Further results on the Bellman equation for exit time optimal control problems with nonnegative Lagrangians: The case of Fuller's Problem

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1. Introduction

The theory of viscosity solutions forms the basis for much current work in optimal control and numerical analysis (cf. [1, 2, 3]). In two recent papers (cf. [4], [5]), we proved theorems characterizing the value function in deterministic optimal control as the unique viscosity solution of the Bellman equation that satisfies appropriate side conditions. The results applied to a very general class of problems whose dynamics and Lagrangians can be unbounded and fully nonlinear, including cases where the targets are unbounded and the dynamics are continuous but not Lipschitz continuous. In particular, the results apply to Fuller's Problem (cf. [8] and §2 below). Uniqueness characterizations of this kind have been used extensively to study singular perturbations, synthesis of optimal controls, convergence of numerical algorithms for approximating value functions and optimal trajectories, and much more (cf. [1], [3], and the hundreds of references therein). However, these results do not apply to exit time problems whose dynamics and Lagrangians are unbounded and whose Lagrangians vanish at some points outside the target for certain control values. In fact, one easily finds exit time problems with vanishing Lagrangians whose Bellman equations admit more than one proper viscosity solution that vanishes on the target, where by properness of a function w, we mean the condition $w(x) \to +\infty$ as $||x|| \to \infty$. Indeed, recall the following case from [4]:

Example Use the dynamics $\dot{x} = u \in [-1, +1]$ and use $\ell(x, a) = (x-2)^2(x-1)^2x^2(x+1)^2(x+2)^2$ as the Lagrangian. We consider the problem

$$\text{Minimize } \int_0^{t^\star} \ell(y_x(s, u), u(s))ds \ \text{ over } \mathcal{U}$$

for each $x \in \mathbb{R}^2$, where $\mathcal{U}$ is the set of all measurable functions $u : [0, \infty) \to [-1, +1]$, $y_x(\cdot, u)$ is the trajectory for the input $u \in \mathcal{U}$ starting at x, and $t^\star$ is the infimum of those times s at which $y_x(s, u) \in \mathcal{T}_1 := \{0\}$. The value function v for this problem, which is defined

$$v(x) := \inf \left\{ \int_o^{t^\star} \ell(y_x(s, u), u(s))ds : \, u \in \mathcal{U}, \, t^\star < \infty \right\},$$

is a proper viscosity solution of the corresponding Bellman equation $|Dv(x)| - \ell(x) = 0$ on $\mathbb{R} \setminus \{0\}$. The value function for the same problem but with the target $\mathcal{T}_1$ replaced by $\mathcal{T}_2 = \{-2, 0, +2\}$ is also a proper viscosity solution of this Bellman equation. □

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The paper [4] gave a uniqueness characterization for viscosity solutions of Bellman equations for exit time problems whose Lagrangians vanish for some points outside the target. The result of that paper applies to a very general class of problems whose dynamics give positive running costs over any interval where the state is outside the target, including Fuller's Problem, and shows that the value function is the unique proper viscosity solution of the Bellman equation which vanishes at the target. For related results, see [6].

This note gives a different approach which improves special cases of the result of [4] by proving that the value function for a class of problems including Fuller's Problem is the unique viscosity solution of the Bellman equation that vanishes at the target and is bounded below. We use the fact that all trajectories of these problems whose total running costs over $[0, \infty)$ are finite tend to the origin (cf. §3). For the preceding example, this condition is of course not satisfied, since the trajectory $x(t) \equiv 1$ generates a zero total running cost without reaching $\mathcal{T}_1$ or $\mathcal{T}_2$.

Remark The paper [7] generalizes the result of this note to variable interest problems with discontinuous exit costs. For such problems, one minimizes

$$\int_0^{t^\star} \ell(y_x(s, \beta), \beta(s))e^{-\delta(s,x,\beta)}ds + e^{-\delta(t^\star,x,\beta)}g(y_x(t^\star, \beta)),$$

over the measurable functions $\beta : [0, \infty) \to A$, where $\delta(s, x, \beta) := \int_0^s h(y_x(r, \beta), \beta(r))dr$, $t^\star$ is the exit time $\inf\{s \geq 0 : y_x(s, \beta) \in \mathcal{T}\}$ to a general closed target $\mathcal{T}$, and $\mathcal{T}$ and the control set A may both be unbounded. The corresponding Bellman equation is

$$\sup_{a \in A}\{-f(x, a) \cdot Dv(x) - \ell(x, a) + h(x, a) \cdot v(x)\} = 0,$$

where f is the dynamics. We can allow cases where h is not bounded below by a positive constant. □

