Further Results on the Bellman Equation for Optimal Control Problems with Exit Times and Nonnegative Instantaneous Costs

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Abstract

We study the Bellman equation for undiscounted exit time optimal control problems for nonlinear systems and nonnegative instantaneous costs using the dynamic programming approach. Using viscosity solution theory, we prove a uniqueness theorem that characterizes the value functions for these problems as the unique viscosity solutions of the corresponding Bellman equations that satisfy appropriate boundary conditions. As a consequence, we show that the value function of 'Fuller's Example' is the unique viscosity solution of the corresponding Bellman equation in the class of functions which are continuous in the plane, null at the origin, and bounded below. This extends a recent result of P. Soravia which shows that the value function for Fuller's problem is the unique nonnegative viscosity solution of the corresponding Bellman equation which is null at the origin and continuous in the plane. Our results also apply to problems with nonconvex, nonsymmetric control sets and unbounded targets.

Key Words: viscosity solutions, optimal control, Fuller Example

1 Introduction

This note is devoted to the study of Hamilton-Jacobi-Bellman equations (HJBE's) for a large class of unbounded optimal control problems for fully nonlinear systems

$$\left\{ \begin{array}{rcl} y'(t) &=& f(y(t), u(t)), \quad t \geq 0, \quad u(t) \in U \\ y(0) &=& x \end{array} \right. \tag{1}$$

with bounded control sets U. The optimal control problems are of the form

$$\text{Minimize} \quad \int_0^{t_x(\alpha)} [R(y_x(t,\alpha)) + Q(\alpha(t))]\, dt \quad \text{over} \quad \mathcal{U}(x), \tag{2}$$

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where

$$\mathcal{U} := \{ \text{ Measurable Functions } [0,+\infty) \to U\},$$

$y_x(\cdot,\alpha)$ is the solution of (1) for each $\alpha \in \mathcal{U}$, $\mathcal{T} \subseteq \mathbb{R}^N$ is a fixed closed target,

$$t_x(\alpha) \;:=\; \inf\{t \geq 0 : y_x(t,\alpha) \in \mathcal{T}\},$$

$\mathcal{U}(x)$ is the set of inputs $\beta \in \mathcal{U}$ for which $t_x(\beta) < \infty$, and R and Q are nonnegative. Our hypotheses allow linear quadratic (LQ) exit time problems, including cases where the instantaneous cost has the form $x'Rx + u'Qu$ with Q positive semidefinite but not positive definite. We also allow cases where the target $\mathcal{T}$ is unbounded.

The value function of (2) will be denoted by v, and $\mathcal{R}$ denotes the set of points that can be brought to $\mathcal{T}$ in finite time using the dynamics (1). Therefore,

$$v(x) := \begin{cases} \inf\limits_{\alpha \in \mathcal{U}(x)} \left(\displaystyle\int_0^{t_x(\alpha)} [R(y_x(t,\alpha)) + Q(\alpha(t))]\, dt \right), & x \in \mathcal{R} \\ +\infty & x \notin \mathcal{R} \end{cases}.$$

Notice that since U is bounded, it need not be the case that $\mathcal{R}$ is all of $\mathbb{R}^N$, even if the dynamics (1) has the form $\dot{x} = Ax + Bu$ and (A,B) is a controllable pair (cf. [12]). The Hamilton-Jacobi-Bellman equation (HJBE) of (2) is

$$-\,R(x) \;+\; \sup_{u \in U}\{-f(x,u) \cdot Dv(x) \;-\; Q(u)\} \;=\; 0. \tag{3}$$

We will study solutions of this equation on $\mathcal{R} \setminus \mathcal{T}$ which vanish on $\mathcal{T}$. Since the value function v for the problem (2) will not be differentiable in general, we will study (3) in the framework of viscosity solution theory (cf. [1] and [7]).

We prove a uniqueness theorem which characterizes v as the unique viscosity solution of (3) on $\mathcal{R} \setminus \mathcal{T}$ that satisfies appropriate boundary conditions. The result applies to Fuller's Problem (cf. [11], [18], [19], and §3 below), problems with nonconvex, nonsymmetric control sets, and problems with unbounded targets. The proof is based on Zorn's Lemma and a variant of Barbălat's Lemma from adaptive control (cf. Lemma 2.3). The result also applies to certain problems with fully nonlinear instantaneous costs $(x,u) \mapsto \ell(x,u)$ (as explained in the remarks below).

Value function characterizations of this kind have been studied and applied by many authors for a large number of stochastic and deterministic optimal control problems and for differential games. Also, uniqueness characterizations form the bases for convergence proofs for many recently developed numerical schemes for approximating value functions. Indeed, one generally proves the convergence of such numerical schemes by showing that the value function v_n of the n-th discretization of the problem converges uniformly on compact sets to a function w which is a viscosity solution of the HJBE as the discretization becomes finer, and then one invokes the uniqueness characterization to show that w is the value function. Recent accounts of work in these areas may be found in [1], [3], [7], and [15]; and the papers [4] and [13] give general uniqueness characterizations of the HJBE for exit time problems whose instantaneous costs ℓ satisfy conditions of the form [1]

$$\ell(x,a) \;\geq\; C_\varepsilon \;>\; 0 \quad \text{for} \quad \text{all} \quad x \notin B(\mathcal{T},\varepsilon) \quad \text{and} \quad a \in U. \tag{4}$$

Since we allow R to vanish outside $\mathcal{T}$, these earlier results cannot be applied to (3). For theorems characterizing v as the unique nonnegative viscosity solution of the HJBE (3) on $\mathcal{R} \setminus \mathcal{T}$ that satisfies appropriate boundary conditions, see [14]. Our work is part of a larger project which generalizes uniqueness characterizations for HJBE's to versions that cover well-known engineering problems with non-Lipschitz dynamics and instantaneous costs which violate the usual boundedness assumptions. Earlier work in this area can be found in [2] (which covers finite horizon LQ problems), [6] (which covers inifinite horizon LQ problems), and [9] (which gives a uniqueness characterization for the HJBE for a class of exit time problems whose dynamics are continuous but not Lipschitz continuous, including the Reflected Brachystochrone Problem from [17]).

