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SHORTEST PATHS FOR THE REEDS-SHEPP CAR: A WORKED OUT EXAMPLE OF THE USE OF GEOMETRIC TECHNIQUES IN NONLINEAR OPTIMAL CONTROL. [1]

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ABSTRACT

We illustrate the use of the techniques of modern geometric optimal control theory by studying the shortest paths for a model of a car that can move forwards and backwards. This problem was discussed in recent work by Reeds and Shepp who showed, by special methods, (a) that shortest path motion could always be achieved by means of trajectories of a special kind, namely, concatenations of at most five pieces, each of which is either a straight line or a circle, and (b) that these concatenations can be classified into 48 three-parameter families. We show how these results fit in a much more general framework, and can be discovered and proved by applying in a systematic way the techniques of Optimal Control Theory. It turns out that the "classical" optimal control tools developed in the 1960's, such as the Pontryagin Maximum Principle and theorems on the existence of optimal trajectories, are helpful to go part of the way and get some information on the shortest paths, but do not suffice to get the full result. On the other hand, when these classical techniques are combined with the use of a more recently developed body of theory, namely, geometric methods based on the Lie algebraic analysis of trajectories, then one can recover the full power of the Reeds-Shepp results, and in fact slightly improve upon them by lowering their 48 to a 46.

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[1] Most of the results and techniques of this paper were presented at a nonholonomic systems day organized by J.-P. Laumond at the Laboratoire d'Automatique et d'Analyse des Systèmes in Toulouse, France, on July 10, 1991, . At this meeting, we learned of a similar effort by A. Boissonat, A. Cérézo and J. Leblond, who also used the Pontryagin Maximum Principle to derive some of the properties of the minimum paths of the Reeds-Shepp car, cf. [7].

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1 Introduction

The purpose of this paper is to provide an introduction to some geometric techniques of modern geometric nonlinear optimal control theory by means of a worked out example. We will study the problem, considered by Reeds and Shepp in their remarkable paper [40], of determining the structure of the shortest paths for a model of a car that is moving in a plane and can go both forwards and backwards. (A 1957 paper by Dubins, [19], had analyzed a similar model for the case when only forward motion is allowed. We discuss Dubins' results in the Appendix.) However, our primary goal is not only to analyze the Reeds-Shepp shortest paths, but to exhibit the power of the tools of modern optimal control theory, and to argue that this power is significantly enhanced by the addition of the relatively newer tools of *geometric control theory*. In particular, we will attempt to show that *the classical techniques of optimal control, as developed in the 1960's and early 1970's, do not by themselves suffice to do the job.* This means that we will simultaneously be making a case for

- the advantages of optimal control theory, and
- the superiority of modern, geometric optimal control over the "classical" theory of the 1960's and early 70's.

Because of the multiple objectives described above, the paper is addressed to three types of readers: (1) *experts in geometric nonlinear control*, who wish to learn about the Reeds-Shepp problem but do not need to be convinced of the importance of optimal control or of geometric methods, (2) *optimal control theorists of a more classical sort*, who are not necessarily familiar with geometric techniques, but wish to learn about them by seeing them at work in a reasonably nontrivial example, (3) *people not necessarily acquainted with optimal control theory.* To make the paper accessible to all three types of readers, we have included explanations of many of the most basic concepts, including some that will probably be well known to many readers. Naturally, these can be skipped by those who are looking for the shortest path to the results.

In [40], Reeds and Shepp showed that any two positions of the car in its state space can be joined by a shortest path of a particularly simple kind, namely, a concatenation of at most five pieces, each of which is either a circle or a straight line. The trajectories they found fall into 48 types characterized by discrete invariants, and each type gives rise to a family that depends on the choice of six parameters, namely, three time parameters plus the initial condition. (Since the state space of the Reeds-Shepp problem is three-dimensional, it is clear that, to be able to join any two arbitrary points p, q, we need at least six independent parameters.) Equivalently, the Reedds-Shepp work singles out *48 three-parameter families of controls* that, together, suffice to join any two states by a shortest path.