2. Statement of Result

Recall that if $F : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is continuous and if $\Omega \subseteq \mathbb{R}^N$ is open, then we say that a continuous function $w : \mathbb{R}^N \to \mathbb{R}$ is a **viscosity solution** of the equation $F(x, Dw(x)) = 0$ on Ω provided these conditions hold:

(1) $F(\bar{x}, D\phi(\bar{x})) \leq 0$ for every C^1 function $\phi : \Omega \to \mathbb{R}$ and for every local maximum $\bar{x}$ of $w - \phi$.

(2) $F(\bar{\bar{x}}, D\mu(\bar{\bar{x}})) \geq 0$ for every C^1 function $\mu : \Omega \to \mathbb{R}$ and for every local minimum $\bar{\bar{x}}$ of $w - \mu$.

We say that a continuous function $\alpha : \mathbb{R} \to [0, \infty)$ is of **class MK** provided $\alpha(0) = 0$ and α is even and strictly increasing on $[0,\infty)$. We will take class MK functions as our Lagrangians. Note that functions of class MK need not be bounded, convex, or Lipschitz.

For each function α of class MK, we consider the following problem, which we denote by P_α:

$$\text{Minimize } \int_0^{t_p(\beta)} \alpha[x_p(t,\beta)]dt \text{ over } \beta\in\mathcal{U} \text{ subject to}$$

$$\dot{x}_p(t,\beta)=y_p(t,\beta),\ \ \dot{y}_p(t,\beta)=\beta(t),\ \ (x_p,y_p)(0)=p,$$

where $t_p(\beta):=\inf\{t\geq 0: x_p(t,\beta)=y_p(t,\beta)=0\}$, $\mathcal{U}$ is as above, the dots are used to indicate time derivatives, and $z_q(t,\beta):=(x_q(t,\beta),y_q(t,\beta))$. The value function of P_α, which we denote by v_α, is defined

$$v_\alpha(p)=\inf\left\{\int_0^{t_p(\beta)}\alpha[x_p(t,\beta)]dt : \beta\in\mathcal{U},\ t_p(\beta)<\infty\right\}.$$

For the case of $\alpha(x)=x^2$, we get Fuller's Problem (FP), whose value function is denoted v_{FP}. We prove the following result on the Bellman equation of P_α:

Theorem *If α is of class MK, and if $w:\mathbb{R}^2\to\mathbb{R}$ is a viscosity solution of the Bellman equation*

$$-x_2D_1w(x_1,x_2)+|D_2w(x_1,x_2)|-\alpha(x_1)=0 \qquad (1)$$

on $\mathbb{R}^2\setminus\{0\}$ which is bounded below and null at $\vec{0}$, then $w(x)=v_\alpha(x)$ for all $x\in\mathbb{R}^2$.

Notice that if α is also convex, then v_α is convex and therefore continuous on $\mathbb{R}^2$. Therefore, v_{FP} is the unique continuous viscosity solution of the corresponding Bellman equation (1) for Fuller's Problem (with $\alpha(x)=x^2$) which is null at 0 and bounded below. The hypotheses of the theorem are minimal, since one can find solutions of (1) which are null at the origin but not bounded below. For details, see [7].

3. Lemmas

To prove our theorem, we need the following converse dynamic programming inequalities and generalization of Barbălat's Lemma:

Lemma 1 *Let α be a function of class MK. Let w be a viscosity solution of (1) on $\mathbb{R}^2\setminus\{0\}$ and $B\subseteq\mathbb{R}^2\setminus\{0\}$ be a bounded open set for which $0\notin\bar{B}$. Let $q\in B$. If $\beta\in\mathcal{U}$ and $0\leq r<\inf\{t\geq 0: z_q(t,\beta)\in\partial B\}$, then*

$$w(q)\leq\int_0^r\alpha[x_q(t,\beta)]dt+w[z_q(r,\beta)]. \qquad (2)$$

Moreover, we have

$$w(q)\geq\inf_{\beta\in\mathcal{U}}\left\{\int_0^{\bar{t}}\alpha[x_q(t,\beta)]dt+w[z_q(\bar{t},\beta)]\right\} \qquad (3)$$

for all $0\leq\bar{t}<\inf\{t\geq 0:\operatorname{dist}(z_q(t,\beta),\partial B)<\delta,\beta\in\mathcal{U}\}$ and all $0<\delta<\operatorname{dist}(q,\partial B)$.