[1] We set $B(S,\varepsilon) := \{p \in \mathbb{R}^N : \inf\{||p-s|| : s \in S\} < \varepsilon\}$ for all $S \subseteq \mathbb{R}^N$ and $\varepsilon > 0$.

2 Definitions, Assumptions, and Main Lemmas

We begin by reviewing the definition of viscosity solutions.

Definition 2.1 Assume that $\mathcal{G} \subseteq \mathbb{R}^N$ is open, that $S \supseteq \mathcal{G}$, and that $F : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ and $w : S \to \mathbb{R}$ are continuous functions. Call w a **viscosity solution** of $F(x, Dw(x)) = 0$ on $\mathcal{G}$ if the following conditions are satisfied: [2]

(C_1) If $\gamma : \mathcal{G} \to \mathbb{R}$ is C^1 and x_o is a local minimum of $w - \gamma$, then $F(x_o,\, D\gamma(x_o)) \geq 0$.

(C_2) If $\lambda : \mathcal{G} \to \mathbb{R}$ is C^1 and x_1 is a local maximum of $w - \lambda$, then $F(x_1,\, D\lambda(x_1)) \leq 0$.

This definition is equivalent to the definition of viscosity solutions based on semidifferentials used in [6] (cf. [1]). By a denseness argument, the definition is unchanged if we replace C^1 with the set C^∞ of infinitely differentiable functions in (C_1) and (C_2). All of what follows is based on the 'test function' viscosity solution definition given above, so we omit the definition based on semidifferentials. We make the following assumptions on the data of our problem (2):

(A_0) The control set U is a compact set.

(A_1) The dynamics $f : \mathbb{R}^N \times U \to \mathbb{R}^N$ is continuous, and there is a constant $L > 0$ such that $(f(x,a) - f(y,a)) \cdot (x - y) \leq L||x - y||^2$ for all $x, y \in \mathbb{R}^N$ and $a \in U$.

(A_2) The target $\mathcal{T} \subseteq \mathbb{R}^N$ is closed and $STC\mathcal{T}$ holds.

(A_3) The instantaneous cost summands $R : \mathbb{R}^N \to \mathbb{R}$ and $Q : U \to \mathbb{R}$ are continuous and nonnegative.

(A_4) If $\beta \in \mathcal{U}$ and $x \in \mathbb{R}^N$, and if $\int_0^\infty R(y_x(s,\beta))\, ds < \infty$, then there is a point $\bar{x} \in \mathcal{T}$ such that $\lim_{s \to \infty} y_x(s, \beta) = \bar{x}$.

By $STC\mathcal{T}$, we mean the condition that $\mathcal{T} \subset \text{Interior}\, \mathcal{R}(\varepsilon)$ for all $\varepsilon > 0$, where

$$\mathcal{R}(\varepsilon) \;:=\; \left\{ x \in \mathbb{R}^N : \exists\, t \in [0, \varepsilon) \;\&\; \alpha \in \mathcal{U} \;\text{ s.t. }\; y_x(t, \alpha) \in \mathcal{T} \right\}.$$

The condition $STC\mathcal{T}$ is satisfied when suitable assumptions are made on the directions of the vector field f and its Lie brackets at $\partial\mathcal{T}$ (cf. [16]). Notice that we allow $\mathcal{T}$ to be unbounded. Condition (A_4) can be relaxed in various ways (as we will see in the remarks below).

By an application of the Schauder Fixed Point Theorem, condition (A_1) guarantees that for each $\alpha \in \mathcal{U}$ and each $x \in \mathbb{R}^N$, there is a unique trajectory for the dynamics (1) which starts at x and which is defined on $[0, \infty)$. This trajectory will be denoted by $y_x(\cdot, \alpha)$. Moreover, one can show that if (A_1) holds, then for all $\alpha \in \mathcal{U}$, the trajectory $y_x(\cdot, \alpha)$ satisfies the condition

$$||y_x(s, \alpha) - x|| \;\leq\; M_x\, t \quad \text{for all} \quad t \;\in\; [0, 1/M_x], \tag{5}$$

where $M_x = \sup\{||f(z,a)|| : a \in U,\, ||z - x|| \leq 1\}$ if the supremum is nonzero and $M_x = 1$ otherwise. The elementary proof is in the appendix of Chapter 3 of [1].

We will give conditions guaranteeing that the value function of (2) is the unique viscosity solution of (3) on $\mathcal{R} \setminus \mathcal{T}$ in classes of functions satisfying boundary conditions of the following type:

$BC(\mathcal{R}, \omega_o)$ $\quad$ $w : \mathcal{R} \to \mathbb{R}$ is bounded below, $\omega_o \in \mathbb{R} \cup \{+\infty\}$, $w(x) < \omega_o$ on $\mathcal{R}$, $w \equiv 0$ on $\mathcal{T}$, and $\lim_{x \to x_o} w(x) = \omega_o$ for all $x_o \in \partial\mathcal{R}$.

[2] By C^1, we mean continuously differentiable. Also, $C(\mathcal{S})$ will denote the set of continuous real-valued functions on a set $\mathcal{S}$.

