that $\mathcal{I}_G$ and Φ_G are regular. This requires that we prove that

(a) the graphs $\mathrm{Gr}(\mathcal{I}_G)$ and $\mathrm{Gr}(\Phi_G)$ are compact,

(b) $\mathcal{I}_G$ and Φ_G can be approximated in the sense of inward graph convergence by sequences of single-valued continuous maps.

Part (a) follows from the fact that $\mathrm{Gr}(G)$ is compact. We now prove part (b). Using the regularity of G, let $\{G_j\}_{j=1}^\infty$ be a sequence of single-valued, everywhere defined continuous maps from $\mathcal{A}(S)$ to $C^0([0,1]; \mathbb{R}^{m\times n}) \times \mathbb{R}^m$ such that $G_j \xrightarrow{\mathrm{igr}} G$ as $j \to \infty$. Then the $\mathcal{I}_{G_j}$ are single-valued, everywhere defined, and continuous.

We show that $\mathcal{I}_{G_j} \xrightarrow{\mathrm{igr}} \mathcal{I}_G$ as $j \to \infty$. Let

$$\delta_j = \sup \Big\{ \mathrm{dist}\Big((\alpha, \beta), \mathrm{Gr}(\mathcal{I}_G)\Big) : (\alpha, \beta) \in \mathrm{Gr}(\mathcal{I}_{G_j}) \Big\} .$$

We want to show that $\delta_j \to 0$ as $j \to \infty$. Assume this is not true. Then there exists an infinite subset J of $\mathbb{N}$ and a strictly positive number θ such that $\delta_j \geq 2\theta$ for all $j \in J$. We can therefore pick members (α_j, β_j) of $\mathrm{Gr}(\mathcal{I}_{G_j})$ for $j \in J$ such that

$$\mathrm{dist}\Big((\alpha_j, \beta_j), \mathrm{Gr}(\mathcal{I}_G)\Big) \geq \theta \text{ whenever } j \in J . \quad (3)$$

If $j \in J$, then $\beta_j \in \mathcal{I}_{G_j}(\alpha_j)$, so we can pick pairs $(h_j, v_j) \in G_j(\alpha_j)$ such that $\beta_j = h_j * \alpha_j + \xi_{v_j}$. Since $G_j \xrightarrow{\mathrm{igr}} G$, we may assume, after making J smaller, if necessary, that the limit

$$(\alpha_\infty, h_\infty, v_\infty) = \lim_{j\to\infty, j\in J} (\alpha_j, h_j, v_j)$$

exists and belongs to $\mathrm{Gr}(G)$. Let $\beta_\infty = h_\infty * \alpha_\infty + \xi_{v_\infty}$. Then $\beta_\infty \in \mathcal{I}_G(\alpha_\infty)$. Lemma 2.4 implies that $\beta_j \to \beta_\infty$ uniformly as $j \to \infty$ via values in J. But then

$$(\alpha_\infty, \beta_\infty) = \lim_{j\to\infty, j\in J} (\alpha_j, \beta_j) .$$

Since $(\alpha_\infty, \beta_\infty) \in \mathrm{Gr}(\mathcal{I}_G)$, we have shown that

$$\lim_{j\to\infty, j\in J} \mathrm{dist}\Big((\alpha_j, \beta_j), \mathrm{Gr}(\mathcal{I}_G)\Big) = 0 ,$$

contradicting (3). Therefore $\delta_j \to 0$ as $j \to \infty$, and we have completed the proof that $\mathcal{I}_G$ is regular.

We must now show that Φ_G is regular. For each $x \in S$, let α_x be the curve given by

$$\alpha_x(t) = tx \quad \text{for} \quad t \in [0,1] .$$

Then $\alpha_x \in \mathcal{A}(S)$, and the map $S \ni x \to \alpha_x \in \mathcal{A}(S)$ is continuous. Define

$$\Phi^j(x) = \mathcal{I}_{G_j}(\alpha_x)(1) \quad \text{for} \quad x \in S .$$

Then Φ^j is a continuous map from S to $\mathbb{R}^m$. (Continuity follows because the map $\mathcal{I}_{G_j} : \mathcal{A}(S) \to C^0([0,1]; \mathbb{R}^m)$ is continuous, and the maps $x \to \alpha_x$ and $\beta \to \beta(1)$ are continuous as well. The continuity of $\mathcal{I}_{G_j}$ follows from Lemma 2.1.)

We now show that $\Phi^j \xrightarrow{\mathrm{igr}} \Phi_G$. Let $(x_j, y_j) \in \mathrm{Gr}(\Phi^j)$. We want to extract a subsequence of $\{(x_j, y_j)\}_{j=1}^\infty$ that converge to a limit $(x, y) \in \mathrm{Gr}(\Phi_G)$. Pick $\beta_j \in \mathcal{I}_{G_j}(\alpha_{x_j})$. Then

$$\mathrm{dist}\Big((\alpha_{x_j}, \beta_j), \mathrm{Gr}(\mathcal{I}_G) \Big) \to 0 \quad \text{as} \quad j \to \infty ,$$

because $\mathcal{I}_{G_j} \xrightarrow{\mathrm{igr}} \mathcal{I}_G$. Since $\mathrm{Gr}(\mathcal{I}_G)$ is compact we may assume, after passing to a susequence, that there exists a pair $(\alpha, \beta) \in \mathrm{Gr}(\mathcal{I}_G)$ such that $\alpha_{x_j} \to \alpha$ and $\beta_j \to \beta$. If we let $x = \alpha(1)$, then $x_j \to x$. Therefore $\alpha_{x_j} \to \alpha_x$, so $\alpha = \alpha_x$. Let $y = \beta(1)$. Then $y \in \Phi_G(x)$, and $(x_j, y_j) \to (x, y)$. So our proof is complete. ♢

3. The main definition

If $n \in \mathbb{Z}_+$, C is a cone in $\mathbb{R}^n$, and $r \in]0, \infty[$, we write $C(r)$ to denote the set $C \cap r\bar{\mathbb{B}}^n$, that is

$$C(r) \stackrel{\mathrm{def}}{=} \{ x \in C : \|x\| \leq r \} .$$

Then $C(r)$ is compact convex if C is a closed convex cone.

Definition 3.1 Let n, m be nonnegative integers, let F be a set-valued map from $\mathbb{R}^n$ to $\mathbb{R}^m$, and let C be a closed convex cone in $\mathbb{R}^n$. We say that Λ is a *path-integral generalized differential of* F *at* $(0,0)$ *in the direction of* C, and write $\Lambda \in PIGD(F, C)$, if (1) Λ is a nonempty compact subset of $\mathbb{R}^{m\times n}$, and (2) for every positive real number δ there exists a number $R \in]0, \infty[$ with the property that for every $r \in]0, R]$ there exists a regular set-valued map $G : \mathcal{A}(C(r)) \rightarrow\!\!\!\!\rightarrow C^0([0,1]; \mathbb{R}^{m\times n}) \times \mathbb{R}^m$ such that

(2a) $h(t) \in \Lambda^\delta$ and $\|v\| \leq \delta r$ whenever $\alpha \in \mathcal{A}(C(r))$, $(h, v) \in G(\alpha)$, $t \in [0,1]$,

(2b) $\mathrm{Gr}(\Phi_G) \subseteq \mathrm{Gr}(F)$. ♢

4. The chain rule

Theorem 4.1 *Let n_1, n_2, n_3 be nonnegative integers, and let F_i be, for $i = 1, 2$, set-valued maps from $\mathbb{R}^{n_i}$ to $\mathbb{R}^{n_{i+1}}$. Assume that*

1. *C_i is a closed convex cone in $\mathbb{R}^{n_i}$ for $i = 1, 2$,*
2. *C_2 is polyhedral,*
3. *$\Lambda_i \in PIGD(F_i, C_i)$ for $i = 1, 2$,*
4. *$F_1(C_1) \subseteq C_2$,*

and

5. *$\Lambda_1 \cdot C_1 \subseteq C_2$ (that is, $L \cdot C_1 \subseteq C_2$ for every $L \in \Lambda_1$).*

Then

$$\Lambda_2 \circ \Lambda_1 \in PIGD(F_2 \circ F_1, C_1) .$$

Proof. The crucial point is that, since the cone C_2 is polyhedral, it is possible to pick a vector $\bar{w} \in \mathrm{Int}_S(C_2)$ and a positive constant $\bar{k}$ such that the following "error correction property" holds

(ECP) *If $m \in \mathbb{N}$, $u_1, \ldots, u_m \in C_2$, $w \in C_2$, and $u_1 + \ldots + u_m = w + \bar{w}$, then there exist vectors $c_1, \ldots, c_m$ such that $\|c_1\| + \ldots + \|c_m\| \leq \bar{k}$, $c_1 + \ldots + c_m = \bar{w}$, and the conditions $\|c_i\| \leq \bar{k}\|u_i\|$ and $u_i - c_i \in C_2$ hold for $i = 1, \ldots, m$.*

(This follows from Theorem 6.7.)

We then may—and will—assume, without loss of generality, that $\|\bar{w}\| \leq 1$. We then fix a number $\rho \in]\,0, \infty\,[$ such that

$$\bar{w} + \rho \bar{\mathbb{B}}^{n_2} \subseteq C_2\,. \tag{4}$$

Then the vector $\bar{w}$ and the number ρ satisfy

$$\Big(s > 0 \wedge u \in S \wedge \|u\| \leq s\Big) \Rightarrow u + \rho^{-1} s \bar{w} \in C_2\,. \tag{5}$$

(Indeed, let $y = \frac{\rho}{s}u$. Then $\|y\| \leq \rho$. Therefore (4) implies $y + \bar{w} \in C_2$. Since $u + \frac{s}{\rho}\bar{w} = \frac{s}{\rho}(y + \bar{w})$, and C_2 is a cone, $u + \frac{s}{\rho}\bar{w} \in C_2$, as stated.)

Write $F = F_2 \circ F_1$, $\Lambda = \Lambda_2 \circ \Lambda_1$, $n = n_1$, $m = n_3$, $C = C_1$, and let

$$\begin{aligned} S &= \text{linear span of } C_2 \text{ in } \mathbb{R}^{n_2}\,, \\ \Pi &= \text{the orthogonal projection from } \mathbb{R}^{n_2} \text{ to } S\,, \\ \kappa_i &= \sup\{\, \|L\| : L \in \Lambda_i \,\} \quad \text{for} \quad i = 1, 2\,. \end{aligned}$$

Then $F : \mathbb{R}^n \rightarrow\!\!\!\rightarrow \mathbb{R}^m$, and Λ is a nonempty compact subset of $\mathbb{R}^{m \times n}$.

Fix a positive real number δ. We want to find a number $R \in]\,0, \infty\,[$ satisfying Property 2# of Definition 3.1.

Define a function $\Psi : [0, \infty\,[\times [0, \infty\,[\to [0, \infty\,[$ by

$$\begin{aligned} \Psi(\delta_1, \delta_2) \;&\stackrel{\text{def}}{=}\; (1 + \bar{k})\Big(\kappa_1 + 2\delta_1(1 + \rho^{-1})\Big)\delta_2 \\ &+ (\kappa_2 + \delta_2)\delta_1 + 2\delta_1\Big(1 + \rho^{-1}(1 + \bar{k})\Big)(\kappa_2 + \delta_2)\,. \end{aligned} \tag{6}$$

Then

$$\lim_{(\delta_1, \delta_2) \to (0,0)} \Psi(\delta_1, \delta_2) = 0\,. \tag{7}$$

We let δ_1, δ_2, be positive numbers such that

$$\Psi(\delta_1, \delta_2) \leq \frac{\delta}{2}\,. \tag{8}$$

For $i = 1, 2$, using the fact that $\Lambda_i \in PIGD(F_i, C_i)$, choose $R_i \in]\,0, \infty\,[$ with the property that for every $r_i \in]\,0, R_i]$ there exists G_i such that

(1) $G_i \in \mathrm{REG}(\mathcal{A}(C_i(r_i)); C^0(\,[0,1]\,; \mathbb{R}^{n_{i+1} \times n_i}\,) \times \mathbb{R}^{n_{i+1}})$,

(2) $\mathrm{Gr}(\Phi_{G_i}) \subseteq \mathrm{Gr}(F_i)\,$,

(3) $(h_i(t), v_i) \in \Lambda_i^{\delta_i} \times (\delta_i r_i \bar{\mathbb{B}}^{n_{i+1}})$ whenever $t \in [0,1]$, $\zeta \in \mathcal{A}(C_i(r_i))$, and $(h_i, v_i) \in G_i(\zeta)$.

Inequality (8) implies in particular that the inclusion

$$\Lambda_2^{\delta_2} \circ \Lambda_1^{\delta_1} \subseteq \Lambda^{\delta} \tag{9}$$

holds. (*Proof.* If $L_i \in \Lambda_i^{\delta_i}$ for $i = 1, 2$, and $L = L_2 \circ L_1$, then we can write $L_i = L_i^0 + E_i$, where $L_i^0 \in \Lambda_i$ and $\|E_i\| \leq \delta_i$. Let $L^0 = L_2^0 \circ L_1^0$, so $L^0 \in \Lambda$. Then $L = L^0 + E$, where $E = L_2^0 \circ E_1 + E_2 \circ L_1^0 + E_2 \circ E_1$, so

$$\|E\| \leq \kappa_2\delta_1 + \delta_2\kappa_1 + \delta_2\delta_1\,.$$

It follows easily from (8) that

$$\kappa_2\delta_1 + \delta_2\kappa_1 + \delta_2\delta_1 \leq \delta\,. \tag{10}$$

Therefore $\|E\| \leq \delta$, showing that $L \in \Lambda^{\delta}$, as stated.)

We let

$$\theta_0 = \kappa_1 + 2\delta_1\Big(1 + \frac{1}{\rho}\Big)\,, \qquad \theta = (1 + \bar{k})\theta_0\,,$$

and choose

$$R = \min\Big(\,R_1, \frac{R_2}{\theta}\,\Big)\,.$$

We then have to show that, with this choice of R, the property of Definition 3.1 is satisfied. For this purpose, we pick $r \in \mathbb{R}$ such that $0 < r \leq R$, and prove the existence of a G satisfying the conditions of Definition 3.1.

Let

$$\begin{aligned} r_1 &= r\,, \\ r_2 &= \theta r\,, \end{aligned}$$

and observe that $0 < r_1 \leq R_1$ and $0 < r_2 \leq R_2$. Pick G_1, G_2, such that (1)-(2)-(3) hold.