Proof. This is a special case of the proof of Theorem III.2.32 of [1]. □

Lemma 2 *Let α be of class MK. If $\phi:[0,\infty)\to\mathbb{R}$ is differentiable and ϕ' is Lipschitz continuous, and if $\int_0^\infty\alpha[\phi(s)]\,ds<\infty$, then $\lim_{t\to\infty}\phi(t)=\lim_{t\to\infty}\phi'(t)=0$.*

Proof. It suffices to check that $\lim_{t\to\infty}\phi'(t)=0$, since then ϕ is Lipschitz, and then the result follows by Barbălat's Lemma. We suppose that $\limsup_{t\to\infty}\phi'(t)>\gamma\in(0,1)$ wlog. (Otherwise, apply the same argument to $-\phi$.) By passing to a subsequence, we can choose $t_k\uparrow+\infty$ for which $t_{k+1}-t_k\geq 1$ and $\phi'(t_k)>\gamma$ for all k. Choose a Lipschitz constant $C>1$ for ϕ', pick $0<\delta<\gamma/(2C)$, and set $I_k=[t_k-\delta,t_k+\delta]$. Then

$$\phi'(t)=\phi'(t)-\phi'(t_k)+\phi'(t_k)\geq -C\,\delta+\gamma\geq\gamma/2 \quad (4)$$

for all $t\in I_k$. Set $v_k=\inf\{|\phi(t)|:t\in I_k\}$ for all k, and choose $s_k\in I_k$ so that $|\phi(s_k)|=v_k$.

If $\phi(s_k)>0$, then (4) gives $s_k=t_k-\delta$, so since $\phi>0$ on I_k, we get $\phi(t)\geq\gamma[t-(t_k-\delta)]/2$ for all $t\in I_k$, so

$$\int_{I_k}\alpha[\phi(s)]ds\ \geq\ 2/\gamma\int_0^{\gamma\delta/4}\alpha(u)du \qquad (5)$$

by changing variables. This last inequality remains true if $\phi(s_k)<0$ (in which case $s_k=t_k+\delta$ and we have $-\phi(t)\geq\gamma[t_k+\delta-t]/2$ on I_k) or if $\phi(s_k)=0$ (in which case we argue as in the $\phi(s_k)>0$ case on $[s_k,t_k+\delta]$ and as in the $\phi(s_k)<0$ case on $[t_k-\delta,s_k]$). Since $\delta<1/2$ and $t_{k+1}-t_k\geq 1$ for all k, the I_k's are disjoint, so we sum on k in (5) to contradict $\int_0^\infty\alpha[\phi(s)]ds<\infty$. □

4. Proof of Theorem

Let α and w be as in the hypotheses. The fact that $w\leq v$ on $\mathbb{R}^2$ follows easily from a repeated application of the inequality (2) (cf. [1]). To prove the reverse inequality, fix $\bar{x}\in\mathbb{R}^2$ and $\varepsilon>0$. We use Zorn's Lemma to find a $\bar{\beta}\in\mathcal{U}$ such that $t_{\bar{x}}(\bar{\beta})<\infty$ and

$$w(\bar{x})\geq\int_0^{t_{\bar{x}}(\bar{\beta})}\alpha[x_{\bar{x}}(t,\bar{\beta})]dt-\varepsilon \qquad (6)$$

and let $\varepsilon\downarrow 0$ to conclude. In what follows, we set $f(x,y,a)=(y,a)$, $\ell(x)=\alpha(x_1)$ for all $x\in\mathbb{R}^2$, and

$$E_j(t)\equiv\varepsilon/4\left[e^{-(j-1)}-e^{-(t+j-1)}\right]$$

for all $t\geq 0$ and $j\in\mathbb{N}$. (Our arguments easily generalize to cases where the instantaneous cost ℓ also depends on the control value.) Consider

$$\mathcal{Z}_1=\left\{\begin{array}{l}(t,\gamma):\ \ 0\leq t\leq 1,\ \text{ and }\ \gamma:[0,t]\to\mathbb{R}^2\text{ is a}\\ \text{trajectory for }\dot{z}=f(z,\beta)\text{ for some }\beta\in\mathcal{U}\\ \text{for which }\gamma(0)=\bar{x}\text{ with the property that}\\ w(\bar{x})\ \geq\ \int_0^t\ell(\gamma(s))ds\ +\ w(\gamma(t))\ -\ E_1(t)\,.\end{array}\right\}.$$