Notice that for cases where $\mathcal{R} = \mathbb{R}^N$, the limit condition in $BC(\mathcal{R}, \omega_o)$ is satisfied vacuously. Note in particular that there are no growth restrictions on functions w which satisfy the condition $BC(\mathbb{R}^N, +\infty)$. Our uniqueness results are based on the following representation lemma from [1]:

Lemma 2.2 Let (A_0)-(A_3) hold, let $B \subseteq \mathbb{R}^N$ be a bounded open set, and let $u \in C(\bar{B})$ be a viscosity solution of the HJBE (3) on B. For each $q \in B$, each $\beta \in \mathcal{U}$, and each [3] $\delta \in (0, \mathrm{dist}\,(p, \partial B)/2)$, set

$$\tau_q(\beta) \ = \ \inf\{\, t \geq 0 : \ y_q(t, \beta) \ \in \ \partial B\} \quad \text{and} \quad T_\delta(q) = \inf_{\alpha \in \mathcal{A}} \{t : \ \mathrm{dist}\ (y_q(t, \alpha), \partial B) \leq \delta\}.$$

Then the following inequalities hold for all $q \in B$, $\beta \in \mathcal{U}$, $r \in [0, \tau_q(\beta))$, and $t \in [0, T_\delta(q))$:

$$u(q) \ \leq \ \int_0^r [R(y_q(s, \beta)) \ + \ Q(\beta(s))] \ ds \ + \ u(y_q(r, \beta)) \tag{6}$$

$$u(q) \ \geq \ \inf_{\alpha \in \mathcal{U}} \left[\int_0^t [R(y_q(s, \alpha)) \ + \ Q(\alpha(s))] \ ds \ + \ u(y_q(t, \alpha)) \right] \tag{7}$$

We also use the following variant of Barbălat's Lemma from adaptive control (cf. [5]):

Lemma 2.3 Let $\phi : [0, \infty) \to \mathbb{R}$ be square integrable and continuously differentiable, and assume that ϕ' is Lipschitz continuous. Then $\lim\limits_{t \to \infty} \phi(t) = \lim\limits_{t \to \infty} \phi'(t) = 0$.

Proof. We first prove that $\lim_{t \to \infty} \phi'(t) = 0$. Suppose this were not the case. We could then find a sequence $t_k \uparrow \infty$ and a $\gamma > 0$ such that for all k, we have either

$$(i) \quad \phi'(t_k) \ \geq \ \gamma \quad \text{or} \quad (ii) \quad \phi'(t_k) \leq -\gamma.$$

Passing to a subsequence as necessary, we can assume that

$$t_{k+1} \ - \ t_k \ \geq \ 1 \quad \text{for all} \quad k. \tag{8}$$

Let C be any Lipschitz constant for ϕ'. First consider the case where (i) holds for all indices k in an infinite index set I_+. Choose

$$\delta \ \in \ \left(0, \ \frac{\gamma}{2C} \wedge \frac{1}{2}\right).$$

Since $\delta < \gamma/(2C)$, we get

$$\phi'(t) \ = \ \phi'(t) \ - \ \phi'(t_k) \ + \ \phi'(t_k) \ \geq \ -C\,|t - t_k| \ + \ \gamma \ \geq \ -C\,\delta \ + \ \gamma \ \geq \ \gamma/2 \ > \ 0$$

for all $t \in [t_k - \delta, t_k + \delta]$ and all $k \in I_+$. Setting

$$v_k \ = \ \inf\{|\phi(t)| : t_k - \delta \leq t \leq t_k + \delta\} \ \ \forall k,$$

pick s_k's so that

$$t_k - \delta \ \leq \ s_k \ \leq \ t_k + \delta \quad \text{and} \quad |\phi(s_k)| \ = \ v_k \quad \text{for all} \quad k.$$

We consider the following three subcases for indices $k \in I_+$:

(1) If $\phi(s_k) > 0$, then $s_k = t_k - \delta$ (since $\phi' > 0$ on $[t_k - \delta, t_k + \delta]$). Therefore, if $t \in [t_k - \delta, t_k + \delta]$, then we get

$$\phi(t) \ \geq \ \phi(t_k - \beta) \ + \ \gamma/2(t - (t_k - \delta)),$$

which gives

$$\int_{t_k - \delta}^{t_k + \delta} \phi^2(s)\, ds \ \geq \ \frac{\gamma^2}{4} \int_{t_k - \delta}^{t_k + \delta} [t - (t_k - \delta)]^2 \ dt \ = \ \frac{\gamma^2}{4} \frac{(2\delta)^3}{3}. \tag{9}$$

[3] We set $\mathrm{dist}(q, P) := \inf\{||q - p|| : p \in P\}$.

(2) If $\phi(s_k) < 0$, then we get $s_k = t_k + \delta$ and $\phi < 0$ on $[t_k - \delta, t_k + \delta]$. Hence, if $t \in [t_k - \delta, t_k + \delta]$, then

$$-\phi(t) \;=\; -\phi(s_k) \;+\; \int_t^{s_k} \phi'(u)du \;>\; v_k \;+\; \gamma/2(s_k - t),$$

so we get

$$\int_{t_k-\delta}^{t_k+\delta} \phi^2(s)\,ds \;\geq\; \frac{\gamma^2}{4} \int_{t_k-\delta}^{t_k+\delta} (t_k + \delta - s)^2\,ds \;=\; \frac{\gamma^2}{12}(2\delta)^3,$$

so we again have (9).