The results of [40] were derived by means of an impressive array of techniques especially developed for the study of this particular problem, without making any use of optimal control theory. In their discovery of the results, and also in the proof, the use of the computer played a fundamental role. According to Reeds and Shepp, "we used a computer to empirically determine a sufficient list of words," that they call W, until "we eventually arrived at and then convinced ourselves that we had a minimal sufficient set" and then, "once we had guessed at W, we used the computer again to help do the extensive algebra in the large number of cases involved to verify that a rigorous proof could be given by the method outlined above. Finally, we found that the proof could be simplified, so that it can easily be followed by an ordinary human without a computer to check the details." But they then go on to point out that "we think that we could never have found the right set of words without using a computer."

We will show how the techniques and concepts of modern optimal control theory make it possible to give a simple and complete derivation of the results of Reeds and Shepp, and in fact slightly lower their 48 to a 46 (cf. Remark 17), showing that their "minimal sufficient set" is not minimal after all, and that human reason, by developing powerful theories with a wide range of applicability, can both *discover* and *verify* results such those of [40] *without any help from a computer*, and can not only do as well but in fact slightly better than the computer. If one just applies in a systematic way the Pontryagin Maximum Principle, together with other well known necessary conditions for optimality that have been available in the literature for several years, and that *were discovered in the course of dealing with other optimal control problems*, it turns out that one is led exactly to a set of "words" which is in fact slightly better that the one of Reeds and Shepp. In our view, this shows that the systematic use of already existing theory provides a very powerful tool that sometimes can do at least as well —and, in fact, slightly better— than the computer. As will be clear from our argument, the application of the Maximum Principle immediately narrows down the set of possible optimal trajectories to a much smaller class, and tells us what to look at in order to produce further improvements. One can then apply other techniques, such as the theory of envelopes that we developed for other purposes in [58] and [65], to exclude most possibilities still permitted by the Maximum Principle. For instance, the Maximum Principle allows for concatenations of an arbitrarily large number N of circles, provided that all of them except for the first and last ones have the same length. If one then applies the theory of envelopes, and carries out the calculation prescribed by the theory, then one finds that as soon as there are three consecutive pieces of equal length, followed or preceded by a fourth piece, then the trajectory is not optimal. We emphasize that both the application of the Maximum Principle as well as the use of the theory of envelopes later on, are not just "verification techniques," i.e. methods for proving something after it has been discovered. They are first and foremost *discovery techniques*, in

that one needs to do nothing other than mechanically apply them, without any guessing, and then the answer pops out right away. (First, the Maximum Principle *forces us* to restrict ourselves to the case of circles with equal length, and then the theory of envelopes tells out what to compute in order to exclude too many circles. The calculation is elementary, involves no more than computing some first partial derivatives, and can be done by hand. The theory of envelopes tells us in advance that, if we succeed in writing a certain vector field as a linear combination of two others, then a certain identity will automatically follow, and this identity implies non-optimality. To stress the simplicity of the method, and to make our paper self-contained, we will even abstain from invoking the general envelope theorem, and will instead perform the calculations directly and verify the envelope identity in our special case.)

The Reeds-Shepp problem turns out to be very much unlike the ones that have been studied in classical optimal control theory, in that (i) existence of time-optimal trajectories is not known *a priori* by the standard convexity arguments, and (ii) the Pontryagin Maximum Principle by itself does not provide enough information, and has to be supplemented by a number of other considerations. This will provide us with an opportunity to exhibit some of the variety of concepts and techniques that have been introduced in recent years. The Reeds-Shepp problem (henceforth abbreviated as "RS"), happens to lend itself particularly well to such an undertaking, because its solution requires the use of many different optimal control tools, but the calculations needed to effectively apply these tools are quite simple. We will thus be able to display the richness and power of the theory without being bothered by cumbersome technicalities.

Acknowledgements. We thank Jean-Paul Laumond and Shankar Sastry for having brought the Reeds-Shepp problem to the attention of the first author, and for their helpful comments. We are greatly indebted to J. Reeds and L. Shepp both for having broken new ground with their paper, and for the discussions they have had with us.