Then $\mathcal{Z}_1$ is partially ordered by the relation $\sim$ defined by $(t_1,\gamma_1)\sim(t_2,\gamma_2)\iff(t_1\leq t_2\ \&\ \gamma_2\lceil[0,t_1]\equiv\gamma_1)$. Moreover, one can use standard arguments to check that every totally ordered subset of $\mathcal{Z}_1$ has an upper bound in $\mathcal{Z}_1$. (Let $\{(s_j, \mu_j)\}_j \subset \mathcal{Z}_1$ be totally ordered, set $s_o = 0$ and $\bar{s} = \sup_j s_j$, and define $\bar{\mu} : [0, \bar{s}) \to \mathbb{R}^N$ by $\bar{\mu}(p) = \mu_j(p)$ for any j such that $s_j > p$. Let $\beta_j : (s_{j-1}, s_j] \to [-1, +1]$ be a control for the trajectory $\mu_j\lceil(s_{j-1}, s_j]$, and define $\beta^\dagger : (0, \infty) \to [-1, +1]$ by $\beta^\dagger(s) := \beta_j(s)$ on $(s_{j-1}, s_j]$ for $j \geq 1$ and $\beta^\dagger(s) = 1$ for $s \geq \bar{s}$. Then $\beta^\dagger$ is measurable and admits a trajectory $z_{\bar{x}}(\cdot, \beta^\dagger)$ on $[0, \infty)$. Since E_1 and w are continuous, the desired upper bound is $(\bar{s}, z_{\bar{x}}(\cdot, \beta^\dagger)) \in \mathcal{Z}_1$.)

It follows from Zorn's Lemma that $\mathcal{Z}_1$ contains a maximal element which we will denote by $(\bar{t}, \bar{\gamma})$. Wlog, $\bar{\gamma}$ does not reach 0 on $[0, \bar{t}]$. (Otherwise the requirement (6) is satisfied if we use the input for $\bar{\gamma}$). We show that $\bar{t} = 1$. Let B be an open set containing $\bar{\gamma}(\bar{t}) =: \hat{x}$ with $0 \notin \bar{B}$. Suppose $\bar{t} < 1$. Set $\tau = \inf\{t : \text{dist}(z_{\hat{x}}(t, \beta), \partial B) \leq \text{dist}(\hat{x}, \partial B)/2, \beta \in \mathcal{U}\}$. By standard estimates (cf. [1], Chapter 3), one checks that $\tau > 0$. By Lemma 1, it follows that there are $t \in (0, 1 - \bar{t})$ and $\beta \in \mathcal{U}$ so that

$$\begin{aligned} w(\hat{x}) \;\; \geq \;\; & \int_0^t \ell(z_{\hat{x}}(s, \beta))\, ds \;\; + \;\; w[z_{\hat{x}}(t, \beta)] \\ & - \;\; E_1(\bar{t} + t) \;\; + \;\; E_1(\bar{t}) \end{aligned} \tag{7}$$

and so that $z_{\hat{x}}(\cdot, \beta)$ remains in B on $[0, t]$. Let $\tilde{\alpha}$ denote the control for $\bar{\gamma}$. Since $(\bar{t}, \bar{\gamma}) \in \mathcal{Z}_1$, it follows that

$$w(\bar{x}) - w(\hat{x}) \;\; \geq \;\; \int_0^{\bar{t}} \ell(z_{\bar{x}}(s, \tilde{\alpha}))\, ds \;\; - \;\; E_1(\bar{t}). \tag{8}$$

Let $\beta^\sharp$ denote the concatenation of the input $\tilde{\alpha}\lceil[0, \bar{t}]$ followed by the input β. Since $t + \bar{t} < 1$, we can add (7) and (8) to conclude that $(t + t, z_{\bar{x}}(\cdot, \beta^\sharp)) \in \mathcal{Z}_1$. This contradicts the maximality of $(\bar{t}, \bar{\gamma})$, so $\bar{t} = 1$.