(3) Assume $\phi(s_k) = 0$. Since $\phi' > 0$ on $[t_k - \delta, t_k + \delta]$, this subcase is covered by applying the method of subcase (1) to $[s_k, t_k + \delta]$ and the method of subcase (2) to $[t_k - \delta, s_k]$ to get $\phi^2(t) > \gamma^2/4(t - s_k)^2$ for all $t \in [t_k - \delta, t_k + \delta]$, so

$$\int_{t_k-\delta}^{t_k+\delta} \phi^2(s)\,ds \;\;\geq\;\; \frac{\gamma^2}{4} \int_{t_k-\delta}^{t_k+\delta} (s_k - s)^2\,ds \;\;\geq\;\; \frac{\gamma^2}{6}\delta^3.$$

Since $\delta < 1/2$ and (8) holds, it follows that the intervals $[t_k - \delta, t_k + \delta]$'s are disjoint. Since

$$\int_0^\infty \phi^2(s)\,ds \;\;<\;\; \infty,$$

it follows that only finitely many of the k satisfy conditon (i), which is a contradiction. By applying the previous argument to $-\phi$, we can also show that only finitely many of the k's satisfy condition (ii) as well. Since $\phi'(t) \to 0$ as $t \to +\infty$, it follows that ϕ is also Lipschitz continuous. The fact that $\phi(t) \to 0$ as $t \to +\infty$ now follows from the well-known version of Barbălat's Lemma as stated in [5]. ∎

3 Statement of Main Result and Remarks

We will prove the following theorem:

Theorem 1 *Assume conditions* (A_0)-(A_4) *are satisfied. If* $w \in C(\mathcal{R})$ *is a viscosity solution of (3) on* $\mathcal{R} \setminus \mathcal{T}$ *which satisfies* $BC(\mathcal{R}, \omega_o)$ *for some* $\omega_o \in \mathbb{R} \cup \{+\infty\}$*, then* $w \equiv v$ *on* $\mathcal{R}$*.*

Notice that for cases where the value function v of (2) is known to satisfy $BC(\mathcal{R}, \nu_o)$ for some $\nu_o \in \mathbb{R} \cup \{+\infty\}$, this theorem becomes a uniqueness characterization for the value function of (2). This follows from the fact (cf. [1]) that v is a viscosity solution of the HJBE (3) on $\mathcal{R} \setminus \mathcal{T}$. Before proving Theorem 1, we will show how to apply the theorem to get a uniqueness characterization for the Fuller Problem (FP). Recall that the FP is

$$\text{Infimize} \quad \int_0^{t_p(\alpha)} (y_{1,p}(t,\alpha))^2\,dt \quad \text{over all} \quad \alpha \in \mathcal{U}(p) \tag{10}$$

for each $p \in \mathbb{R}^2$, where $t \mapsto y_p(t,u) := (y_{1,p}(t,u), y_{2,p}(t,u))'$ is the trajectory of

$$\begin{cases} \dot{x}(t) \;=\; y(t), \;\; \dot{y}(t) \;=\; u(t) \;\in\; U := [-1, +1] \\ (x(0), y(0)) \;=\; p \end{cases}$$

and $t_p(u)$ is the first time this trajectory reaches the target $\mathcal{T} := \{\vec{0}\} \subseteq \mathbb{R}^2$. The FP admits optimal controls for each initial position (cf. [19]). In particular, the FP data give $\mathbb{R}^2 = \mathcal{R}$. Using this fact and the convexity of $x \mapsto x^2$, one easily checks that the value function v_{FP} for the FP is strictly convex on $\mathbb{R}^2$ and therefore continuous (cf. p. 81 of [10]). Taking $Q \equiv 0$ and $R((x,y)') = x^2$, it follows from Lemma 2.3 above that the FP satisfies (A_0)-(A_4). Since v_{FP} satisfies $BC(\mathbb{R}^2, +\infty)$, we conclude as follows:

Corollary 3.1 The Fuller Problem value function is the unique viscosity solution of

$$-\ y(Dw((x,y)'))_1 \ + \ |(Dw((x,y)'))_2| \ - \ x^2 \ = \ 0, \quad (x,y)' \in \mathbb{R}^2 \setminus \{\vec{0}\} \tag{11}$$

in the class of functions in $C(\mathbb{R}^2)$ which are bounded below and null at $\vec{0} \in \mathbb{R}^2$.

Notice that this extends a result in [14] which characterizes the FP value function as the unique viscosity solution of (11) in the class of nonnegative functions $w \in C(\mathbb{R}^2)$ which vanish at the origin. More generally, we get a uniqueness result for the problems

$$\text{Infimize} \quad \int_0^{t_x(\alpha)} \sum_{j=1}^{n-1} x_j^2(t,\alpha)\, dt \quad \text{over} \quad \mathcal{U}(x) \quad \text{subject} \quad \text{to}$$

$$\begin{array}{rcl} \dot{x}_1(t,\alpha) & = & x_2(t,\alpha) \\ \dot{x}_2(t,\alpha) & = & x_3(t,\alpha) \\ & \vdots & \\ \dot{x}_{n-1}(t,\alpha) & = & x_n(t,\alpha) \\ \dot{x}_n(t,\alpha) & = & \alpha(t) \in U \end{array}$$

with the target $\{\vec{0}\} \subset \mathbb{R}^n$ for any bounded set $U \subset \mathbb{R}$. This follows from a repeated application of Lemma 2.3 which we omit.

Remark 3.2 The requirement that the instantaneous cost has the form $R(x) + Q(u)$ can be relaxed in several ways. One way to do this is to allow general nonnegative continuous instantaneous costs ℓ that satisfy the condition

$$(A_4') \qquad \left[\, \beta \in \mathcal{U} \ \ \& \ \ \int_0^\infty \ell(y_x(s,\beta), \beta(s))\, ds \ < \infty \,\right] \ \Rightarrow \ \left[\, \exists x_o \in \mathcal{T} \ \ \text{s.t.} \ \ \lim_{s\to\infty} y_x(s,\beta) = x_o \,\right],$$

in which case the HJBE becomes

$$\sup_{u\in U} \{-f(x,u)\cdot Dv(x) \ - \ \ell(x,u)\} \ = \ 0. \tag{12}$$

Alternatively, we can allow Q to depend on x as well as u. In fact, the only properties of the function Q needed in the proof of Theorem 1 are nonnegativity and continuity.