2 Outline

In this section we give a first outline of the strategy to be followed, and discuss some general issues that occur when trying to apply optimal control theory to particular problems.

The main tool for our analysis will be the Pontryagin Maximum Principle (PMP) which, since the publication of the book [39], has been justly regarded as one of the fundamental results of optimal control theory. The PMP provides a necessary condition for a trajectory to be optimal. Sometimes this condition can be applied to derive very strong conclusions about a problem, e.g. that all optimal trajectories necessarily are of a very special kind. But often the conclusions derived from the PMP do not suffice to solve the truly interesting problem which is, typically, that of finding an *optimal synthesis* or, at least, a nice family of trajectories

which is *sufficient for optimality.* (The definitions are as follows: a collection $\mathcal{F}$ of trajectories is *sufficient for optimality* if, given any two points p, q, there exists a trajectory in $\mathcal{F}$ that goes from p to q and is optimal. Notice that it is not required that all the members of $\mathcal{F}$ be optimal, only that $\mathcal{F}$ contain sufficiently many optimal trajectories. The result of [40] is precisely the determination of a particularly nice such $\mathcal{F}$. The concept of "sufficient family" was introduced in [60], [61], [62]. An *optimal synthesis* is a selection of an optimal trajectory for each initial point, for the problem of optimally reaching a given target set, having some extra regularity properties. Various definitions of optimal synthesis —with different technical requirements— and of "regular synthesis" —i.e. a synthesis whose trajectories fit together in some particularly smooth or piecewise smooth way— have been proposed in the literature, e.g. in [6], [16], [17], [55], [59], [60], [61], [62], [66]. Usually, a synthesis is required to be "memoryless," i.e. such that, if $\gamma : [a, b] \to M$ is the chosen trajectory for a given initial state p, and $q = \gamma(t)$, then the trajectory chosen from q is precisely the restriction of γ to $[t, b]$. That is, once we find ourselves at a point q, having started earlier at p, the instructions given by the synthesis are the same as those it would give if we were starting anew at q. For the more general concept of a trajectory selection where this memoryless property is not required, the name "presynthesis" was proposed in [66].)

It turns out that, for the search for sufficient families and for the synthesis problem, the information obtained from the PMP is usually insufficient by itself, and has to be supplemented with other considerations. This is so because at least four other issues need to be resolved:

I1 the PMP by itself does not guarantee the *existence* of optimal trajectories;

I2 the necessary condition of the PMP is often too weak, and is satisfied by too many trajectories other than the optimal ones;

I3 extracting concrete information from the PMP can be difficult;

I4 in many cases, the class of optimal trajectories is too large, so to get a synthesis or a nice sufficient family one has to restrict it further, by proving the existence of nice selections.

To resolve Issue I1, *existence theorems for optimal trajectories* are needed. The existence question was extensively investigated in the 1960's, mostly by means of functional analytic arguments, and can be considered well understood by now (cf. [5], [18]).

Remark 1 Existence theorems often involve a *controllability* assumption. For instance, if it is desired to show that for any given pair (p, q) of initial and terminal states there is an optimal trajectory from p to q, this is typically done by picking a sequence of trajectories γ_j from p to q whose costs converge to the infimum, and then somehow extracting a convergent subsequence. This requires knowing, to begin with, that there exists at least one trajectory

from p to q, i.e. that q is *reachable* from p. The study of which points are reachable from which points is known as *reachability theory.* A system is *completely controllable* if for every p, q it follows that q is reachable from p. If we want existence of optimal trajectories for all p, q, then the following "zero-th issue" arises:

I0 Controllability has to be established separately. ∎

Issue I2 poses a much more serious difficulty, and has led to the development of many *high order necessary conditions for optimality.* (The PMP is usually regarded as the "first order" condition.) The search for such conditions was a very active field in the 1960's and early 1970's, after which progress slowed down, until the advent of geometric methods provided a new perspective that led to the discovery of many new conditions (cf. [1], [2], [3], [4], [8], [9], [10], [11], [28], [29], [42], [43], [44], [45], [52], [53], [54], [58], [60], [61], [62], [65]).