Let

$$\begin{aligned} \omega &= 2\delta_1 r_1\,, \\ \tilde{r} &= \kappa_1 r\,, \\ \hat{r} &= \theta_0 r_1\,, \end{aligned}$$

so that

$$r_2 = (1 + \bar{k})\hat{r}\,. \tag{11}$$

Let K be the ω-neighborhood of $C_2(\tilde{r})$, that is,

$$K = \{\, x \in \mathbb{R}^{n_2} : \mathrm{dist}(x, C_2(\tilde{r})) \leq \omega \,\}\,.$$

We claim that

$$\mathcal{I}_{G_1}\Big(\,\mathcal{A}(C_1(r_1))\,\Big) \subseteq \mathcal{A}(K)\,. \tag{12}$$

To see this, let $\beta \in \mathcal{I}_{G_1}\Big(\,\mathcal{A}(C_1(r_1))\Big)$. Pick $\alpha \in \mathcal{A}(C_1(r_1))$ such that $\beta \in \mathcal{I}_{G_1}(\alpha)$. Then choose $(h_1, v_1) \in G_1(\alpha)$ such that $\beta = h_1 * \alpha + \xi_{v_1}$. It is clear that $\beta(0) = 0$, and β is absolutely continuous. So (12) will be proved if we show that $\dot{\beta}(t) \in K$ for almost all $t \in [0,1]$.

Let E be a subset of $[0,1]$ such that $\mathrm{meas}(E) = 1$, having the property that for every $t \in E$

(I) the derivative $\dot{\alpha}(t)$ exists and belongs to $C_1(r_1)$,

(II) $\dot{\beta}(t)$ exists and is equal to $h_1(t) \cdot \dot{\alpha}(t) + v_1$.

Let $t \in E$. Since $h_1(t) \in \Lambda_1^{\delta_1}$, we can write $h_1(t) = L + \tilde{L}$, with $L \in \Lambda_1$ and $\|\tilde{L}\| \leq \delta_1$. Then

$$\dot{\beta}(t) = L \cdot \dot{\alpha}(t) + \tilde{L} \cdot \dot{\alpha}(t) + v_1\,,$$

and

$$\|\tilde{L} \cdot \dot{\alpha}(t) + v_1\| \leq 2\delta_1 r_1 = \omega\,.$$

We now use the hypothesis that $\Lambda_1 C_1 \subseteq C_2$ to conclude that $L \cdot \dot{\alpha}(t) \in C_2$. Moreover, $\|L \cdot \dot{\alpha}(t)\| \leq \kappa_1 r_1 = \tilde{r}$. Therefore $L \cdot \dot{\alpha}(t) \in C_2(\tilde{r})$. It then follows that

$$\mathrm{dist}(\,\dot{\beta}(t), C_2(\tilde{r})\,) \leq \omega\,. \tag{13}$$

So $\dot{\beta}(t) \in K$. Since this is true for all $t \in E$, we have shown that $\dot{\beta}(t) \in K$ for almost all t, completing the proof of (12).

We are now ready to begin the long process of defining the set-valued map

$$G : \mathcal{A}(C(r)) \rightrightarrows C^0([0,1]\,;\mathbb{R}^{m\times n}) \times \mathbb{R}^m .$$

The first step will be to assign, to each triple (α, h_1, v_1) such that $\alpha \in \mathcal{A}(C(r))$ and $(h_1, v_1) \in G_1(\alpha)$, a curve $\beta_{\alpha,h_1,v_1} : [0,1] \to \mathbb{R}^{n_2}$.

Pick $\alpha \in \mathcal{A}(C(r)) = \mathcal{A}(C_1(r_1))$, and $(h_1, v_1) \in G_1(\alpha)$. Let

$$\beta_{\alpha,h_1,v_1} \stackrel{\text{def}}{=} h_1 * \alpha + \xi_{v_1} .$$

Let $\beta_{\alpha,h_1,v_1} \stackrel{\text{def}}{=} h_1 * \alpha + \xi_{v_1}$. Then $\beta_{\alpha,h_1,v_1} \in \mathcal{I}_{G_1}(\alpha)$, so (12) implies that $\beta_{\alpha,h_1,v_1} \in \mathcal{A}(K)$. Moreover,

$$\beta_{\alpha,h_1,v_1}(1) \in C_2 , \tag{14}$$

because

$$\beta_{\alpha,h_1,v_1}(1) \in \Phi_{G_1}(\alpha(1)) \subseteq F_1(\alpha(1)) \subseteq C_2 .$$

The second step is to correct the error arising from the fact that $\beta \in \mathcal{A}(K)$ rather than in $\mathcal{A}(C_2(\tilde{r}))$. For this purpose we define, whenever α belongs to $\mathcal{A}(C(r))$ and $(h_1, v_1) \in G_1(\alpha)$, a new curve $\gamma_{\alpha,h_1,v_1} \in C^0([0,1]\,;\mathbb{R}^{n_2})$ by letting $\gamma_{\alpha,h_1,v_1}(t) = \Pi(\beta_{\alpha,h_1,v_1}(t)) + \frac{\omega t}{\rho}\bar{w}$ for $t \in [0,1]$.

We claim that

$$\gamma_{\alpha,h_1,v_1} \in \mathcal{A}(C_2(\hat{r})) . \tag{15}$$

To see this, write $\beta = \beta_{\alpha,h_1,v_1}$, $\gamma = \gamma_{\alpha,h_1,v_1}$, and observe that γ is an absolutely continuous curve, and $\gamma(0) = 0$. In addition, for almost all $t \in [0,1]$, $\dot{\gamma}(t)$ exists and is equal to $\Pi(\dot{\beta}(t)) + \frac{\omega}{\rho}\bar{w}$, and $\dot{\beta}(t) \in K$. Let E be the set of all $t \in [0,1]$ for which this is true. Then $\mathrm{meas}(E) = 1$. Moreover,

$$\dot{\gamma}(t) \in C_2(\hat{r}) \quad \text{whenever} \quad t \in E . \tag{16}$$

(*Proof.* Fix $t \in E$. Since $\dot{\beta}(t) \in K$, we can write $\dot{\beta}(t) = b_1 + b_2$, with $b_1 \in C_2(\tilde{r})$ and $\|b_2\| \leq \omega$. Then $\Pi(\dot{\beta}(t)) = \Pi(b_1) + \Pi(b_2) = b_1 + \Pi(b_2)$, since $b_1 \in S$. Moreover, $\|\Pi(b_2)\| \leq \omega$, because $\|b_2\| \leq \omega$ and Π is an orthogonal projection. Then (5) implies that $\Pi(b_2) + \rho^{-1}\omega\bar{w} \in C_2$. Since $b_1 \in C_2$, and $\dot{\gamma}(t) = b_1 + \Pi(b_2) + \rho^{-1}\omega\bar{w}$, we conclude that $\dot{\gamma}(t) \in C_2$. Furthermore,

$$\begin{aligned} \|\dot{\gamma}(t)\| &\leq \|\Pi(\dot{\beta}(t))\| + \rho^{-1}\omega\|\bar{w}\| \leq \|\dot{\beta}(t)\| + \rho^{-1}\omega \\ &\leq \tilde{r} + \omega + \rho^{-1}\omega = (\kappa_1 + 2\delta_1 + 2\rho^{-1}\delta_1)r_1 \\ &= \theta_0 r_1 = \hat{r} . \end{aligned}$$

So $\dot{\gamma}(t) \in C_2(\hat{r})$, and the proof of (16) is complete.) Therefore (15) holds.

The third step is to make a piecewise linear approximation of the curves γ_{α,h_1,v_1}, by first choosing a large positive integer N as follows. The fact that G_2 is regular implies that the set $G_2(\mathcal{A}(C_2(r_2)))$ is compact in $C^0([0,1]\,;\mathbb{R}^{n_3\times n_2}) \times \mathbb{R}^{n_3}$. Let H be the set of those $h_2 \in C^0([0,1]\,;\mathbb{R}^{n_3\times n_2})$ such that $(h_2, v_2) \in G_2\Big(\mathcal{A}(C_2(r_2))\Big)$ for some $v_2 \in \mathbb{R}^{n_3}$. Then H is compact as well. Hence H is uniformly equicontinuous, so we can choose $N \in \mathbb{N}$ such that $\|h_2(t) - h_2(s)\| \leq \frac{\delta}{4\theta_0}$ whenever $h_2 \in H$, $t, s \in [0,1]$, and $|t - s| \leq \frac{1}{N}$. With this choice of N, we define

$$\begin{aligned} \eta_{\alpha,h_1,v_1}(t) &= (j - Nt)\gamma_{\alpha,h_1,v_1}(N^{-1}(j-1)) \\ &+ (Nt + 1 - j)\gamma_{\alpha,h_1,v_1}(N^{-1}j) \end{aligned} \tag{17}$$

whenever $\alpha \in \mathcal{A}(C(r))$, $(h_1, v_1) \in G_1(\alpha)$, $\frac{j-1}{N} \leq t \leq \frac{j}{N}$, $j \in \mathbb{N}$, $1 \leq j \leq N$.

Then $\eta_{\alpha,h_1,v_1} \in C^0([0,1]\,;\mathbb{R}^{n_2})$, and

$$\eta_{\alpha,h_1,v_1}(0) = 0 , \tag{18}$$

$$\begin{aligned} \eta_{\alpha,h_1,v_1}(1) &= \gamma_{\alpha,h_1,v_1}(1) \\ &= \beta_{\alpha,h_1,v_1}(1) + \rho^{-1}\omega\bar{w} \\ &\in C_2 + \rho^{-1}\omega . \end{aligned} \tag{19}$$

Moreover, the map η_{α,h_1,v_1} is linear on each interval $I_j = [N^{-1}(j-1), N^{-1}j]$. The derivative $\dot{\eta}_{\alpha,h_1,v_1}(t)$ of η_{α,h_1,v_1} is equal, for $t \in I_j$, to $u^N_{\alpha,h_1,v_1,j}$, where

$$\begin{aligned} u^N_{\alpha,h_1,v_1,j} &= N\gamma_{\alpha,h_1,v_1}(N^{-1}j) - N\gamma_{\alpha,h_1,v_1}(N^{-1}(j-1)) \\ &= N\int_{\frac{j-1}{N}}^{\frac{j}{N}} \dot{\gamma}_{\alpha,h_1,v_1}(t)\,dt . \end{aligned} \tag{20}$$

Since $\dot{\gamma}_{\alpha,h_1,v_1}(t) \in C_2(\hat{r})$ for almost all t, the vectors $u^N_{\alpha,h_1,v_1,j}$ belong to $C_2(\hat{r})$ as well. So $\dot{\eta}_{\alpha,h_1,v_1}(t) \in C_2(\hat{r})$ for almost all $t \in [0,1]$, and then (18) implies that

$$\eta_{\alpha,h_1,v_1} \in \mathcal{A}(C_2(\hat{r})) . \tag{21}$$

The fourth step is to take care of the undesirable fact that η_{α,h_1,v_1} satisfies (19), and produce a curve whose terminal point is $\beta_{\alpha,h_1,v_1}(1)$ rather than $\beta_{\alpha,h_1,v_1}(1) + \rho^{-1}\omega\bar{w}$. For this purpose, we define $\tilde{u}^N_{\alpha,h_1,v_1,j} = \frac{\rho}{N\omega}u^N_{\alpha,h_1,v_1,j}$. Then

$$\sum_{j=1}^{N} \tilde{u}^N_{\alpha,h_1,v_1,j} = \frac{\rho}{\omega}\beta_{\alpha,h_1,v_1}(1) + \bar{w} .$$

It then follows from (ECP) that there exists an N-tuple $\tilde{\mathbf{c}} = (\tilde{c}_1, \ldots, \tilde{c}_N)$ of vectors that satisfies

$$\tilde{u}^N_{\alpha,h_1,v_1,j} - \tilde{c}_j \in C_2 \tag{22}$$

$$\tilde{c}_1 + \cdots + \tilde{c}_N = \bar{w} , \tag{23}$$

$$\|\tilde{c}_1\| + \cdots + \|\tilde{c}_N\| \leq \bar{k} , \tag{24}$$

$$\|\tilde{c}_j\| \leq \bar{k}\|\tilde{u}^N_{\alpha,h_1,v_1,j}\| \quad , \tag{25}$$

for $j = 1, \ldots, N$. Define $c_j = \rho^{-1}N\omega\tilde{c}_j$ and $\mathbf{c} = (c_1, \ldots, c_N)$. Then, for $j = 1, \ldots, N$,

$$u^N_{\alpha,h_1,v_1,j} - c_j \in C_2 \quad , \tag{26}$$

$$c_1 + \cdots + c_N = \rho^{-1}N\omega\bar{w} , \tag{27}$$

$$\|c_1\| + \cdots + \|c_N\| \leq \rho^{-1}N\omega\bar{k} , \tag{28}$$

$$\|c_j\| \leq \bar{k}\|u^N_{\alpha,h_1,v_1,j}\| \quad . \tag{29}$$

Let $\mu_{\mathbf{c}}$ be the function such that

$$\mu_{\mathbf{c}} \in C^0([0,1]\,;\mathbb{R}^{n_2}) , \tag{30}$$

$$\mu_{\mathbf{c}}(0) = 0 , \tag{31}$$

$$\dot{\mu}_{\mathbf{c}} \equiv c_j \quad \text{on} \quad I_j \quad \text{for} \quad j = 1, \ldots, N . \tag{32}$$

Then

$$\int_0^1 \|\dot{\mu}_{\mathbf{c}}(t)\|\,dt = \frac{1}{N}(\,\|c_1\| + \cdots + \|c_N\|\,) \leq \frac{\omega\bar{k}}{\rho} \tag{33}$$

and

$$\mu_{\mathbf{c}}(1) = \frac{1}{N}(\,c_1 + \cdots + c_N\,) = \frac{\omega}{\rho}\bar{w} . \tag{34}$$

Define a curve $\zeta_{\alpha,h_1,v_1,\mathbf{c}}$ by letting

$$\zeta_{\alpha,h_1,v_1,\mathbf{c}}(t) = \eta_{\alpha,h_1,v_1}(t) - \mu_{\mathbf{c}}(t) \quad \text{for} \quad t \in [0,1] . \tag{35}$$

Then $\zeta_{\alpha,h_1,v_1,\mathbf{c}}$ satisfies

$$\zeta_{\alpha,h_1,v_1,\mathbf{c}} \in C^0([0,1];\mathbb{R}^{n_2}), \tag{36}$$

$$\zeta_{\alpha,h_1,v_1,\mathbf{c}}(0) = 0, \tag{37}$$

$$\begin{aligned}\zeta_{\alpha,h_1,v_1,\mathbf{c}}(1) &= \eta_{\alpha,h_1,v_1}(1) - \mu_{\mathbf{c}}(1) \\ &= \beta_{\alpha,h_1,v_1}(1).\end{aligned} \tag{38}$$

Moreover, $\zeta_{\alpha,h_1,v_1,\mathbf{c}}$ is linear on each interval I_j, and the derivative $\dot\zeta_{\alpha,h_1,v_1,\mathbf{c}}(t)$ of $\zeta_{\alpha,h_1,v_1,\mathbf{c}}$ is equal, for $t \in I_j$, to $u^N_{\alpha,h_1,v_1,j} - c_j$. It follows from (26) that the vectors $v_j = u^N_{\alpha,h_1,v_1,j} - c_j$ belong to C_2. Moreover, the bound (29) implies that

$$\|v_j\| \le (1+\bar k)\|u^N_{\alpha,h_1,v_1,j}\| \le (1+\bar k)\hat r = r_2.$$

Therefore

$$\dot\zeta_{\alpha,h_1,v_1,\mathbf{c}} \in C_2(r_2) \quad \text{for a.e. } t \in [0,1], \tag{39}$$

and then (37) implies that

$$\zeta_{\alpha,h_1,v_1,\mathbf{c}} \in \mathcal{A}(C_2(r_2)). \tag{40}$$

We have now finally succeeded in producing, for each curve $\alpha \in \mathcal{A}(C(r))$ and each pair $(h_1,v_1) \in G_1(\alpha)$, a curve $\zeta_{\alpha,h_1,v_1,\mathbf{c}} \in \mathcal{A}(C_2(r_2))$ whose terminal point is exactly $\beta_{\alpha,h_1,v_1}(1)$. Moreover, this curve is "close" to β_{α,h_1,v_1}, in the sense that it is close to η_{α,h_1,v_1}, which is close to γ_{α,h_1,v_1}, which is close to β_{α,h_1,v_1}.