Remark 3.3 Notice that if the functions R and Q in Theorem 1 are such that

$$\forall\ \varepsilon > 0, \ \ \exists\ C_\varepsilon > 0 \quad \text{s.t.} \quad R(x) \ + \ Q(u) \ > \ C_\varepsilon \quad \forall x \in \mathbb{R}^N \setminus B(\mathcal{T},\varepsilon) \ \ \& \ \ \forall u \in U, \tag{13}$$

then it follows from the uniqueness characterizations in [13] that v is the only possible viscosity solution of the HJBE (3) on $\mathcal{R}\backslash\mathcal{T}$ which is continuous on $\mathcal{R}$ and which satisfies $BC(\mathcal{R},\omega_o)$ for some $\omega_o \in \mathbb{R}\cup\{+\infty\}$. On the other hand, condition (13) could very well fail, even if $R(x) \equiv x'Rx$ and $Q(u) \equiv u'Qu$ and Q is positive definite, since we allow $\vec{0} \in U$. Therefore, our uniqueness characterization does not follow from the earlier work.

Remark 3.4 The paper [8] proves a local uniqueness result for (12) which shows as a specical case that the FP value function v_{FP} is the unique viscosity solution of the corresponding HJBE (11) on $\mathbb{R}^2 \setminus \{\vec{0}\}$ in the class of functions w which vanish at $\vec{0}$, are continuous on $\mathbb{R}^2$, and satisfy a growth condition which is a generalization of properness (where properness of a function v is the condition that $\lim_{||x||\to\infty} v(x) = +\infty$). This growth condition can be satisfied by functions which are not bounded below. The main argument of that paper is a generalization of the uniqueness arguments for free boundary problems given in Chapter 4 of [1]. The novelty of Corollary 3.1 above is that it gives a uniqueness characterization for the FP value function within a class of functions which includes functions which violate the growth conditions of [8].

4 Proof of Main Result

Let w satisfy the hypotheses of Theorem 1, and let $\bar{x} \in \mathcal{R} \setminus \mathcal{T}$ be given. The fact that $v(\bar{x}) \geq w(\bar{x})$ is an easy consequence of the first part of Lemma 2.2 which is obtained by fixing an input $\alpha \in \mathcal{U}$ that drives $\bar{x}$ to $\mathcal{T}$ in finite time, taking B to be any open tube containing the trace of $y_{\bar{x}}(\cdot, \alpha)$ on $[0, t_{\bar{x}}(\alpha)]$, and then infimizing over all α. Such a tube exists since trajectories that reach $\mathcal{T}$ in finite time remain in $\mathcal{R}$ on $[0, t_{\bar{x}}(\alpha)]$ (because points which are outside $\mathcal{R}$ cannot be brought to $\mathcal{T}$ in finite time using the dynamics (1)). We therefore only show that $w(\bar{x}) \geq v(\bar{x})$.

Choose $\varepsilon > 0$ such that

$$w(\bar{x}) \;<\; \omega_o \;-\; \varepsilon. \tag{14}$$

This is possible since $BC(\mathcal{R}, \omega_o)$ is satisfied. (We allow $\omega_o = +\infty$, in which case (14) holds for all $\varepsilon > 0$.) The inequality $w(\bar{x}) \geq v(\bar{x})$ will evidently follow if we find an input $\bar{\alpha} \in \mathcal{U}$ that drives $\bar{x}$ to $\mathcal{T}$ in finite time and which is such that

$$w(\bar{x}) \;\geq\; \int_0^{t_{\bar{x}}(\bar{\alpha})} \{R(y_x(s, \bar{\alpha})) \;+\; Q(\bar{\alpha}(s))\}\, ds \;-\; \varepsilon, \tag{15}$$

since that would give $w(\bar{x}) \geq v(\bar{x}) - \varepsilon$, and then we can let $\varepsilon \downarrow 0$. We now prove the existence of such a $\bar{\alpha}$ by using Zorn's Lemma. In what follows, we define the functions $E_1, E_2, \ldots$ by

$$E_j(t) \;\equiv\; \frac{\varepsilon}{4}\left[e^{-(j-1)} - e^{-(t+j-1)}\right] \quad \text{for each} \quad t > 0$$

and we set

$$\ell(x, u) \;=\; R(x) \;+\; Q(u).$$

Set

$$\mathcal{Z}_1 \;=\; \left\{ \begin{array}{l} (t, \gamma): \quad 0 \leq t \leq 1, \;\; \text{and } \gamma : [0, t] \to \mathcal{R} \;\text{ is a trajectory} \\ \text{for } \dot{x} \;=\; f(x, \alpha) \;\text{ for some }\; \alpha \in \mathcal{U} \;\text{ with } \gamma(0) \;=\; \bar{x} \\ \text{so that } w(\bar{x}) \;\geq\; \int_0^t \ell(\gamma(s), \alpha(s))\, ds + w(\gamma(t)) - E_1(t). \end{array} \right\}.$$

Then $\mathcal{Z}_1$ is partially ordered by the relation

$$(t_1, \gamma_1) \;\sim\; (t_2, \gamma_2) \;\;\text{iff}\;\; [\;\; t_1 \leq t_2 \;\;\text{and}\;\; \gamma_2 \lceil [0, t_1] \equiv \gamma_1 \;\;] \tag{16}$$

Moreover, one can use standard arguments to check that every totally ordered subset of $\mathcal{Z}_1$ has an upper bound in $\mathcal{Z}_1$.