We now extend $\bar{\gamma}$ to get the desired trajectory. Set

$$\mathcal{Z}_2 = \left\{ \begin{array}{l} (t, \gamma) : \;\; 0 \leq t \leq 1, \;\; \text{and} \;\; \gamma : [0, t] \to \mathbb{R}^2 \text{ is} \\ \text{a trajectory for } \dot{z} = f(z, \beta) \text{ for some input} \\ \beta \in \mathcal{U} \text{ which is such that } \gamma(0) = \bar{\gamma}(1) \text{ and} \\ w(\bar{\gamma}(1)) \;\geq\; \int_0^t \ell(\gamma(s))ds + w(\gamma(t)) - E_2(t)\,. \end{array} \right\}.$$

and partially order $\mathcal{Z}_2$ as before. By the preceding, we know that $\mathcal{Z}_2$ contains a maximal element $(1, \tau_2)$. Let $u_2 : [0, 1] \to [-1, +1]$ denote the input for τ_2. We can assume wlog that τ_2 does not reach 0 on $[0, 1]$. Indeed, if there is an $s_2 \in [0, 1]$ such that $\tau_2(s_2) = 0$, then we can satisfy the requirement by concatenating $\tilde{\alpha}$ followed by u_2, where $\tilde{\alpha}$ is the control for $\bar{\gamma}$ on $[0, 1]$. Let c_2 denote the concatenation of $\tilde{\alpha}\lceil[0, 1]$ followed by $u_2\lceil[0, 1]$, and let γ_2 denote the corresponding concatenated trajectory for c_2 from $\bar{x}$ to $\tau_2(1)$.

Now reapply the procedure to $\gamma_2(2)$ to get a longer trajectory that reaches 0 or runs for three time units. The procedure is iterated, and it results in a sequence of terminal points $p_n := \phi_n(n)$ for trajectories ϕ_n which begin at $\bar{x}$, are defined on $[0, n]$, and wlog do not reach 0. It follows that for all $n \in \mathbb{N}$, we would have

$$w(\bar{x}) \;\; \geq \;\; \int_0^n \ell(\phi_n(s, c_n))\, ds \; + \; w(\phi_n(n)) \; - \; \frac{\varepsilon}{2}. \tag{9}$$

Setting $\hat{\alpha}(s) = c_n(s)$ if $n - 1 < s \leq n$, letting $\hat{\phi}$ denote the corresponding trajectory, and letting b denote a lower bound for w, a passage to the limit as $n \to \infty$ in (9) gives $\int_o^\infty \ell(\hat{\phi}(s))ds \leq w(\bar{x}) - b + \varepsilon/2 < \infty$, since $\hat{\phi} \equiv \phi_n$ on $(n-1, n]$. It follows from Lemma 2 that $\lim_{s\to\infty} \hat{\phi}(s) = 0$. Since the FP dynamics is controllable to the origin and w is continuous, standard estimates (cf. [1]) guarantee $m \in \mathbb{N}$ and $\beta_m \in \mathcal{U}$ for which $w(\hat{\phi}(m)) \geq -\varepsilon/4$, $t_{p_m}(\beta_m) < \infty$, and $\int_0^{t_{p_m}(\beta_m)} \ell(z_{p_m}(s, \beta_m))ds < \varepsilon/4$. We then satisfy (6) by taking $\bar{\beta}$ to be $\hat{\alpha}\lceil[0, m]$ followed by β_m. □

Remark One can prove a variant of the theorem for solutions of (1) on open sets $\Omega \subseteq \mathbb{R}^2$ with $\{0\}$ replaced by a general closed target $\mathcal{T}$. To do this, we add the condition $\gamma : [0, 1] \to \Omega$ in the $\mathcal{Z}_i$'s and assume

$$w(x) < \bar{w} \;\; \forall x \in \Omega \quad \text{and} \quad \lim_{x \to x_o} w(x) = \bar{w} \;\; \forall x_o \in \partial\Omega$$

for some $\bar{w} \in \mathbb{R} \cup \{+\infty\}$ (cf. [1]). The proof of the inequality $w \leq v_\alpha$ is more involved since one must consider trajectories which exit Ω before reaching $\mathcal{T}$. To cover cases where $\mathcal{T} \not\subseteq \Omega$, we assume $\partial\Omega \subseteq \mathcal{T}$. For details, see [7]. □

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