This curve need not be unique, because $\mathbf{c}$ may fail to be unique, so this nonuniqueness will have to be taken care of. As a first step in that direction, we introduce the notation $\mathbf{C}_{\alpha,h_1,v_1}$ to refer to the set of all N-tuples $\mathbf{c} = (c_1,\ldots,c_N)$ that belong to $(\mathbb{R}^{n_2})^N$ and satisfy (26), (27), (28), and (29).

Given a curve $\alpha \in \mathcal{A}(C(r))$, a pair $(h_1,v_1) \in G_1(\alpha)$, and a $\mathbf{c} \in \mathbf{C}_{\alpha,h_1,v_1}$, pick $(h_2,v_2) \in G_2(\zeta_{\alpha,h_1,v_1,\mathbf{c}})$. Define

$$\begin{aligned} h_{\alpha,h_1,v_1,\mathbf{c},h_2,v_2}(t) &= h_2(t)\cdot h_1(t) \ \text{ for } t \in [0,1], \\ v_{\alpha,h_1,v_1,\mathbf{c},h_2,v_2} &= v_2 + \Big(\int_0^1 h_2(t)\,dt\Big)\cdot v_1 \\ &\quad - \int_0^1 h_2(t)\cdot(\dot\beta(t) - \dot\zeta(t))\,dt. \end{aligned}$$

We let $G(\alpha)$ be the set of all pairs $(h_{\alpha,h_1,v_1,\mathbf{c},h_2,v_2}, v_{\alpha,h_1,v_1,\mathbf{c},h_2,v_2})$, as α varies over $\mathcal{A}(C(r))$, (h_1,v_1) varies over all members of $G_1(\alpha)$, $\mathbf{c}$ varies over all members of $\mathbf{C}_{\alpha,h_1,v_1}$, and the pair (h_2,v_2) varies over all members of $G_2(\zeta_{\alpha,h_1,v_1,\mathbf{c}})$.

With this definition, it is clear that G is a set-valued map from $\mathcal{A}(C(r))$ to $C^0([0,1];\mathbb{R}^{m\times n}) \times \mathbb{R}^m$. Moreover, if $(h,v) \in G(\alpha)$ for an $\alpha \in \mathcal{A}(C(r))$, then the following three facts are true.

(F1) *The matrix-valued function h takes values in Λ^δ.*

(F2) *If $x = \alpha(1)$, $\sigma = h * \alpha + \xi_v$, and $z = \sigma(1)$, then $z \in F(x)$.*

(F3) $\|v\| \le \delta r$.

Proof of (F1). Write

$$(h,v) = (h_{\alpha,h_1,v_1,\mathbf{c},h_2,v_2},\, v_{\alpha,h_1,v_1,\mathbf{c},h_2,v_2}). \tag{41}$$

Then, if $t \in [0,1]$, (9) implies that $h_2(t)\cdot h_1(t) \in \Lambda^\delta$, since $h_1(t) \in \Lambda_1^{\delta_1}$ and $h_2(t) \in \Lambda_2^{\delta_2}$.

Proof of (F2). Write $\beta = \beta_{\alpha,h_1,v_1}$, $\gamma = \gamma_{\alpha,h_1,v_1}$, $\eta = \eta_{\alpha,h_1,v_1}$, $\zeta = \zeta_{\alpha,h_1,v_1,\mathbf{c}}$, $\lambda = \beta - \zeta$, $y = \beta(1) = \zeta(1)$, $\nu = h_2 * \zeta + \xi_{v_2}$.

Then $y \in \Phi_{G_1}(x)$, because $\beta \in \mathcal{I}_{G_1}(\alpha)$, so $\beta(1) \in \Phi_{G_1}(x)$. Therefore y belongs to $F_1(x)$, because $\mathrm{Gr}(\Phi_{G_1}) \subseteq \mathrm{Gr}(F_1)$. Now, $z = \int_0^1 h(t)\cdot\dot\alpha(t)\,dt + v$, so

$$\begin{aligned} z &= \int_0^1 h_2(t)\cdot h_1(t)\cdot\dot\alpha(t)\,dt + \int_0^1 (h_2(t)\cdot v_1)\,dt + v_2 \\ &\quad - \int_0^1 h_2(t)\cdot\dot\lambda(t)\,dt \\ &= \int_0^1 h_2(t)\cdot\Big(h_1(t)\cdot\dot\alpha(t) + v_1\Big)\,dt + v_2 \\ &\quad - \int_0^1 h_2(t)\cdot\dot\lambda(t)\,dt \\ &= \int_0^1 h_2(t)\cdot\dot\beta(t)\,dt + v_2 \\ &\quad - \int_0^1 h_2(t)\cdot(\dot\beta(t) - \dot\zeta(t))\,dt \\ &= \int_0^1 h_2(t)\cdot\dot\zeta(t)\,dt + v_2 \\ &= \nu(1). \end{aligned}$$

But $\nu \in \mathcal{I}_{G_2}(\zeta)$, because $\nu = h_2 * \zeta + \xi_{v_2}$ and $(h_2,v_2) \in G_2(\zeta)$. Therefore

$$z = \nu(1) \in \Phi_{G_2}(\zeta(1)) = \Phi_{G_2}(y).$$

Hence z belongs to $F_2(y)$, because $\mathrm{Gr}(\Phi_{G_2}) \subseteq \mathrm{Gr}(F_2)$. Since $y \in F_1(x)$ and $z \in F_2(y)$, the conclusion that $z \in F(x)$ follows.

Proof of (F3). Using the notations introduced above, we have

$$v = v_2 + \Big(\int_0^1 h_2(t)\,dt\Big)\cdot v_1 + \int_0^1 h_2(t)\cdot\dot\lambda(t)\,dt, \tag{42}$$

so

$$\begin{aligned} \|v\| &\le \delta_2 r_2 + (\kappa_2+\delta_2)\delta_1 r_1 + E \\ &\le \Big(\theta\delta_2 + (\kappa_2+\delta_2)\delta_1\Big) r + E, \end{aligned} \tag{43}$$

where

$$E = \Big\|\int_0^1 h_2(t)\cdot\dot\lambda(t)\,dt\Big\|. \tag{44}$$

To estimate E, we write

$$\lambda = (\beta - \beta^*) + (\beta^* - \gamma) + (\gamma - \eta) + (\eta - \zeta) \tag{45}$$

where $\beta^* = \Pi\circ\beta$. Then $\dot\beta^* = \Pi\circ\dot\beta$, because Π is linear. It follows from (13) that

$$\|\dot\beta(t) - \dot\beta^*(t)\| \le \omega \quad \text{for a.e. } \ t \in [0,1],$$

and then

$$\begin{aligned} \Big\|\int_0^1 h_2(t)\cdot(\dot\beta(t) - \dot\beta^*(t))\,dt\Big\| &\le (\kappa_2+\delta_2)\omega \\ &= 2\delta_1(\kappa_2+\delta_2) r. \end{aligned} \tag{46}$$

The function $\dot\beta^* - \dot\gamma$ has a constant value, equal to $\frac{\omega}{\rho}\bar w$. Therefore

$$\begin{aligned} \Big\|\int_0^1 h_2(t)\cdot(\dot\beta^*(t) - \dot\gamma(t))\,dt\Big\| &\le \rho^{-1}(\kappa_2+\delta_2)\omega\|\bar w\| \\ &\le 2\rho^{-1}\delta_1(\kappa_2+\delta_2) r. \end{aligned} \tag{47}$$

The function $\eta - \zeta$ is $\mu_{\mathbf{c}}$. Then (33) implies the bound

$$\Big\| \int_0^1 h_2(t)\cdot(\dot\eta(t)-\dot\zeta(t))\,dt\Big\| \le \rho^{-1}\bar k(\kappa_2+\delta_2)\omega \le 2\rho^{-1}\delta_1\bar k(\kappa_2+\delta_2)r\,. \quad (48)$$

Finally, we have to estimate the integral $\sigma = \int_0^1 h_2(t)\cdot(\dot\gamma(t)-\dot\eta(t))\,dt$. Let $\hat h_2$ be the piecewise constant function on $[0,1]$ whose value on each interval I_j is equal to $h_2(\frac{j-1}{N})$. Then

$$\sigma = \int_0^1 (h_2(t)-\hat h_2(t))\cdot(\dot\gamma(t)-\dot\eta(t))\,dt + \int_0^1 \hat h_2(t)\cdot(\dot\gamma(t)-\dot\eta(t))\,dt\,.$$

Since γ and η belong to $\mathcal{A}(C_2(\hat r))$, the estimate

$$\|\dot\gamma(t)-\dot\eta(t)\| \le 2\hat r \quad (49)$$

is true for almost all t. Our choice of N implies the estimate

$$\|h_2(t)-\hat h_2(t)\| \le \frac{\delta}{4\theta_0} = \frac{\delta r}{4\hat r}\,. \quad (50)$$

Then (49) and (50) imply

$$\left\|\int_0^1 (h_2(t)-\hat h_2(t))\cdot(\dot\gamma(t)-\dot\eta(t))\,dt\right\| \le \frac{\delta r}{2}\,. \quad (51)$$

Finally,

$$\int_0^1 \hat h_2(t)\cdot(\dot\gamma(t)-\dot\eta(t))\,dt = 0\,, \quad (52)$$

because $\hat h_2$ is constant on each interval I_j, and $\dot\eta$ is also constant on I_j, and equal to the average of $\dot\gamma$ over I_j.

If we combine (43), (44), (45), (46), (47), (48), (51), and (52), we end up with the bound

$$\|v\| \le \Big(\bar\psi + \frac{\delta}{2}\Big)r\,, \quad (53)$$

where

$$\begin{aligned}\bar\psi &= \theta\delta_2 + (\kappa_2+\delta_2)\delta_1 + 2\delta_1\left(1+\frac{1+\bar k}{\rho}\right)(\kappa_2+\delta_2)\\ &= (1+\bar k)\left(\kappa_1 + 2\delta_1\Big(1+\frac{1}{\rho}\Big)\right)\delta_2 + (\kappa_2+\delta_2)\delta_1\\ &\quad +2\delta_1\left(1+\frac{1+\bar k}{\rho}\right)(\kappa_2+\delta_2)\\ &= \Psi(\delta_1,\delta_2)\\ &\le \frac{\delta}{2}\,.\end{aligned}$$

The proof of (F3) is thus complete.

In view of (F1), (F2) and (F3), our conclusion will follow if we prove that G is regular. To prove the regularity of G, we express G as a composite of regular maps.

We define

$$\begin{aligned}\mathcal{U} &= C^0([0,1];\mathbb{R}^{n_2\times n})\,,\\ \hat{\mathcal{U}} &= \Big\{h_1\in\mathcal{U} : \|h_1(t)\|\le\kappa_1+\delta_1 \text{ for all } t\in[0,1]\Big\}\,,\\ \mathcal{Y} &= C^0([0,1];\mathbb{R}^{n_2})\,,\\ \mathcal{B} &= \Big\{\beta\in\mathcal{Y} : (\kappa_1+2\delta_1)\|\beta(t)-\beta(s)\|\le|t-s| \text{ whenever } t,s\in[0,1]\Big\}\,,\\ \tilde{\mathcal{Y}} &= (\mathbb{R}^{n_2})^N\,,\\ \mathcal{Z} &= \Big\{\eta\in\mathcal{Y} : \eta \text{ is linear on } I_j \text{ for } j=1,\ldots,N\Big\}\,,\\ \tilde{\mathcal{Z}} &= \mathcal{Z}\cap\mathcal{A}(C_2(r_2))\,,\\ \mathcal{W} &= C^0([0,1];\mathbb{R}^{m\times n_2})\,,\\ \hat{\mathcal{W}} &= \Big\{h_2\in\mathcal{W} : \|h_2(t)\|\le\kappa_2+\delta_2 \text{ for all } t\in[0,1]\Big\}\,,\\ \mathcal{Q} &= C^0([0,1];\mathbb{R}^{m\times n})\,,\\ \mathcal{X}_1 &= \mathcal{A}(C(r))\times\hat{\mathcal{U}}\times\mathbb{R}^{n_2}\,,\\ \mathcal{X}_2 &= \mathcal{A}(C(r))\times\hat{\mathcal{U}}\times\mathbb{R}^{n_2}\times\mathcal{B}\times\mathcal{Y}\times\mathcal{Z}\,,\\ \mathcal{X}_3 &= \mathcal{A}(C(r))\times\hat{\mathcal{U}}\times\mathbb{R}^{n_2}\times\mathcal{B}\times\mathcal{Y}\times\mathcal{Z}\times\tilde{\mathcal{Y}}\,,\end{aligned}$$

We then let $\Gamma_1 : \mathcal{A}(C(r)) \to\!\!\!\to \mathcal{X}_1$ be the set-valued map that sends $\alpha\in\mathcal{A}(C(r))$ to the set $\{\alpha\}\times G_1(\alpha)$, so that

$$\Gamma_1(\alpha) = \Big\{(\alpha,h_1,v_1) : (h_1,v_1)\in G_1(\alpha)\Big\} \text{ if } \alpha\in\mathcal{A}(C(r)).$$

Then

$$\Gamma_1 \in \mathrm{REG}(\mathcal{A}(C(r));\mathcal{X}_1)\,, \quad (54)$$

because of the identity $\Gamma_1 = (\mathbb{I}_{\mathcal{A}(C(r))}\times G_1)\circ\Delta_1$, where $\Delta_1 : \mathcal{A}(C(r))\to\mathcal{A}(C(r))\times\mathcal{A}(C(r))$ is the diagonal map (i.e., the map that sends $\alpha\in\mathcal{A}(C(r))$ to the pair (α,α)).