Indeed, let $\{(s_j, \mu_j)\}_j$ be a totally ordered subset of $\mathcal{Z}_1$, set $\bar{s} = \sup_j s_j$, and then define the trajectory $\bar{\mu} : [0, \bar{s}) \to \mathbb{R}^N$ by $\bar{\mu}(p) = \mu_j(p)$ for any j so that $s_j > p$. (This is valid since the s_j's increase and μ_j extends μ_k for $k \leq j$.) Let $\beta_j : (s_{j-1}, s_j] \to U$ be a control for the trajectory $\mu_j \lceil (s_{j-1}, s_j]$, fix $\bar{u} \in U$, and define $\beta^\dagger : (0, \infty) \to U$ by $\beta^\dagger(s) := \beta_j(s)$ on $(s_{j-1}, s_j]$ for $j \geq 1$ and $\beta^\dagger(s) = \bar{u}$ for $s \geq \bar{s}$ (where we set $s_0 = 0$ and assume wlog that $s_j < \bar{s}$ for all j). Then $\beta^\dagger$ is measurable and therefore admits a corresponding trajectory $y_{\bar{x}}(\cdot, \beta^\dagger)$ on $[0, \infty)$. Moreover, $y_{\bar{x}}(\cdot, \beta^\dagger)$ coincides with $\bar{\mu}$ on $[0, \bar{s})$. Setting $\bar{\mu}(\bar{s}) = y_{\bar{x}}(\bar{s}, \beta^\dagger)$, it follows that $(\bar{s}, \bar{\mu})$ is an upper bound for $\{(s_j, \mu_j)\}_j$ in the relation (16). Also, the choice of $\varepsilon > 0$ guarantees that $\bar{\mu}(\bar{s}) \in \mathcal{R}$, which implies that $(\bar{s}, \bar{\mu}) \in \mathcal{Z}_1$.

Indeed, if this were not true, and if we use α_j to denote the control for μ_j for all j, then we would get

$$\mu_j(s_j) \;\;\to\;\; \bar{\mu}(\bar{s}) \in \partial\mathcal{R}.$$

It would then follow from the assumption $BC(\mathcal{R}, \omega_o)$ and the nonnegativity of ℓ that

$$\begin{aligned} w(\bar{x}) \;&\geq\; \int_0^{s_j} \ell(\mu_j(u), \alpha_j(s))\, du \;+\; w(\mu_j(s_j)) - E_1(s_j) \\ &\geq\; w(\mu_j(s_j)) - E_1(s_j) \;\;\geq\;\; w(\mu_j(s_j)) - \varepsilon \;\;\to\;\; \omega_o - \varepsilon, \end{aligned}$$

so $w(\bar{x}) \geq \omega_o - \varepsilon$. This stands in contradiction to (14), so we conclude that $\bar{\mu}(\bar{s}) \in \mathcal{R}$. Since w is continuous at $\bar{\mu}(\bar{s}) \in \mathcal{R}$, it follows that $(\bar{s}, \bar{\mu}) \in \mathcal{Z}_1$ is an upper bound for $\{(s_j, \mu_j)\}$.

It follows from Zorn's Lemma that $\mathcal{Z}_1$ contains a maximal element which we will denote by $(\bar{t}, \bar{\gamma})$. We will assume wlog that $\bar{\gamma}$ does not reach $\mathcal{T}$ on $[0, \bar{t}]$ (since otherwise (15) is satisfied if we use the input for $\bar{\gamma}$). We will now show that $\bar{t} = 1$. Let B be an open set containing $\bar{\gamma}(\bar{t})$ whose closure lies in $\mathcal{R} \setminus \mathcal{T}$. Such as set exists since our hypothesis $STC\mathcal{T}$ implies that $\mathcal{R}$ is open (cf. [1]) and we are assuming that $\mathcal{T}$ is closed. Suppose that $\bar{t} < 1$, set $q = \bar{x}$, and pick

$$\delta \;\in\; [0, \mathrm{dist}(\bar{x}, \partial B)/2).$$

Set

$$T_\delta(\bar{x}) \;=\; \inf_\alpha \left\{ t : \mathrm{dist}(y_{\bar{x}}(t,\alpha), \partial B) \leq \delta \right\}.$$

Taking $x = \bar{x}$ in (5), it follows that $T_\delta(\bar{x}) > 0$.

Indeed, for each $\tilde{\delta} > 0$, we can use (5) to find a $\tilde{t} > 0$ so that $y_{\bar{x}}(t, \alpha) \in B(\{\bar{x}\}, \tilde{\delta})$ for all $\alpha \in \mathcal{U}$ and all $t \in [0, \tilde{t}]$. Choosing $\tilde{\delta} = 1/4\, \mathrm{dist}(\bar{x}, \partial B)$, it follows that

$$\begin{aligned} \mathrm{dist}(y_{\bar{x}}(t,\alpha), \partial B) \;&\geq\; \mathrm{dist}(\bar{x}, \partial B) - ||y_{\bar{x}}(t,\alpha) - \bar{x}|| \\ &\geq\; \mathrm{dist}(\bar{x}, \partial B) - \frac{1}{4}\mathrm{dist}(\bar{x}, \partial B) \\ &\geq\; \frac{1}{2}\mathrm{dist}(\bar{x}, \partial B) \quad > \quad \delta \end{aligned}$$

for all $t \in [0, \tilde{t}]$ and $\alpha \in \mathcal{U}$. In particular, $T_\delta(\bar{x}) \geq \tilde{t} > 0$.

By the conclusion (7) of Lemma 2.2, it follows that there is a $t \in (0, 1 - \bar{t})$ and a $\beta \in \mathcal{U}$ so that

$$w(\bar{\gamma}(\bar{t})) \;\geq\; \int_0^t \ell(y_{\bar{\gamma}(\bar{t})}(s,\beta), \beta(s))\, ds \;+\; w(y_{\bar{\gamma}(\bar{t})}(t,\beta)) \;-\; E_1(\bar{t} + t) \;+\; E_1(\bar{t}) \tag{17}$$

and so that $y_{\bar{\gamma}(\bar{t})}(\cdot, \beta)$ remains in B on $[0, t]$. Let $\tilde{\alpha}$ denote the control for $\bar{\gamma}$. Since $(\bar{t}, \bar{\gamma}) \in \mathcal{Z}_1$, it follows that

$$w(\bar{\gamma}(0)) \;-\; w(\bar{\gamma}(\bar{t})) \;\geq\; \int_0^{\bar{t}} \ell(y_{\bar{x}}(s, \tilde{\alpha}), \tilde{\alpha}(s))\, ds \;-\; E_1(\bar{t}). \tag{18}$$