We then let $\Gamma_2 : \mathcal{X}_1\to\mathcal{X}_2$ be the ordinary map that sends each triple $(\alpha,h_1,v_1)\in\mathcal{X}_1$ to the 6-tuple

$$(\alpha,h_1,v_1,\beta,\gamma,\eta)\in\mathcal{X}_2\,,$$

where

$$\begin{aligned}\beta &= h_1 * \alpha + \xi_{v_1}\,,\\ \gamma(t) &= \Pi(\beta(t)) + \frac{\omega t}{\rho}\bar w \quad \text{for} \quad t\in[0,1]\,,\end{aligned}$$

and

$$\eta(t) = (j-Nt)\gamma\Big(\frac{j-1}{N}\Big) + (Nt+1-j)\gamma\Big(\frac{j}{N}\Big) \quad (55)$$

whenever $\frac{j-1}{N}\le t\le\frac{j}{N}$, $j\in\mathbb{N}$, and $1\le j\le N$.

We then let $\Gamma_3^0 : \mathcal{Z}\to\tilde{\mathcal{Y}}$ be the map that sends each $\eta\in\mathcal{Z}$ to the N-tuple

$$\Gamma_3^0(\eta) = (u_1,\ldots,u_N)\in\tilde{\mathcal{Y}}\,, \quad (56)$$

where

$$u_j = N\left(\eta\Big(\frac{j}{N}\Big) - \eta\Big(\frac{j-1}{N}\Big)\right) \text{ for } j=1,\ldots,N\,. \quad (57)$$

Next, we let $\Gamma_3 : \mathcal{X}_2\to\mathcal{X}_3$ be the map that sends each 6-tuple $(\alpha,h_1,v_1,\beta,\gamma,\eta)$ to the 7-tuple

$$\Gamma_3(\alpha,h_1,v_1,\beta,\gamma,\eta) = (\alpha,h_1,v_1,\beta,\gamma,\eta,\Gamma_3^0(\eta))\,. \quad (58)$$

Next, we define $\tilde{\mathcal{K}}_0 = C_2(\hat r)^N$, and let $\tilde{\mathcal{K}}$ be the set of all $(u_1,\ldots,u_N)\in\tilde{\mathcal{K}}_0$ such that $N(u_1+\ldots+u_N) = w_0 + \frac{\omega}{\rho}\bar w$ for some $w_0\in C_2$. Then $\tilde{\mathcal{K}}_0$ and $\tilde{\mathcal{K}}$ are compact convex subsets of $\tilde{\mathcal{Y}}$.

We let Γ_4^0 be a continuous retraction from $\tilde{\mathcal{Y}}$ onto $\tilde{\mathcal{K}}$, and define

$$\mathcal{X}_4 = \mathcal{A}(C(r))\times\hat{\mathcal{U}}\times\mathbb{R}^{n_2}\times\mathcal{B}\times\mathcal{Y}\times\mathcal{Z}\times\tilde{\mathcal{K}}\,.$$

We then let $\Gamma_4 : \mathcal{X}_3\to\mathcal{X}_4$ be the map that sends each 7-tuple $(\alpha,h_1,v_1,\beta,\gamma,\eta,\mathbf{u})$ to the 7-tuple

$$\Gamma_4(\alpha,h_1,v_1,\beta,\gamma,\eta,\mathbf{u}) = (\alpha,h_1,v_1,\beta,\gamma,\eta,\Gamma_4^0(\mathbf{u}))\,. \quad (59)$$

We now define a set-valued map $\Gamma_5^0 : \tilde{\mathcal{K}} \rightarrow\!\!\!\rightarrow \tilde{\mathcal{Y}}$, by letting $\Gamma_5^0(u_1,\ldots,u_N)$ be, if $(u_1,\ldots,u_N) \in \tilde{\mathcal{K}}$, he set of all N-tuples $(c_1,\ldots,c_N)$ that satisfy, for $j = 1,\ldots,N$, the conditions

$$u_j - c_j \in C_2 \quad , \tag{60}$$

$$c_1 + \cdots + c_N = \frac{N\omega}{\rho}\bar{w}\,, \tag{61}$$

$$\|c_1\| + \cdots + \|c_N\| \le \frac{N\omega\bar{k}}{\rho}\,, \tag{62}$$

$$\|c_j\| \le \bar{k}\|u_j\| \quad . \tag{63}$$

Then Γ_5^0 has convex values and a compact graph. Moreover, (ECP) implies that the values of Γ_5^0 are nonempty. It then follows from Theorem 5.2 of [1] that

$$\Gamma_5^0 \in \mathrm{REG}(\tilde{\mathcal{K}};\tilde{\mathcal{Y}})\,. \tag{64}$$

We then define

$$\mathcal{X}_5 = \mathcal{A}(C(r)) \times \hat{\mathcal{U}} \times \mathbb{R}^{n_2} \times \mathcal{B} \times \mathcal{Y} \times \mathcal{Z} \times \tilde{\mathcal{K}} \times \tilde{\mathcal{Y}}\,,$$

and let $\Gamma_5 : \mathcal{X}_4 \rightarrow\!\!\!\rightarrow \mathcal{X}_5$ be the set-valued map that sends each 7-tuple $(\alpha, h_1, v_1, \beta, \gamma, \eta, \mathbf{u}) \in \mathcal{X}_4$ to the set

$$\Gamma_5(\alpha, h_1, v_1, \beta, \gamma, \eta, \mathbf{u}) = \Big\{ (\alpha, h_1, v_1, \beta, \gamma, \eta, \mathbf{u}, \mathbf{c}) : \mathbf{c} \in \Gamma_5^0(\mathbf{u}) \Big\} \subseteq \mathcal{X}_5\,. \tag{65}$$

Then

$$\Gamma_5 = \Big(1\!\!1_{\mathcal{X}_4} \times \Gamma_5\Big) \circ \Delta_2\,, \tag{66}$$

where $\Delta_2 : \mathcal{X}_4 \rightarrow \mathcal{X}_4 \times \tilde{\mathcal{K}}$ is the map that sends a 7-tuple $(\alpha, h_1, v_1, \beta, \gamma, \eta, \mathbf{u})$ to the 8-tuple $(\alpha, h_1, v_1, \beta, \gamma, \eta, \mathbf{u}, \mathbf{u})$. It follows from (64) and (66) that

$$\Gamma_5 \in \mathrm{REG}(\mathcal{X}_4;\mathcal{X}_5)\,. \tag{67}$$

Next, define

$$\mathcal{X}_6 = \mathcal{A}(C(r)) \times \hat{\mathcal{U}} \times \mathbb{R}^{n_2} \times \mathcal{B} \times \mathcal{Y} \times \mathcal{Z} \times \tilde{\mathcal{K}} \times \tilde{\mathcal{Y}} \times \mathcal{Z} \times \mathcal{Z}\,,$$

and let $\Gamma_6 : \mathcal{X}_5 \rightarrow \mathcal{X}_6$ be the ordinary map that sends each 8-tuple $(\alpha, h_1, v_1, \beta, \gamma, \eta, \mathbf{u}, \mathbf{c}) \in \mathcal{X}_5$ to the 10-tuple

$$\Gamma_6(\alpha, h_1, v_1, \beta, \gamma, \eta, \mathbf{u}, \mathbf{c}) = (\alpha, h_1, v_1, \beta, \gamma, \eta, \mathbf{u}, \mathbf{c}, \mu_{\mathbf{c}}, \eta - \mu_{\mathbf{c}}) \in \mathcal{X}_6\,.$$

We now observe that the real linear space $\mathcal{Z}$ is finite-dimensional, and $\tilde{\mathcal{Z}}$ is a nonempty compact convex subset of $\mathcal{Z}$. Let Γ_7^0 be a continuous retraction from $\mathcal{Z}$ onto $\tilde{\mathcal{Z}}$. Define

$$\mathcal{X}_7 = \mathcal{A}(C(r)) \times \hat{\mathcal{U}} \times \mathbb{R}^{n_2} \times \mathcal{B} \times \mathcal{Y} \times \mathcal{Z} \times \tilde{\mathcal{K}} \times \tilde{\mathcal{Y}} \times \mathcal{Z} \times \tilde{\mathcal{Z}}\,,$$

and let $\Gamma_7 : \mathcal{X}_6 \rightarrow \mathcal{X}_7$ be the ordinary map that sends each 10-tuple $(\alpha, h_1, v_1, \beta, \gamma, \eta, \mathbf{u}, \mathbf{c}, \mu, \zeta) \in \mathcal{X}_6$ to the 10-tuple

$$\Gamma_7(\alpha, h, v, \beta, \gamma, \eta, \mathbf{u}, \mathbf{c}, \mu, \zeta) = (\alpha, h_1, v_1, \beta, \gamma, \eta, \mathbf{u}, \mathbf{c}, \mu, \Gamma_7^0(\zeta)) \in \mathcal{X}_7\,. \tag{68}$$

Next, we let Γ_8^0 be the set-valued map $\tilde{\mathcal{Z}} \rightarrow\!\!\!\rightarrow \hat{\mathcal{W}} \times \mathbb{R}^m$ that sends $\zeta \in \tilde{\mathcal{Z}}$ to the set $G_2(\zeta) \subseteq \hat{\mathcal{W}} \times \mathbb{R}^m$. Then

$$\Gamma_8^0 = G_2 \circ \iota_{\mathcal{A}(C_2(r_2)),\tilde{\mathcal{Z}}}\,,$$

where $\iota_{\mathcal{A}(C_2(r_2)),\tilde{\mathcal{Z}}}$ is the inclusion map from $\tilde{\mathcal{Z}}$ to $\mathcal{A}(C_2(r_2))$. Therefore $\Gamma_8^0 \in \mathrm{REG}(\tilde{\mathcal{Z}};\hat{\mathcal{W}} \times \mathbb{R}^m)$.

Define $\mathcal{X}_8$ to be the product $\mathcal{A}(C(r)) \times \hat{\mathcal{U}} \times \mathbb{R}^{n_2} \times \mathcal{B} \times \mathcal{Y} \times \mathcal{Z} \times \tilde{\mathcal{K}} \times \tilde{\mathcal{Y}} \times \mathcal{Z} \times \tilde{\mathcal{Z}} \times \hat{\mathcal{W}} \times \mathbb{R}^m$, and let $\Gamma_8 : \mathcal{X}_7 \rightarrow\!\!\!\rightarrow \mathcal{X}_8$ be the set-valued map that sends each 10-tuple $(\alpha, h_1, v_1, \beta, \gamma, \eta, \mathbf{u}, \mathbf{c}, \mu, \zeta) \in \mathcal{X}_7$ to the set

$$\begin{aligned}\Gamma_8(\alpha, h_1, v_1, \beta, \gamma, \eta, \mathbf{u}, \mathbf{c}, \mu, \zeta) &= \Big\{ (\alpha, h_1, v_1, \beta, \gamma, \eta, \mathbf{u}, \mathbf{c}, \mu, \zeta) \Big\} \times G_2(\zeta) \subseteq \mathcal{X}_8 \\ &= \Big\{ (\alpha, h_1, v_1, \beta, \gamma, \eta, \mathbf{u}, \mathbf{c}, \mu, \zeta, h_2, v_2) : (h_2, v_2) \in G_2(\zeta) \Big\}\,.\end{aligned}$$

It is then clear that $\Gamma_8 = \Big(1\!\!1_{\mathcal{X}_7} \times G_2\Big) \circ \Delta_3$, where $\Delta_3 : \mathcal{X}_7 \rightarrow \mathcal{X}_7 \times \tilde{\mathcal{Z}}$ is the map that sends a 10-tuple $(\alpha, h_1, v_1, \beta, \gamma, \eta, \mathbf{u}, \mathbf{c}, \mu, \zeta) \in \mathcal{X}_7$ to the 11-tuple

$$(\alpha, h_1, v_1, \beta, \gamma, \eta, \mathbf{u}, \mathbf{c}, \mu, \zeta, \zeta) \in \mathcal{X}_7 \times \tilde{\mathcal{Z}}\,.$$

Therefore

$$\Gamma_8 \in \mathrm{REG}(\mathcal{X}_7;\mathcal{X}_8)\,. \tag{69}$$

Finally, we define an ordinary map $\Gamma_9 : \mathcal{X}_8 \rightarrow \mathcal{Q} \times \mathbb{R}^m$ by letting

$$\Gamma_9(\alpha, h_1, v_1, \beta, \gamma, \eta, \mathbf{u}, \mathbf{c}, \mu, \zeta, h_2, v_2) = v_2 + \Big(\int_0^1 h_2(t)\,dt\Big) \cdot v_1 - \Big(h_2 * (\beta - \zeta)\Big)(1)\,.$$

We have thus defined nine set-valued maps

$$\begin{aligned}\Gamma_1 &\in \mathrm{REG}(\mathcal{A}(C(r));\mathcal{X}_1)\,,\\ \Gamma_2 &\in C^0(\,\mathcal{X}_1\,;\,\mathcal{X}_2\,)\,,\\ \Gamma_3 &\in C^0(\,\mathcal{X}_2\,;\,\mathcal{X}_3\,)\,,\\ \Gamma_4 &\in C^0(\,\mathcal{X}_3\,;\,\mathcal{X}_4\,)\,,\\ \Gamma_5 &\in \mathrm{REG}(\mathcal{X}_4;\mathcal{X}_5)\,,\\ \Gamma_6 &\in C^0(\,\mathcal{X}_5\,;\,\mathcal{X}_6\,)\,,\\ \Gamma_7 &\in C^0(\,\mathcal{X}_6\,;\,\mathcal{X}_7\,)\,,\\ \Gamma_8 &\in \mathrm{REG}(\mathcal{X}_7;\mathcal{X}_8)\,,\\ \Gamma_9 &\in C^0(\,\mathcal{X}_8\,;\,\mathcal{Q} \times \mathbb{R}^m\,)\,.\end{aligned}$$

(The regularity of Γ_1, Γ_5 and Γ_8 has already been proved, cf. (54), (67), (69). The continuity of Γ_2, Γ_3, Γ_4, Γ_6 and Γ_7 follows trivially from their definitions. The continuity of Γ_9 follows from Lemma 2.1 because, if $\{\xi^j\}_{j\in\mathbb{N}}$ is a sequence of points in $C^0(\,\mathcal{X}_8\,;\,\mathcal{Q} \times \mathbb{R}^m\,)$ that converges in $C^0(\,\mathcal{X}_8\,;\,\mathcal{Q} \times \mathbb{R}^m\,)$ to a limit ξ^∞, and we write

$$\xi^j = (\alpha^j, h_1^j, v_1^j, \beta^j, \gamma^j, \eta^j, \mathbf{u}^j, \mathbf{c}^j, \mu^j, \zeta^j, h_2^j, v_2^j)\,,$$

then the β_j belong to $\mathcal{B}$ and the ζ_j belong to $\tilde{\mathcal{Z}}$, so the sequence $\{\beta^j - \zeta^j\}_{j\in\mathbb{N}}$ is uniformly Lipschitz, and then the facts that $\beta^j \rightarrow \beta^\infty$, $\zeta^j \rightarrow \zeta^\infty$, $h_2^j \rightarrow h_2^\infty$, imply that $h_2^j * (\beta^j - \zeta^j) \rightarrow h_2^\infty * (\beta^\infty - \zeta^\infty)$.)

On the other hand,

$$G = \Gamma_9 \circ \Gamma_8 \circ \Gamma_7 \circ \Gamma_6 \circ \Gamma_5 \circ \Gamma_4 \circ \Gamma_3 \circ \Gamma_2 \circ \Gamma_1\,.$$

Therefore G is regular, and our proof is complete. ◇

5. The open mapping theorem

We now show that path-integral generalized differentials have the directional open mapping property.

Theorem 5.1 *Assume that* $n, m \in \mathbb{Z}_+$, $F \in \mathcal{SVM}(\mathbb{R}^n, \mathbb{R}^m)$, $\bar{w} \in \mathbb{R}^m$, *C is a polyhedral convex cone in* $\mathbb{R}^n$, Λ *belongs to* $PIGD(F;0,0;C)$, *and* $\bar{w} \in \bigcap_{L\in\Lambda} \mathrm{Int}(LC)$. *Then there exists a closed convex cone D in* $\mathbb{R}^m$ *such that* $\bar{w} \in \mathrm{Int}(D)$, *having the property that for every* $\delta \in]0,\infty[$ *there exists an* $\varepsilon(\delta) \in]0,\infty[$ *such that*

$$\boxed{D \cap \{y \in \mathbb{R}^m : \|y\| \le \varepsilon(\delta)\} \subseteq F(C \cap \{x \in \mathbb{R}^n : \|x\| \le \delta\})}\,.$$

Proof. Let us assume that $\bar{w} \neq 0$. Pick a closed convex cone $\hat{D}$ in $\mathbb{R}^n$ such that $\bar{w} \in \mathrm{Int}(\hat{D})$, a compact neighborhood Λ' of Λ such that $\hat{D} \subseteq LC$ for every $L \in \Lambda'$, and a continuous map $\Lambda' \times \hat{D} \ni (L, y) \mapsto \eta(L, y) \in C$ which is positively homogeneous of degree 1 with respect to y and such that $L \cdot \eta(L, y) = y$ for all $(L, y) \in \Lambda' \times \hat{D}$. Pick θ such that $0 < \theta < \|\bar{w}\|$ and the ball $\{y \in \mathbb{R}^m : \|y - \bar{w}\| \leq 2\theta\}$ is entirely contained in $\hat{D}$. Let D be the smallest closed convex cone that contains the ball $\{y \in \mathbb{R}^m : \|y - \bar{w}\| \leq \theta\}$. Then D satisfies:

$$\left(y \in D \wedge \|z\| \leq \frac{\theta\|y\|}{\|\bar{w}\| + \theta}\right) \Rightarrow y + z \in \hat{D}\,. \tag{70}$$

(Indeed, if $y = 0$ then $z = 0$, so $y + z = 0 \in \hat{D}$. Assume that $y \neq 0$. Then we can write $y = su$ with $s > 0$ and $\|u - \bar{w}\| \leq \theta$. Then $\|u\| \leq \|\bar{w}\| + \|u - \bar{w}\| \leq \|\bar{w}\| + \theta$. Therefore $\|y\| \leq s(\|\bar{w}\|+\theta)$, so $s \leq \frac{\|y\|}{\|\bar{w}\|+\theta}$. Furthermore, $y + z = s\tilde{u}$, where $\tilde{u} = u + \frac{z}{s}$. Then

$$\|\tilde{u}-\bar{w}\| = \|u-\bar{w}+\frac{z}{s}\| \leq \|u-\bar{w}\|+\frac{\|z\|}{s} \leq \theta+\frac{\|z\|}{\|y\|}(\|\bar{w}\|+\theta)\,,$$

so $\|\tilde{u} - \bar{w}\| \leq 2\theta$, and then $y + z \in \hat{D}$.)

Let $M = \sup\{\|\eta(L, y)\| : L \in \Lambda', y \in \hat{D}, \|y\| \leq 1\}$. Then $\|\eta(L, y)\| \leq M\|y\|$ whenever $L \in \Lambda'$ and $y \in \hat{D}$.

Fix a $\delta \in]0, \infty[$. Let δ' be such that $0 < \delta' \leq \delta$, $\Lambda^{\delta'} \subseteq \Lambda'$, $2M\delta' \leq \frac{\theta}{\|\bar{w}\|+\theta}$. Then choose $R \in \mathbb{R}$ such that $R > 0$ and a family $\{G_r : 0 < r \leq R\}$ of regular set-valued maps $G_r : \mathcal{A}(C(r)) \twoheadrightarrow C^0(\,[0, 1]\,; \mathbb{R}^{m\times n}\,) \times \mathbb{R}^m$ such that

(a) $h(t) \in \Lambda^{\delta'}$ and $\|v\| \leq \delta' r$ whenever $\alpha \in \mathcal{A}(C(r))$, $(h, v) \in G_r(\alpha)$, $t \in [0, 1]$,

(b) $\mathrm{Gr}(\Phi_{G_r}) \subseteq \mathrm{Gr}(F)$.

By making R smaller, if necessary, we may assume that $R \leq \delta$.

We choose ε such that $2M\varepsilon < R$, and show that this choice of ε satisfies our requirements.

Fix y such that $0 < \|y\| \leq \varepsilon$. Let $\rho = \|y\|$, and choose $r = 2M\rho$. Then $r < R$. We will show that there is an $x \in C$ such that $\|x\| \leq \delta$ and $y \in F(x)$. For this purpose, it suffices to find a triple $(\alpha, h, v) \in \mathrm{Gr}(G_r)$ such that $v + \int_0^1 h(t) \cdot \dot{\alpha}(t)\, dt = y$ and $\|\alpha(1)\| \leq \delta$. The first equality, in turn, will follow if α satisfies

$$h(t) \cdot \dot{\alpha}(t) = y - v \quad \text{for a.e.} \quad t\,, \tag{71}$$

as well as $\|\alpha(1)\| \leq \delta$. If $y - v \in D$, then (71) will follow if $\dot{\alpha}(t) = \eta(h(t), y - v)$ for a.e. t, i.e., if

$$\alpha(t) = \int_0^t \eta(h(s), y - v)\, ds \quad \text{for all} \quad t\,. \tag{72}$$

Let Σ be the set-valued map on $\mathcal{A}(C(r))$ that asigns to each $\alpha \in \mathcal{A}(C(r))$ the set $\Sigma(\alpha)$ of all paths $\beta = \beta_{\alpha,h,v}$ for all $(h, v) \in G_r(\alpha)$, where $\beta_{\alpha,h,v}(t) = \int_0^t \eta(h(s), y - v)\, ds$. Then Σ is well defined and takes values in $\mathcal{A}(C(r))$. (*Proof.* Let $\alpha \in \mathcal{A}(C(r))$ and $(h, v) \in G_r(\alpha)$. Then $\|v\| \leq \delta' r = 2M\delta'\rho = 2M\delta'\|y\| \leq \frac{\theta\|y\|}{\|\bar{w}\|+\theta}$. Therefore (70) implies that $y - v \in \hat{D}$. Since $h(t) \in \Lambda^{\delta'} \subseteq \Lambda'$ for each t, $\eta(h(t), y - v)$ is defined for each t. Since the map $t \mapsto \eta(h(t), y - v)$ is continuous, $\beta_{\alpha,h,v}$ is well defined. Moreover, $\dot{\beta}_{\alpha,h,v}(t) = \eta(h(t), y - v) \in C$. On the other hand, $\|\dot{\beta}_{\alpha,h,v}(t)\| = \|\eta(h(t), y - v)\| \leq M\|y - v\|$. But $\|y - v\| \leq \|y\| + \|v\| \leq \rho + \delta' r$, and this implies that $\|\dot{\beta}_{\alpha,h,v}(t)\| \leq M\rho + M\delta' r$. But $M\rho = \frac{r}{2}$, and we know that $M\delta' r \leq \frac{\theta r}{2(\|\bar{w}\|+\theta)} < \frac{r}{2}$. So $\|\dot{\beta}_{\alpha,h,v}(t)\| \leq r$. Therefore $\dot{\beta}_{\alpha,h,v}(t) \in C(r)$. Hence $\beta_{\alpha,h,v} \in \mathcal{A}(C(r))$.)

It is easy to see that Σ is regular. Hence Σ is a regular map from the compact convex set $\mathcal{A}(C(r))$ to itself. By the obvious extension of Schauder's fixed point theorem to regular maps, Σ has a fixed point α_*. Then, for some $(h, v) \in G_r(\alpha_*)$, we have $h(t) \cdot \dot{\alpha}_*(t) = y - v$ for almost all t. Hence if we let $x = \alpha(1)$, we have $y = v + \int_0^1 h(t) \cdot \dot{\alpha}_*(t)\, dt$, so that $y \in \Phi_{G_r}(x)$, and then $y \in F(x)$. Finally the fact that $\alpha \in \mathcal{A}(C(r))$ implies that $\|x\| \leq r = 2M\rho \leq 2M\varepsilon < R \leq \delta$. This completes the proof. ♢

6. A property of polyhedral cones

Let C be a closed convex cone in a finite-dimensional normed linear space X. Let S_C be the linear subspace of X spanned by C, so the interior $\mathrm{Int}_{S_C}(C)$ of C relative to S_C is nonempty.

Suppose we are trying to add several vectors $u_1, \ldots, u_m$ belonging to C so as to obtain a vector $w \in C$, but instead of achieving this desired result we produce a sum $\tilde{w}$, which is "larger" than the target value w in the sense of the partial ordering induced by the cone, i.e., such that $\tilde{w} = w + e$ for some $e \in C$. We would like to correct this error e by subtracting correction terms c_i—not necessarily belonging to C—to the vectors u_i, in such a way that the new vectors $v_i = u_i - c_i$ belong to C and add up to w. Moreover, we want to be able to do this while keeping the total error norm—i.e., the sum $E = \|c_1\| + \ldots + \|c_m\|$—bounded by a fixed constant $\bar{k}$, independently of m, the u_i's, and w. And, in addition, we want the c_i's to be bounded by the u_i's. Equivalently, we want to subtract off the "error" e from the sum $u_1 + \ldots + u_m$ by expressing e as a sum $e = c_1 + \ldots + c_m$ in such a way that after subtracting each c_i from its corresponding u_i the resulting vectors still belong to C, and we want to do this with a bound on the sum E, and with the the c_i's bounded by the u_i's.

Whether this "error correction" is possible for a particular choice of $e \in C$ is a property—the "error-correcting property," abbreviated "ECP"—of the pair (C, e). Naturally, the ECP holds if $e = 0$ but, as we shall see, it is important for the ECP to hold for some $e \in \mathrm{Int}_{S_C}(C)$, so the only case when $e = 0$ is an acceptable choice is when C is a linear subspace, i.e., $C = S_C$.

The purpose of this subsection is to show that e can be chosen to belong to $\mathrm{Int}_{S_C}(C)$ if C is a polyhedral cone. We will also exhibit an example showing that for more general cones it may happen that there do not exist any $e \in C\backslash\{0\}$ such that (C, e) has the ECP.

It will be clear, both from the proof of the positive result for polyhedral cones, and from the counterexample for a cone which is not polyhedral, that the crucial property of polyhedral cones, for the purpose of establishing the ECP, is the fact that *the set of all possible tangent cones T_xC, as x varies over C, is finite.*

First, we give a formal definition of the ECP.

Definition 6.1 Let X be a finite-dimensional normed linear space, and let C be a closed convex cone in X.

Let $e \in C$. Then

1. If $\bar{k} \in \mathbb{R}$, $\bar{k} \geq 0$, we say that the pair (C, e) has the *error-correction property*—abbr. "ECP"—*with constant* $\bar{k}$ if

(ECP) for every $m \in \mathbb{N}$, every $w \in C$, and every m-tuple $(u_1, \ldots, u_m)$ of members of C such that

$$u_1 + \ldots + u_m = w + e$$

there exist vectors $c_1, \ldots, c_m$ in X such that

$$\begin{aligned} u_i - c_i &\in C \quad \text{for } i = 1, \ldots, m\,, && (73)\\ c_1 + \ldots + c_m &= e\,, && (74)\\ \|c_1\| + \ldots + \|c_m\| &\leq \bar{k}\,, && (75)\\ \|c_i\| &\leq \bar{k}\|u_i\| \quad \text{for } i = 1, \ldots, m\,. && (76) \end{aligned}$$

2. We say that (C, e) has the *error-correction property*—abbr. "ECP"—if there exists a constant $\bar{k}$ such that (C, e) has the error-correction property with constant $\bar{k}$.