Let $\beta^\sharp$ denote the concatenation of the restriction of the input $\tilde{\alpha}$ to $[0, \bar{t}]$ followed by the input β. If we now add the inequalities (17) and (18), then we get

$$w(\bar{\gamma}(0)) \;\geq\; \int_0^{\bar{t}+t} \ell(y_{\bar{x}}(s, \beta^\sharp), \beta^\sharp(s))\, ds \;-\; E_1(\bar{t} + t). \tag{19}$$

Since t was chosen so that $\bar{t} + t < 1$ and $B \subseteq \mathcal{R}$, we conclude from (19) that $\big(\bar{t} + t, y_{\bar{x}}(\cdot, \beta^\sharp)\big) \in \mathcal{Z}_1$. This contradicts the maximality of the pair $(\bar{t}, \bar{\gamma})$. Therefore, $\bar{t} = 1$.

We will now extend $\bar{\gamma} : [0,1] \to \mathcal{R}$ to get the desired trajectory. Set

$$\mathcal{Z}_2 \;=\; \left\{ \begin{array}{l} (t, \gamma) : \quad 0 \,\leq\, t \,\leq\, 1, \;\text{ and } \gamma : \; [0,t] \to \mathcal{R} \;\text{ is a trajectory} \\ \text{for } \dot{x} \;=\; f(x, \alpha) \;\text{ for some }\; \alpha \,\in\, \mathcal{U} \;\text{ with }\; \gamma(0) \;=\; \bar{\gamma}(1) \\ \text{so that } w(\bar{\gamma}(1)) \;\geq\; \int_0^t \ell(\gamma(s), \alpha(s))\, ds + w(\gamma(t)) - E_2(t). \end{array} \right\},$$

and partially order $\mathcal{Z}_2$ by the relation (16). By the preceding argument, we know that $\mathcal{Z}_2$ contains a maximal element $(1, \tau_2)$. Let $u_2 : [0,1] \to U$ denote the input for τ_2. We can assume wlog that τ_2 does not reach $\mathcal{T}$ on $[0,1]$. Indeed, if there is an $s_2 \in [0,1]$ such that $\tau_2(s_2) \in \mathcal{T}$, then we can satisfy the requirement (15) by setting

$$\bar{\alpha}(t) = \left\{ \begin{array}{ll} \tilde{\alpha}(t), & 0 \leq t \leq 1 \\ u_2(t-1), & 1 \leq t \leq 1 + s_2 \end{array} \right. ,$$

where $\tilde{\alpha}$ is the control for $\bar{\gamma}$ on $[0,1]$. Let c_2 denote the concatenation of $\tilde{\alpha}\lceil[0,1]$ followed by $u_2\lceil[0,1]$, and let γ_2 denote the corresponding concatenated trajectory for c_2 from $\bar{x}$ to $\tau_2(1)$.

Now reapply the procedure to $\gamma_2(2)$ to get a longer trajectory that reaches $\mathcal{T}$ or runs for three time units without leaving $\mathcal{R}$. The procedure is iterated, and it results in a sequence of terminal points $\phi_n(n)$ for trajectories ϕ_n which begin at $\bar{x}$, are defined on $[0,n]$, and remain in $\mathcal{R}\setminus\mathcal{T}$ wlog. It follows that for all $n\in\mathbb{N}$, we would have

$$w(\bar{x}) \;\geq\; \int_0^n \ell(\phi_n(s,c_n),c_n(s))\,ds \;+\; w(\phi_n(n)) \;-\; \frac{\varepsilon}{2}(1-e^{-n}). \tag{20}$$

Recall from assumption (A_3) that Q is nonnegative. Setting $\hat{\alpha}(s)=c_n(s)$ if $n-1<s\leq n$, letting $\hat{\phi}$ denote the corresponding trajectory, and letting b denote a lower bound for w, a passage to the limit as $n\to\infty$ in (20) gives

$$\int_0^\infty R(\hat{\phi}(s))\,ds \;\leq\; w(\bar{x}) \;-\; b \;+\; \frac{\varepsilon}{2} \;<\; \infty,$$

since $\hat{\phi}\equiv\phi_n$ on $(n-1,n]$. It follows from the assumption (A_4) that $\lim_{s\to\infty}\hat{\phi}(s)$ exists and lies in the target $\mathcal{T}$.

Since we were assuming that U is compact, and since ℓ is continuous, we conclude that

$$\sup_{s\in[0,\infty)} \ell(\hat{\phi}(s),\hat{\alpha}(s))\,ds \;\leq\; M \;<\; \infty$$

Since condition $STC\mathcal{T}$ is satisfied, it follows that there is an input $\tilde{\beta}$ and an $m\in\mathbb{N}$ so that

$$\left\{\int_0^{t_{\phi_m(m)}(\tilde{\beta})} \ell(y_{\phi_m(m)}(s,\tilde{\beta}),\tilde{\beta}(s))\,ds \;\bigvee\; |w(\phi_m(m))|\right\} \;<\; \varepsilon/4. \tag{21}$$