3. We say that C is *ECP-good* if there exists a vector e belonging to the interior $\mathrm{Int}_{S_C}(C)$ of C relative to the linear span $S_C = \mathrm{span}(C)$ such that the pair (C, e) has the ECP. ♢

Remark 6.2 The ECP property does not depend on the choice of the norm on X. Moreover, *the ECP property is an intrinsic property of the cone C, and is independent of the space X in which C is embedded.* (Precisely, this means that C *is ECP-good as a cone in X if and only if it is ECP-good as a cone in S_C*. The proof is straightforward. First, it is trivial that if C is ECP-good as a cone in S_C then it is ECP-good as a cone in X. To establish the reverse implication we observe that, whenever m, w, $u_1, \ldots, u_m$ are given as in Definition 6.1, if it is possible to choose $c_1, \ldots, c_m$ in X such that (73), (74), (75), (76) hold, then the c_i can also be chosen to belong to S_C, since one can replace the c_i by their projections $\Pi(c_i)$, where Π is the orthogonal projection from X onto S_C relative to some inner product on X.) ♢

Remark 6.3 If X, C, e, w, m, $u_1, \ldots, u_m$ are as in Definition 6.1, then it is always possible to choose correction terms c_i such that $c_1 + \ldots + c_m = e$ and $u_i - c_i \in C$. For example, one can choose $c_i = u_i - \lambda_i w$, where the λ_i are arbitrary nonnegative numbers such that $\lambda_1 + \ldots + \lambda_m = 1$. Then, if we let $v_i = u_i - c_i$, we have $v_i = \lambda_i w$, so the v_i belong to C and $v_1 + \ldots + v_m = w$. It follows that $c_1 + \ldots + c_m = e$, so (73) and (74) are true. This, however, does not suffice to guarantee that the ECP holds, because the error bounds (75) and (76) need not be satisfied. ♢

It is easy to see that

Proposition 6.4 *If X is a finite-dimensional normed linear space, C is a closed convex cone in X, and $e = 0$, then (C, e) has the error-correction property. In particular, if C is a linear subspace of X, then C is ECP-good.*

Proof. If $e = 0$, and w, m, $u_1, \ldots, u_m$ are as in Definition 6.1, then we can choose $c_i = 0$, and all the conditions of the definition hold, with $\bar{k} = 0$.

If C is a linear subspace, then $0 \in \mathrm{Int}_{S_C}(C)$, and $(C, 0)$ has the ECP. Therefore C is ECP-good. ♢

Remark 6.5 The following example shows that *not all closed convex cones are ECP-good.*

Let $X = \mathbb{R}^3$, and let C be the cone

$$C = \{\, (x, y, z) \in \mathbb{R}^3 : z \geq 0 \wedge z^2 \geq x^2 + y^2 \,\}\,.$$

Then C *is not ECP-good.* To see this, we will actually show that *the pair (C, e) does not have the ECP for any $e \in C$ such that $e \neq 0$.*

Fix $e \in C$ such that $e \neq 0$. Let $e = (\alpha, \beta, \gamma)$, so $\gamma \geq 0$, $\gamma^2 \geq \alpha^2 + \beta^2$, and $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. After a rotation of the x and y axes, we may assume that $\alpha = 0$ and $\beta \geq 0$. Then $e = (0, \beta, \gamma)$, $\gamma \geq \beta \geq 0$, and $\gamma > 0$.

For $N \in \mathbb{R}$, $N > 0$, let $w^N = (2N, 0, 2N)$, so $w^N \in C$. Then all the w^N belong to the ray

$$R = \{\, (r, 0, r) : r \geq 0 \,\}\,.$$

Clearly, $R \subseteq C$, and R is an extreme ray (that is, if $z_1 \in C$, $z_2 \in C$, and $z_1 + z_2 \in R$, then $z_1 \in R$ and $z_2 \in R$).

Let $m = 2$, and let

$$u_1^N = \Big(N, g_N, N + \frac{\gamma}{2}\Big)\,, \quad u_2^N = \Big(N, h_N, N + \frac{\gamma}{2}\Big)\,,$$

where

$$g_N = \frac{1}{2}\sqrt{\gamma^2 + 4\gamma N}\,, \qquad h_N = \beta - g_N\,.$$

Then

$$\begin{aligned} N^2 + g_N^2 &= N^2 + \frac{1}{4}(\gamma^2 + 4\gamma N)\\ &= N^2 + \gamma N + \frac{\gamma^2}{4}\\ &= \Big(N + \frac{\gamma}{2}\Big)^2\,, \end{aligned}$$

and

$$\begin{aligned} N^2 + h_N^2 &= N^2 + (\beta - g_N)^2\\ &= N^2 + \beta^2 + g_N^2 - 2\beta g_N\\ &= \Big(N + \frac{\gamma}{2}\Big)^2 + \beta^2 - 2\beta g_N\\ &= \Big(N + \frac{\gamma}{2}\Big)^2 + \beta(\beta - 2g_N)\\ &\leq \Big(N + \frac{\gamma}{2}\Big)^2\,, \end{aligned}$$

where the last inequality follows because

$$0 \leq \beta \leq \gamma \leq \sqrt{\gamma^2 + 4\gamma N} = 2g_N\,.$$

Therefore $u_1^N \in C$ and $u_2^N \in C$. Moreover, it is clear that

$$u_1^N + u_2^N = w^N + e\,.$$

Fix N, and let c_1, c_2 be such that $v_1 = u_1^N - c_1 \in C$, $v_2 = u_2^N - c_2 \in C$, and $c_1 + c_2 = e$. Then $v_1 + v_2 = w^N$. Since the ray R is extreme, the vectors v_1 and v_2 must belong to R. Therefore, if we write $v_i = (v_{i,1}, v_{i,2}, v_{i,3})$, $c_i = (c_{i,1}, c_{i,2}, c_{i,3})$, we must have $v_{i,2} = 0$, and then $c_{1,2} = g_N$, $c_{2,2} = h_N$. Therefore

$$\|c_1\| + \|c_2\| \geq g_N + |h_N|\,.$$

Since N is arbitrary, and the quantity $g_N + |h_N|$ goes to $+\infty$ as $N \uparrow +\infty$ (because $\gamma > 0$), the constant $\bar{k}$ of Definition 6.1 cannot exist. Therefore the pair (C, e) does not have the ECP. ♢

Recall that

- A *convex polyhedron* in a linear space X is a subset of X which is the convex hull of a finite set.
- The *convex cone generated by a subset* S of a linear space X is the smallest convex cone in X that contains S.
- The *closed convex cone generated by a subset* S of a linear space X is the smallest closed convex cone in X that contains S.
- A *polyhedral cone* in X is a cone in X which is the convex cone generated by a finite subset of X.
- If K is a compact convex subset of a normed linear space X, and $x \in X$, the *tangent cone to* K *at* x is the closed convex cone in X generated by the set of all vectors $y - x$, $y \in K$.

If X, K, x are as above, we use T_xK to denote the tangent cone to K at x. The following facts are then well known and easy to prove.

Lemma 6.6 *Let X be a normed linear space. Then*

(1) Every convex polyhedron in X is compact.

(2) Every polyhedral cone in X is closed.

(3) If C is a polyhedral cone F, and F is a finite subset of X such that C is the convex cone generated by F, then there exists a positive constant κ such that every $v \in C$ can be expressed as a linear combination

$$v = \sum_{f \in F} \alpha_f f\,, \quad 0 \le \alpha_f \le \kappa\|v\| \quad \text{for all} \quad f \in F\,. \tag{77}$$

(4) If K is a convex polyhedron, and $x \in K$, then

(4.a) T_xK is a polyhedral cone,

(4.b) there exists a $\rho \in \mathbb{R}$ such that $\rho > 0$ and $x + v \in K$ whenever $v \in T_xK$ and $\|v\| \le \rho$.

(5) If K is a convex polyhedron, then $\{T_xK : x \in K\}$ is a finite set. $\diamondsuit$

Theorem 6.7 *Let X be a finite-dimensional real linear space, and let C be a polyhedral cone in X. Then C is ECP-good.*

Proof. Recall that a cone D is *pointed* if there is no nonzero vector v such that $v \in D$ and $-v \in D$.

We first show that

(*) *we may assume, without loss of generality, that C is pointed and has nonempty interior.*

To prove (*), we assume that every pointed polyhedral cone with nonempty interior in any finite-dimensional real linear space Y is ECP-good, and prove that if C is an arbitrary polyhedral cone in X then C is ECP-good. For this purpose, since the ECP property does not depend on the choice of norm, we may assume that X is Euclidean.

Let C be a polyhedral cone in $\mathbb{R}^n$. Let E be the "edge" of C, that is, the set of all $v \in C$ such that $-v \in C$. Then E is a linear subspace of X. Let E' be the orthogonal complement of E in X, and let $C' = C \cap E'$. Then C' is a closed convex cone. Moroever, C' is clearly polyhedral and pointed.

Let E'' be the linear span of C', so $E'' \subseteq E'$. Then, as a subset of E'', C' is a pointed polyhedral cone with nonempty interior. So C' is ECP-good. Let $e \in \mathrm{Int}_{E''}(C')$ be such that (C', e) has the ECP. We show that $e \in \mathrm{Int}_{S_C}(C)$ and (C, e) has the ECP.

First, we observe that $C \subseteq E + E''$. (*Proof.* If $v \in C$, then $v = w + w'$, $w \in E$, $w' \in E'$. Moreover, $-w \in E \subseteq C$, so $w' = v - w \in C + C$, and then $w' \in C$. Then $w' \in E' \cap C' = E''$, so $v \in E + E''$.)

Since $E \subseteq C$, the cone C is in fact equal to the sum $E + (C \cap E'')$. Also,

$$C \cap E'' = C \cap E'' \cap E' = C' \cap E'' = C'\,,$$

since $C' \subseteq E'' \subseteq E'$. So

$$C = E + C'\,.$$

This clearly implies that

$$S_C = E + E''\,.$$

Moreover, the sum $E + E''$ is orthogonal, so $E + E''$ is isomorphic to the product $E \times E''$, under the map $\mu : E \times E'' \to E + E''$ given by $\mu(v, v'') = v + v''$. Clearly, $\mu(E \times C') = E + C' = C$. Since $e \in \mathrm{Int}_{E''}(C')$ and $0 \in \mathrm{Int}_E(E)$, we have $(0, e) \in \mathrm{Int}_{E \times E''}(E \times C')$, so $\mu(0, e) \in \mathrm{Int}_{\mu(E \times E'')}(\mu(E \times C'))$, that is,

$$e \in \mathrm{Int}_{E+E''}(C) = \mathrm{Int}_{S_C}(C)\,.$$

Now let $\bar{k}$ be such that (C', e) has the ECP with constant $\bar{k}$. Let $m \in \mathbb{N}$, $w \in C$, $(u_1, \ldots, u_m) \in C^m$ be such that

$$u_1 + \ldots + u_m = w + e\,.$$

Write

$$u_j = \tilde{u}_j + u_j''\,, \quad \tilde{u}_j \in E\,, \quad u_j'' \in C'\,,$$

and also

$$w = \tilde{w} + w''\,, \quad \tilde{w} \in E\,, \quad w'' \in C'\,.$$

Then

$$\begin{aligned} \tilde{u}_1 + \ldots + \tilde{u}_m &= \tilde{w}\,, \\ u_1'' + \ldots + u_m'' &= w'' + e\,. \end{aligned}$$

Since (C', e) has the ECP with constant $\bar{k}$, there exist vectors $c_1, \ldots, c_m$ in E'' such that

$$u_i'' - c_i \in C' \quad \text{for} \quad i = 1, \ldots, m\,, \tag{78}$$
$$c_1 + \ldots + c_m = e\,, \tag{79}$$
$$\|c_1\| + \ldots + \|c_m\| \le \bar{k}\,, \tag{80}$$
$$\|c_i\| \le \bar{k}\|u_i''\| \quad \text{for} \quad i = 1, \ldots, m\,. \tag{81}$$

But then $u_i - c_i = \tilde{u}_i + u_i'' - c_i \in C$, and $\|u_i''\| \le \|u_i\|$, because the sum $E + E''$ is orthogonal. Therefore

$$u_i - c_i \in C \quad \text{for} \quad i = 1, \ldots, m\,, \tag{82}$$
$$\|c_i\| \le \bar{k}\|u_i\| \quad \text{for} \quad i = 1, \ldots, m\,, \tag{83}$$

and (79), (80) hold. Hence (C, e) has the ECP, and the proof of (*) is complete.

From now on, we assume that C is pointed and has a nonempty interior in X. We choose an arbitrary interior point e of C, and prove that (C, e) has the ECP.

Let $\bar{\zeta} \in X^\dagger$ be a nontrivial linear functional such that

$$\bar{\zeta}(v) > 0 \quad \text{whenever} \quad v \in C\,,\ v \ne 0\,.$$

(*Proof of the existence of* $\bar{\zeta}$. This follows from the fact that C is pointed. Indeed, let $C^\dagger$ be the polar cone of C. Then $C^\dagger$ has nonempty interior in $X^\dagger$, because otherwise there would exist a nontrivial linear functional ξ on $X^\dagger$ that vanishes identically on $C^\dagger$, and this would yield—using the canonical identification of X with $X^{\dagger\dagger}$ and the identity $C^{\dagger\dagger} = C$—a nonzero vector $v \in X$ such that $\zeta(v) = 0$ for all $\zeta \in C^\dagger$. But then $v \in C$ and $-v \in C$, contradicting the fact that (C, e) is pointed. If ζ is an interior point of $C^\dagger$, and $v \in C$, $v \neq 0$, then $\zeta(v) < 0$, because we know that $\zeta(v) \leq 0$, and if $\zeta(v) = 0$ then the nontrivial linear map $X^\dagger \ni z \to z(v) \in \mathbb{R}$ would have a local maximum at ζ. So $\zeta(v) < 0$ for all $v \in C\backslash\{0\}$, and then $\bar{\zeta} = -\zeta$ has the desired property.)

Since $e \neq 0$ and $e \in C$, we may normalize $\bar{\zeta}$ so that $\bar{\zeta}(e) = 1$. Since the ECP property does not depend on the norm, we are entitled to assume that X is Euclidean, and we may choose the inner product so that $\|e\| = 1$ and e is orthogonal to the kernel of $\bar{\zeta}$. Under the canonical identification of X with $X^\dagger$ arising from the inner product, $\bar{\zeta}$ will then correspond to a vector $\bar{z} \in X$ such that $e \perp \bar{z}^\perp$. But then $e = r\bar{z}$ for some $r \in \mathbb{R}$, and the facts that $\bar{\zeta}(e) = 1$ and $\|e\| = 1$ imply $1 = \langle \bar{z}, e\rangle = r\|e\|^2 = r$, so $r = 1$ and then $\bar{z} = e$.