Indeed, the continuity of w on $\mathcal{R}$ and the hypothesis that $w\equiv 0$ on $\mathcal{T}\subseteq\mathcal{R}$ guarantee that $w(\phi_n(n))\to 0$ as $n\to\infty$, and the $STC\mathcal{T}$ condition ensures that we can drive $\phi_n(n)$ to $\mathcal{T}$ in time $<\varepsilon/[4(M+1)]$ when n is large enough. Fixing $m\in\mathbb{N}$ for which there is a $\tilde{\beta}$ satisfying (21), and letting $\bar{\alpha}$ denote the concatenation of the input $c_m\lceil[0,m]$ followed by the input $\tilde{\beta}$, we combine (20) and (21) to get

$$\begin{aligned}
w(\bar{x}) \;&\geq\; \int_0^m \ell(\phi_m(s),c_m(s))\,ds \;+\; w(\phi_m(m)) \\
&\quad+\; \int_0^{t_{\phi_m(m)}(\tilde{\beta})} \ell(y_{\phi_m(m)}(s,\tilde{\beta}),\tilde{\beta}(s))\,ds \;-\; \frac{\varepsilon}{2}(1-e^{-m}) \;-\; \frac{\varepsilon}{4} \\
&\geq\; \int_0^m \ell(\phi_m(s),c_m(s))\,ds + \int_0^{t_{\phi_m(m)}(\tilde{\beta})} \ell(y_{\phi_m(m)}(s,\tilde{\beta}),\tilde{\beta}(s))\,ds - \frac{\varepsilon}{2}(1-e^{-m}) - \frac{\varepsilon}{2} \\
&\geq\; \int_0^{t_{\bar{x}}(\bar{\alpha})} \ell(y_{\bar{x}}(s,\bar{\alpha}),\bar{\alpha}(s))\,ds - \frac{\varepsilon}{2}(2-e^{-m}).
\end{aligned}$$

Therefore, $\bar{\alpha}$ satisfies the requirement (15). This proves Theorem 1.

Remark: As we mentioned in Remark 3.2, Theorem 1 goes through for general nonlinear instantaneous costs of the form $\ell(x,a)=R(x)+Q(x,a)$ if Q and R are continuous and nonnegative and R satisfies condition (A_4). This follows since Lemma 2.2 goes through for fully nonlinear instantaneous costs which are bounded below (cf. Chapter 3 of [1]). The preceding argument shows that for cases where $\omega_o=+\infty$, we can allow Q to take negative values as well if we require the condition

$$[\,\hat{\alpha}\in\mathcal{U} \quad \& \quad y_x(s,\hat{\alpha})\in\mathcal{R}\;\forall s\geq 0\,] \;\Rightarrow\; \left[\lim_{\tau\to+\infty}\int_0^\tau Q(y_x(s,\hat{\alpha}),\hat{\alpha}(s))\,ds \;\geq\; 0\right]$$

and if we assume that the other hypotheses of the theorem are satisfied. Of course, we need Q to be bounded below for the HJBE (3) to be defined for all $x\in\mathcal{R}\setminus\mathcal{T}$.

References

[1] Bardi, M., and I. Capuzzo-Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Birkhäuser, Boston, 1997.

[2] Bardi, Martino, and Francesca Da Lio, "On the Bellman equation for some unbounded control problems," *NODEA Nonlinear Differential Equations Appl.*, **4**(1997), pp. 276-285.

[3] Bardi, M., M. Falcone, and P. Soravia, "Numerical methods for pursuit-evasion games and viscosity solutions," in *Stochastic and Differential Games: Theory and Numerical Methods*, M. Bardi, T. E. S. Raghavan, and T. Parthasarathy, Eds., Birkhäuser, Boston, 1999.

[4] Bardi, Martino, and Pierpaolo Soravia, "Hamilton-Jacobi equations with singular boundary conditions on a free boundary and applications to differential games," *Trans. Amer. Math. Soc.*, **325**(1991), pp. 205-229.

[5] Desch, W., H. Longemann, E.P. Ryan, and E.D. Sontag, "Meagre functions and asymptotic behavior of dynamical systems," preprint. (To appear in *Nonlinear Analysis*)

[6] Da Lio, F., "On the Bellman equation for infinite horizon problems with unbounded cost functional," *Appl. Math. Optim.*, **41**(1999), pp. 171-197.

[7] Fleming, W.H., and H.M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer, New York, 1993.

[8] Malisoff, M., "Viscosity solutions of the Bellman equation for exit time optimal control problems with vanishing Lagrangians," submitted for publication.

[9] Malisoff, M., "A remark on the Bellman equation for optimal control problems with exit times and noncoercing dynamics," in *Proc. 38th IEEE CDC*, IEEE, New York, 1999, pp. 877-881.

[10] Rockafeller, R.T., *Convex Analysis*, Princeton University Press, Princeton, NJ, 1970.

[11] Piccoli, B., and H. Sussmann, "Regular synthesis and sufficient conditions for optimality," to appear in *SIAM J. Control*.

[12] Sontag, E.D., *Mathematical Control Theory, Deterministic Finite Dimensional Systems, Second Edition*, Springer-Verlag, New York, 1998.

[13] Soravia, P., "Pursuit evasion problems and viscosity solutions of Isaacs equations," *SIAM J. Control Opt.*, **31**(1993), pp. 604-23.

[14] Soravia, P., "Optimality principles and representation formulas for viscosity solutions of Hamilton-Jacobi equations I: Equations of unbounded and degenerate control problems without uniqueness," *Adv. Differential Equations*, **4**(1999), pp. 275-296.

[15] Souganidis, Panagiotis, "Two-player, zero-sum differential games and viscosity solutions," in *Stochastic and Differential Games: Theory and Numerical Methods*, M. Bardi, T. E. S. Raghavan, and T. Parthasarathy, Eds., Birkhäuser, Boston, 1999.

[16] Sussmann, H.J., "A general theorem on local controllability," *SIAM J. Control Optim.,* **25**(1987), pp. 158-194.

[17] Sussmann, H.J., "From the Brachystochrone problem to the maximum principle," in *Proc. 35th IEEE CDC*, IEEE, New York, 1996, pp. 1588-1594.

[18] Sussmann, H.J., and B. Piccoli, "Regular presynthesis and synthesis, and optimality of families of extremals," in *Proc. 38th IEEE CDC*, IEEE, New York, 1999, pp. 3352-7.

[19] Zelikin, M.I., and V.F. Borisov, *Theory of Chattering Control with Applications to Astronautics, Robotics, Economics, and Engineering*, Birkhäuser, Boston, 1994.