Let $n = \dim(X)$. We may assume that $n > 0$, because if $n = 0$ then our conclusion is trivial. Write $n = \nu + 1$, $\nu \in \mathbb{Z}_+$. By choosing an orthonormal basis of X whose first member is e, we may identify X with $\mathbb{R} \times \mathbb{R}^\nu$ in such a way that $e = (1, 0)$. Then C is a polyhedral cone in $\mathbb{R} \times \mathbb{R}^\nu$, having the property that

$$(r, x) \in C\backslash\{0\} \Longrightarrow r > 0\,.$$

(This follows because, if $v = (r, x) \in C\backslash\{0\}$, then $r = \langle e, v\rangle = \langle \bar{z}, v\rangle = \bar{\zeta}(v) > 0$.)

Since C is polyhedral, we may fix a finite set $P = \{p_1, \ldots, p_N\}$ of points of C such that C is the smallest convex cone containing P, i.e., the set of all linear combinations $v = \alpha_1 p_1 + \ldots + \alpha_N p_N$ such that $\alpha_i \geq 0$ for $i = 1, \ldots, N$. Clearly, we may assume that all the p_i are nonzero, and then we can write

$$p_i = (\rho_i, q_i)\,, \quad \rho_i > 0\,, \quad q_i \in \mathbb{R}^\nu\,.$$

Then, after multiplication by ρ_i^{-1}, we may assume that $\rho_i = 1$ for $i = 1, \ldots, N$. Let

$$K = \{\, x \in \mathbb{R}^\nu : (1, x) \in C\}\,, \quad Q = \{q_1, \ldots, q_N\}\,.$$

It then follows that

$$K = \text{convex hull of } Q\,, \tag{84}$$

$$0 \in \mathrm{Int}_{\mathbb{R}^\nu} K\,, \tag{85}$$

$$C = \Big\{\, (r, rx) : x \in K\,,\, r \geq 0 \Big\}\,. \tag{86}$$

Let $\mathbb{T}$ be the set of all the tangent cones $T_x K$, as x varies over all points of K. Then $\mathbb{T}$ is a finite set by Lemma 6.6. Let M be the number of members of $\mathbb{T}$. For each $k \in \mathbb{N}$, let $\mathcal{T}_k$ be the set of all k-tuples $\mathbf{C} = (C_1, \ldots, C_k)$ of different members of $\mathbb{T}$ such that

$$C_1 + \ldots + C_k = \mathbb{R}^\nu\,. \tag{87}$$

Then $\mathcal{T}_k = \emptyset$ if $k > M$. Let $\mathcal{T} = \cup_{k\in\mathbb{N}} \mathcal{T}_k$. Then the set $\mathcal{T}$ is finite.

Choose, for each $\mathbf{C} = (C_1, \ldots, C_k) \in \mathcal{T}$, a continuous map

$$\mathbb{R}^\nu \ni x \mapsto \Theta^{\mathbf{C}}(x) = (\theta_1^{\mathbf{C}}(x), \ldots, \theta_k^{\mathbf{C}}(x)) \in C_1 \times \cdots \times C_k$$

which is positively homogeneous of degree 1 and satisfies

$$\theta_1^{\mathbf{C}}(x) + \cdots + \theta_k^{\mathbf{C}}(x) = x \quad \text{for all} \quad x \in \mathbb{R}^\nu\,.$$

(*Proof that* $\Theta^{\mathbf{C}}$ *exists.* Fix $\mathbf{C}$. Let $(b_1, \ldots, b_\nu)$ be the canonical basis of $\mathbb{R}^n$, so b_i is the vector $(b_i^1, \ldots, b_i^\nu)$, where $b_i^j = 0$ if $i \neq j$, $b_i^i = 1$. Let $b_0 = -(b_1 + \cdots + b_\nu)$. Then every vector $x \in \mathbb{R}^\nu$ can be expressed in a unique way as an affine combination of $b_0, b_1, \ldots, b_\nu$, i.e., as a linear combination

$$x = \alpha_0 b_0 + \alpha_1 b_1 + \cdots + \alpha_\nu b_\nu\,,\ \ \alpha_0 + \alpha_1 + \cdots + \alpha_\nu = 1\,, \tag{88}$$

as can be seen by observing that, if $\tilde{b}_i = (b_i, 1) \in \mathbb{R}^{\nu+1}$, $\tilde{x} = (x, 1) \in \mathbb{R}^{\nu+1}$, then $(\tilde{b}_0, \tilde{b}_1, \cdots, \tilde{b}_\nu)$ is a basis of $\mathbb{R}^{\nu+1}$, and (88) holds if and only if

$$\tilde{x} = \alpha_0 \tilde{b}_0 + \alpha_1 \tilde{b}_1 + \cdots + \alpha_\nu \tilde{b}_\nu\,.$$

For $x \in \mathbb{R}^\nu$, let $\alpha_i(x)$, $i = 0, 1, \ldots, \nu$, be the unique coefficients α_i such that (88) holds. Then the functions $\alpha_i : \mathbb{R}^\nu \to \mathbb{R}$ are obviously continuous. Clearly,

$$\alpha_0(0) = \alpha_1(0) = \cdots = \alpha_\nu(0) = \frac{1}{\nu + 1}\,.$$

So the continuity of the α_i implies that there exists $r \in \mathbb{R}$ such that $r > 0$ and $\alpha_i(x) \geq 0$ whenever $i \in \{0, 1, \ldots, \nu\}$, $x \in \mathbb{R}^\nu$ and $\|x\| \leq r$. Define, for $i = 1, \ldots, \nu$,

$$\beta_i(x) = \begin{cases} \frac{\|x\|}{r} \cdot \alpha_i\Big(\frac{rx}{\|x\|}\Big) & \text{if} \quad x \in \mathbb{R}^\nu\backslash\{0\}\,, \\ 0 & \text{if} \quad x = 0 \in \mathbb{R}^\nu\,, \end{cases}$$

so the β_i are continuous real-valued functions on $\mathbb{R}^\nu$ that are everywhere nonnegative and positively homogeneous of degree 1. Using (87), write

$$b_i = c_{i,1} + c_{i,2} + \cdots + c_{i,k}\,,\ c_{i,j} \in C_j \text{ for, } j = 1, 2, \ldots, k\,.$$

Then, let $\theta_j(x) = \beta_0(x) c_{0,j} + \beta_1(x) c_{1,j} + \cdots + \beta_\nu(x) c_{\nu,j}$ for $x \in \mathbb{R}^\nu$, $j = 1, \ldots, k$. Then each θ_j is a continuous map from $\mathbb{R}^\nu$ to C_j, which is positively homogeneous of degree 1. Moreover, if $x \in \mathbb{R}^\nu$ and $x \neq 0$, then

$$\sum_{j=1}^k \theta_j(x) = \sum_{j=1}^k \sum_{i=1}^\nu \beta_i(x) c_{i,j} = \sum_{i=1}^\nu \beta_i(x) \Big(\sum_{j=1}^k c_{i,j}\Big)$$

$$= \sum_{i=1}^\nu \beta_i(x) b_i = \frac{\|x\|}{r} \sum_{i=1}^\nu \alpha_i\left(\frac{rx}{\|x\|}\right) b_i = \frac{\|x\|}{r} \cdot \frac{rx}{\|x\|} = x\,.$$

Obviously, the identity $\sum_{j=1}^k \theta_j(x) = x$ is also valid when $x = 0$. Therefore, if we let $\Theta^{\mathbf{C}}$ be the map $x \to (\theta_1(x), \ldots, \theta_k(x))$, then all the desired properties are satisfied.)

Let $\kappa_1 = \max\Big\{\, \|x\| : x \in K \Big\}$, $\kappa_2 = \sqrt{1 + \kappa_1^2}$. For each $\mathbf{C} = (C_1, \ldots, C_k) \in \mathcal{T}$, define

$$\kappa^{\mathbf{C}} = \sup\Big\{\, \|\theta_j^{\mathbf{C}}(x)\| : x \in \mathbb{R}^\nu\,,\, \|x\| = 1\,,\, j = 1, \ldots, k \Big\}\,.$$

Then define $\kappa_3 = \max\Big\{\, \kappa^{\mathbf{C}} : \mathbf{C} \in \mathcal{T} \Big\}$. We now choose the constant $\bar{k}$ by letting $\bar{k} = 1 + \kappa_2 + 2M\kappa_1\kappa_3$, and prove that the pair (C, e) has the ECP with constant $\bar{k}$.

Pick $m \in \mathbb{N}$, $w \in C$, and an m-tuple $(u_1, \ldots, u_m)$ of members of C such that

$$u_1 + \ldots + u_m = w + e\,, \tag{89}$$

and write

$$w = (R, R\omega)\,,\ \ R \geq 0\,,\ \ \omega \in K\,. \tag{90}$$

We have to find vectors $c_1, \ldots, c_m$ in X such that

$$u_i - c_i \in C \qquad \text{for } i = 1, \ldots, m, \tag{91}$$

$$c_1 + \ldots + c_m = e, \tag{92}$$

$$\|c_1\| + \ldots + \|c_m\| \leq \bar{k}, \tag{93}$$

$$\|c_i\| \leq \bar{k}\|u_i\| \quad \text{for } i = 1, \ldots, m. \tag{94}$$

For this purpose, we find a Lipschitz curve

$$J \ni s \to \gamma(s) = (c_1(s), \ldots, c_m(s)) \in (\mathbb{R}^n)^m\,, \qquad (95)$$

defined on $J = [0,1]$ and such that

$$c_i(0) = 0 \quad \text{for} \quad i = 1, \ldots, m\,, \qquad (96)$$

$$u_i - c_i(s) \in C \quad \text{for} \quad i = 1, \ldots, m\,, \quad s \in J\,, \qquad (97)$$

$$c_1(s) + \ldots + c_m(s) = se \quad \text{for} \quad s \in J\,, \qquad (98)$$

$$\langle e, \dot{c}_i(s)\rangle \geq 0 \quad \text{for} \quad i = 1, \ldots, m\,, \quad \text{a.e.} \quad s \in J\,, \qquad (99)$$

$$\|\dot{c}_1(s)\| + \ldots + \|\dot{c}_m(s)\| \leq \bar{k} \quad \text{for} \quad \text{a.e.} \quad s \in J\,. \qquad (100)$$

Let Γ be the set of all pairs (J, γ) such that (1) J is a subinterval of $[0,1]$, (2) $0 \in J$, and (3) γ is a Lipschitz curve of the form (95) such that (96), (97), (98), (99), (100) hold. Partially order Γ by letting $(J_1, \gamma_1) \preceq (J_2, \gamma_2)$ iff $J_1 \subseteq J_2$ and $\gamma_1 = \gamma_2 \lceil J$. It then follows from Zorn's Lemma that Γ has a member (J, γ) which is maximal with respect to $\preceq$. Let $\bar{s} = \sup J$, so $0 \leq \bar{s} \leq 1$, and either $J = [0, \bar{s}[$ or $J = [0, \bar{s}]$. If $J = [0, \bar{s}[$ then, since γ is Lipschitz, it can be extended to a curve $\hat{\gamma}$ defined on $[0, \bar{s}]$, and then the pair $([0, \bar{s}], \hat{\gamma})$ satisfies $([0, \bar{s}], \hat{\gamma}) \in \Gamma$, $(J, \gamma) \preceq ([0, \bar{s}], \hat{\gamma})$, and $(J, \gamma) \neq ([0, \bar{s}], \hat{\gamma})$, thus contradicting the maximality of (J, γ). So the possibility that $J = [0, \bar{s}[$ is excluded. Hence $J = [0, \bar{s}]$.

We now prove that $\bar{s} = 1$. We do this by showing that if $\bar{s} < 1$ then γ can be extended to a curve $\hat{\gamma} : [0, \bar{s} + \varepsilon]$ for some positive ε such that $\varepsilon \leq 1 - \bar{s}$, in such a way that the pair $([0, \bar{s} + \varepsilon], \hat{\gamma})$ still belongs to Γ.

Let $\bar{y} = w + (1 - \bar{s})e$. Then $\bar{y} \in \text{Int}_X(C)$, because $w \in C$, $e \in \text{Int}_X(C)$ and $\bar{s} < 1$. Write $\bar{y} = (\bar{r}, \bar{r}\bar{x})$, with $\bar{r} \geq 0$ and $\bar{x} \in K$. Then $\bar{y} = (\bar{r}, R\omega)$, $\bar{r} = R + 1 - \bar{s} > 0$, $\bar{x} = \frac{R}{\bar{r}}\omega$, and $\bar{x} \in \text{Int}_{\mathbb{R}^\nu} K$. Next, write

$$u_i - c_i(\bar{s}) = \bar{v}_i = (\bar{r}_i, \bar{r}_i\bar{x}_i),$$

with $\bar{r}_i \geq 0$ and $\bar{x}_i \in K$ for $i = 1, \ldots m$. Then

$$\begin{aligned} \bar{v}_1 + \cdots + \bar{v}_m &= \bar{y}\,, \\ \bar{r}_1 + \cdots + \bar{r}_m &= \bar{r}\,, \\ \bar{r}_1\bar{x}_1 + \cdots + \bar{r}_m\bar{x}_m &= \bar{r}\bar{x} = R\omega\,, \end{aligned}$$

so

$$\bar{x} = \bar{\rho}_1\bar{x}_1 + \cdots + \bar{\rho}_m\bar{x}_m\,, \qquad (101)$$

where

$$\bar{\rho}_i = \frac{\bar{r}_i}{\bar{r}} = \frac{\bar{r}_i}{R + 1 - \bar{s}}\,,$$

so that

$$\bar{\rho}_i \geq 0 \quad \text{for} \quad i = 1, \ldots m\,,$$

and

$$\bar{\rho}_1 + \cdots + \bar{\rho}_m = 1\,.$$

It then follows that

$$\bar{\rho}_1 T_{\bar{x}_1} K + \cdots + \bar{\rho}_m T_{\bar{x}_m} K = \mathbb{R}^\nu\,. \qquad (102)$$

(*Proof.* Using the fact that $\bar{x} \in \text{Int}_{\mathbb{R}^\nu} K$, pick a real number δ such that $\delta > 0$ and $\|y\| \leq \delta \Longrightarrow \bar{x} + y \in K$. If $\|y\| \leq \delta$, and we let $z = \bar{x} + y$, then $z \in K$, and

$$\begin{aligned} y &= z - \bar{x} = z - \sum_{i=1}^m \bar{\rho}_i x_i = \sum_{i=1}^m \bar{\rho}_i (z - \bar{x}_i) \\ &\in \bar{\rho}_1 T_{\bar{x}_1} K + \cdots + \bar{\rho}_m T_{\bar{x}_m} K\,. \end{aligned}$$

Therefore the cone $\bar{\rho}_1 T_{x_1} K + \cdots + \bar{\rho}_1 T_{x_m} K$ contains a neighborhood of 0 in $\mathbb{R}^\nu$, and (101) follows.)

Clearly, (102) remains true if we eliminate from the sum those terms $\bar{\rho}_i T_{\bar{x}_i} K$ for which $\bar{\rho}_i = 0$. Once this is done, we can replace each remaining term $\bar{\rho}_i T_{\bar{x}_i} K$ by $T_{\bar{x}_i} K$, because the $T_{\bar{x}_i} K$ are cones. Finally, we can eliminate repeated terms, after which we will be left with at most M terms.

Hence there exist $\mu \in \mathbb{N}$ and $\mathbf{i} = (i_1, \ldots, i_\mu) \in \mathbb{N}^\mu$ such that (i) $1 \leq i_1 < i_2 < \cdots < i_\mu \leq m$, (ii) $T_{\bar{x}_{i_j}} K \neq T_{\bar{x}_{i_\ell}} K$ whenever $j \neq \ell$ and $1 \leq j, \ell \leq \mu$, (iii) $\bar{\rho}_{i_j} > 0$ whenever $1 \leq j \leq \mu$, and (iv) $T_{\bar{x}_{i_1}} K + \cdots + T_{\bar{x}_{i_\mu}} K = \mathbb{R}^\nu$.

Now let

$$y(h) = \bar{y} - he = \Big(\bar{r} - h, R\omega\Big) \quad \text{for} \quad h \geq 0\,.$$

Then

$$\begin{aligned} y(h) &= \Big(\bar{r} - h, (\bar{r} - h)\frac{R}{\bar{r} - h}\omega\Big) \\ &= \bigg(\bar{r} - h, (\bar{r} - h)\Big(\bar{x} + \frac{R}{\bar{r} - h}\omega - \bar{x}\Big)\bigg) \\ &= \bigg(\bar{r} - h, (\bar{r} - h)\Big(\bar{x} + \frac{R}{\bar{r} - h}\omega - \frac{R}{\bar{r}}\omega\Big)\bigg) \\ &= \bigg(\bar{r} - h, (\bar{r} - h)\Big(\bar{x} + \Big(\frac{R}{\bar{r} - h} - \frac{R}{\bar{r}}\Big)\omega\Big)\bigg) \\ &= \bigg(\bar{r} - h, (\bar{r} - h)\Big(\bar{x} + \frac{hR}{\bar{r}(\bar{r} - h)}\omega\Big)\bigg) \\ &= \Big(\bar{r} - h, (\bar{r} - h)x(h)\Big)\,, \end{aligned}$$

where

$$x(h) = \bar{x} + \frac{hR}{\bar{r}(\bar{r} - h)}\omega\,.$$

Let $\mathbf{T}$ be the μ-tuple

$$\mathbf{T} = \Big(T_{\bar{x}_{i_1}} K\,, T_{\bar{x}_{i_2}} K\,, \ldots\,, T_{\bar{x}_{i_\mu}} K\Big)\,.$$

Then $\mathbf{T} \in \mathcal{T}_\mu$, so $\mu \leq M$. Let

$$\mathbb{R}^\nu \ni x \to \Theta(x) = (\theta_1(x), \ldots, \theta_\mu(x))$$

be the map $\Theta^{\mathbf{T}}$, so $\Theta : \mathbb{R}^\nu \mapsto T_{\bar{x}_{i_1}} K \times T_{\bar{x}_{i_2}} K \times \cdots T_{\bar{x}_{i_\mu}} K$. Then Θ is positively homogeneous of degree 1 and such that $\theta_1(x) + \cdots + \theta_\mu(x) = x$ for all $x \in \mathbb{R}^\nu$. Moreover,

$$\begin{aligned} \|\theta_j(x)\| &\leq \kappa^{\mathbf{T}}\|x\| \leq \kappa_3\|x\| \leq \kappa_1\kappa_3 \\ &\text{whenever} \quad j = 1, \ldots, \mu \ \text{and} \ x \in K\,. \end{aligned} \qquad (103)$$

Define

$$\psi_j(h) = \frac{hR}{\bar{\rho}_{i_j}\bar{r}(\bar{r} - h)}\theta_j(\omega) \ \text{ for } \ j = 1, \ldots, \mu\,.$$

Then

$$\bar{\rho}_{i_1}\psi_1(h) + \cdots + \bar{\rho}_{i_\mu}\psi_\mu(h) = \frac{hR}{\bar{r}(\bar{r} - h)}\omega = x(h) - \bar{x}\,. \qquad (104)$$

Next, let

$$r_i(h) = \bar{r}_i - \bar{\rho}_i h = \bar{\rho}_i(\bar{r} - h) \ \text{ if } \ i = 1, \ldots, m\,, \qquad (105)$$

$$x_i(h) = \begin{cases} \bar{x}_{i_j} + \psi_j(h) & \text{if} \quad i = i_j\,,\ j \in \{1, \ldots, \mu\}, \\ \bar{x}_i & \text{if} \quad i \notin \{i_1, \ldots, i_\mu\}\,. \end{cases} \qquad (106)$$

$$v_i(h) = (r_i(h), r_i(h)x_i(h)) \ \text{ if } \ i = 1, \ldots, m\,. \qquad (107)$$

Then

$$\begin{aligned}
&r_1(h)x_1(h)+\cdots+r_m(h)x_m(h)\\
&\quad= (\bar r-h)\Big(\bar\rho_1 x_1(h)+\cdots+\bar\rho_m x_m(h)\Big)\\
&\quad= (\bar r-h)\Big(\bar\rho_1\bar x_1+\cdots+\bar\rho_m\bar x_m\Big)\\
&\qquad +(\bar r-h)\Big(\bar\rho_{i_1}\psi_1(h)+\cdots+\bar\rho_{i_\mu}\psi_\mu(h)\Big)\\
&\quad= (\bar r-h)\bar x+(\bar r-h)(x(h)-\bar x)\\
&\quad= (\bar r-h)x(h)\,,
\end{aligned}$$

and

$$r_1(h)+\cdots+r_m(h)=\bar r-h\,. \tag{108}$$

Therefore

$$v_1(h)+\cdots+v_m(h)=(\bar r-h,(\bar r-h)x(h))=y(h)\,. \tag{109}$$

If $j\in\{1,\ldots,\mu\}$, then Lemma 6.6 implies that there is a positive constant δ_j such that $\bar x_{i_j}+z\in K$ whenever $z\in T_{\bar x_{i_j}}K$ and $\|z\|\le\delta_j$. Since $\psi_j(h)$ is a multiple of $\theta_j(\omega)$, with a factor that goes to zero as $h\downarrow 0$, and $\theta_j(\omega)\in T_{\bar x_{i_j}}K$, there exists a positive constant ε_j such that $\bar x_{i_j}+\psi_j(h)\in K$ whenever $0\le h\le\varepsilon_j$. Let

$$\varepsilon=\min\Big(1-\bar s\,,\ \frac{\bar r}{2}\,,\ \varepsilon_1\,,\ \ldots\,,\ \varepsilon_\mu\Big)\,. \tag{110}$$

Then $\bar s+\varepsilon\le 1$, and

$$x_i(h)\in K \text{ if } i\in\{1,\ldots,m\}\,,\ 0\le h\le\varepsilon\,. \tag{111}$$

We now extend the functions $c_i:[0,\bar s]\to\mathbb{R}^n$ to the interval $[0,\bar s+\varepsilon]$ by letting $c_i(\bar s+h)=c_i(\bar s)+\bar v_i-v_i(h)$ if $i=1,\ldots,m$ and $0\le h\le\varepsilon$. Then

$$\begin{aligned}
&c_1(\bar s+h)+\cdots+c_m(\bar s+h)\\
&\quad= c_1(\bar s)+\cdots+c_m(\bar s)+\bar v_1+\cdots+\bar v_m\\
&\qquad -(v_1(h)+\cdots+v_m(h))\\
&\quad= \bar s e+\bar y-y(h)\\
&\quad= \bar s e+he\\
&\quad= (\bar s+h)e\,,
\end{aligned}$$

so that

$$c_1(\bar s+h)+\cdots+c_m(\bar s+h)=(\bar s+h)e \tag{112}$$

if $0\le h\le\varepsilon$. Moreover, if $h\in[0,\varepsilon]$, and $i\in\{1,\ldots,m\}$, then

$$\begin{aligned}
u_i-c_i(\bar s+h) &= u_i-\Big(c_i(\bar s)+\bar v_i-v_i(h)\Big)\\
&= u_i-c_i(\bar s)-\bar v_i+v_i(h)\\
&= \bar v_i-\bar v_i+v_i(h)\\
&= v_i(h)\\
&= (r_i(h),r_i(h)x_i(h))\\
&= r_i(h)(1,x_i(h))\,.
\end{aligned}$$

Since $x_i(h)\in K$ by (111), the point $(1,x_i(h))$ belongs to C, and then $r_i(h)(1,x_i(h))$ also belongs to C. So we have shown that

$$u_i-c_i(\bar s+h)\in C \quad\text{if}\quad i=1,\ldots,m\,,\quad 0\le h\le\varepsilon\,. \tag{113}$$

We now study the derivatives $\dot c_i$ of the functions c_i on the interval $[\bar s,\bar s+\varepsilon]$. We have (using the fact that $v_i(h)=r_i(h)(1,x_i(h))$)

$$\begin{aligned}
\dot c_i(\bar s+h) &= -\dot v_i(h)\\
&= -\dot r_i(h)(1,x_i(h))-r_i(h)(0,\dot x_i(h))\\
&= \bar\rho_i(1,x_i(h))-r_i(h)(0,\dot x_i(h))\,.
\end{aligned} \tag{114}$$

This proves, first of all, that the first component of $\dot c_i(\bar s+h)$ is equal to the nonnegative number $\bar\rho_i$, so

$$\langle e,\dot c_i(\bar s+h)\rangle\ge 0 \quad\text{for}\quad i=1,\ldots,m\,,\quad h\ge 0\,. \tag{115}$$

Moreover, the norm of the first term of the right-hand side of (114) is bounded by $\bar\rho_i\kappa_2$, since $x_i(h)\in K$.

If $i\notin\{i_1,\ldots,i_\mu\}$, then the second term of the right-hand side of (114) vanishes, and we get the bound

$$\|\dot c_i(\bar s+h)\|\le\bar\rho_i\kappa_2 \ \text{ if }\ i\notin\{i_1,\ldots,i_\mu\}\,. \tag{116}$$

If $i\in\{i_1,\ldots,i_\mu\}$, then the second term of the right-hand side of (114) need not vanish, so we have to estimate it. Assume $i=i_j$, $j\in\{1,\ldots,\mu\}$. Then

$$\dot x_i(h)=\dot\psi_j(h)=\frac{R}{\bar\rho_i\bar r(\bar r-h)}\theta_j(\omega)+\frac{hR}{\bar\rho_i\bar r(\bar r-h)^2}\theta_j(\omega),$$

so (105) implies that

$$\begin{aligned}
r_i(h)\dot x_i(h) &= \frac{\bar\rho_i(\bar r-h)R}{\bar\rho_i\bar r(\bar r-h)}\theta_j(\omega)+\frac{h\bar\rho_i(\bar r-h)R}{\bar\rho_i\bar r(\bar r-h)^2}\theta_j(\omega)\\
&= \Big(\frac{R}{\bar r}+\frac{hR}{\bar r(\bar r-h)}\Big)\theta_j(\omega)\,,
\end{aligned}$$

and then

$$\|r_i(h)\dot x_i(h)\|\le\Big(1+\frac{h}{\bar r-h}\Big)\|\theta_j(\omega)\|\le 2\kappa_1\kappa_3\,,$$

because (1) $\omega\in K$, so (103) can be used to conclude that $\|\theta_j(\omega)\|\le\kappa_1\kappa_3$, (2) $R\le\bar r$, and (3) the bound $h\le\varepsilon\le\frac{\bar r}{2}$ implies $h\le\bar r-h$, and then $\frac{h}{\bar r-h}\le 1$.

This gives the estimate

$$\|\dot c_i(\bar s+h)\|\le\bar\rho_i\kappa_2+2\kappa_1\kappa_3 \ \text{ if }\ i\in\{i_1,\ldots,i_\mu\}\,. \tag{117}$$

Then (116) and (117) imply (since $\bar\rho_1+\cdots+\bar\rho_m=1$) that

$$\sum_{j=1}^{\mu}\|\dot c_j(\bar s+h)\|\le\kappa_2+2\mu\kappa_1\kappa_3\le\kappa_2+2M\kappa_1\kappa_3\le\bar k\,. \tag{118}$$

It follows from (112), (113), (115), and (118), that the new curve $[0,\bar s+\varepsilon]\ni s\to\hat\gamma(s)=(\hat c_1(s),\ldots,\hat c_m(s))$, where the $\hat c_i$ are the components c_i of γ, extended as above to the interval $[0,\bar s+\varepsilon]$, satisfies the conditions (96), (97), (98), (99), (100) that characterize the set Γ. Hence the pair $([0,\bar s+\varepsilon],\hat\gamma)$ belongs to Γ. Therefore the pair $([0,\bar s],\gamma)$ is not a maximal member of Γ, and we have reached a contradiction. This happened because we assumed that $\bar s<1$. Hence $\bar s=1$.

We have therefore established that there exists a curve (95) which is defined on the full interval $J=[0,1]$ and is such that (96), (97), (98), (99), (100) hold. Fix such a curve, and define

$$c_i=c_i(1) \quad\text{for}\quad i=1,\ldots,m\,.$$

Then (97) implies that (91) is satisfied, (98) implies (92), and (100) together with (96) imply (93).

To complete our proof, we have to show that (94) is true as well. At this point, we will use (99). The vector $v_i=u_i-c_i$ belongs to C, and if we write $u_i=(\hat r_i,\hat r_i\hat x_i)$, $v_i=(r_i,r_ix_i)$, with $\hat x_i,x_i\in K$, $\hat r_i\ge 0$, $r_i\ge 0$, then (99) implies that $r_i\le\hat r_i$. Therefore

$$\|v_i\|^2=r_i^2(1+\|x_i\|^2)\le r_i^2\kappa_2^2\le\hat r_i^2\kappa_2^2\le\|u_i\|^2\kappa_2^2\,,$$

since $\hat r_i\le\|u_i\|$. Hence $\|v_i\|\le\kappa_2\|u_i\|$, and then, since $c_i=u_i-v_i$, we have $\|c_i\|\le(1+\kappa_2)\|u_i\|\le\bar k\|u_i\|$, showing that (94) is indeed true, and completing our proof. ♢

References

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