Issues I3 and I4 —which do not appear to have been seriously addressed in the pre-1975 optimal control literature— are also rather delicate. (Notice that Issues I2 and I3 are quite different. Issue I3 has to do with the *kind of information* one gets about a trajectory when one applies the PMP, which is not at all the same as the *strength* of the PMP alluded to in Issue I2.[3]) In recent years, motivated by the desire to study synthesis problems, both issues have attracted a lot of attention in the geometric control literature, and several new methods have been introduced to deal with them.

It turns out that *all four issues listed above arise in the study of the RS problem.* To begin with, no general theorem giving existence of optimal controls is directly applicable to the RS problem, because the set of control values is not convex. Therefore, RS cannot be studied by just using necessary conditions for optimality together with the following naïve reasoning: to prove that every γ can be shortened to a $\hat{\gamma} \in \mathcal{F}$, pick an arbitrary γ with starting point p and terminal point q, then pick an optimal $\hat{\gamma}$ from p to q, so $\hat{\gamma}$ is certainly shorter than γ, and then use necessary conditions to show that $\hat{\gamma} \in \mathcal{F}$, or that $\hat{\gamma}$ can be replaced by a $\tilde{\gamma} \in \mathcal{F}$. (The reasoning fails, because $\hat{\gamma}$ need not exist.) So Reeds and Shepp were forced to use a different approach and make a direct study of the possibility of "shortening" arbitrary trajectories. Starting with some trajectory γ from a point p to another point q, they show that γ can be shortened in various ways, until one ends up with a trajectory that cannot be shortened further and happens to be of the special kind that appears in their main result. This requires the comparison of lengths of various non-optimal paths, which is done by elaborate special arguments.

We will proceed differently, and attempt to follow a time-honored mathematical tradition, based on a three step "E-E-R" (extension, existence, regularity) approach:

[3] A necessary condition for optimality can clearly be at the same time quite strong and totally useless, as shown by the tautological statement that "optimality is a necessary condition for optimality."

EER1 Take a problem, originally specified in a certain space S of objects, for which there is no obvious theorem giving existence of solutions (e.g. the Dirichlet problem in the space of C^∞ functions), and embed it in a problem (e.g. the Dirichlet problem in a Hilbert space such as H_1) in a suitably chosen extended space S_{ext} (usually taken to be the completion of S with respect to some metric).

EER2 Prove existence of solutions for the new, "extended" problem. (Usually, this involves the fact that S_{ext} has sufficiently many compact subsets.)

EER3 Prove *regularity* of the solutions in S_{ext} and show, in particular, that they actually lie in S (e.g. show that every harmonic function is smooth).

In our case, this amounts to studying the *convexified Reeds-Shepp* (henceforth abbreviated as "CRS") model. This is obtained by *convexifying* the RS model, i.e. by replacing the nonconvex control set $\tilde{U}$ of [40] by its convex hull U. In other words, if u and v denote, respectively, the velocity of motion of the car and the angular velocity of its rotation, we will assume in the CRS model that $|u| \leq 1$ and $|v| \leq 1$, whereas in the RS model it is required that $|u| = 1$ and $|v| \leq 1$, so that $\tilde{U}$ is the union of two vertical segments in the (u, v) plane, whereas U is the unit square. (The 1957 work of Dubins [19] studied the case when $u = 1$, i.e. when the car can only move forwards.) In the RS model the car is always moving —forward or backward— with velocity 1, so that the RS paths have a radius of curvature not smaller than 1, which means that "sharp turns" are not allowed. In the CRS model the car can be turning at the maximum angular velocity while moving slowly, or even standing still. So very sharp turns —and even turning in place— are permitted. Since the velocity vector of the RS paths in the plane has length one, these paths are parametrized by arc-length, so length and time are equal. (Naturally, this is no longer true for the CRS model, since now $u = 0$ is allowed.) So the problem formulation of [40], in terms of shortest paths, is completely equivalent to a minimum time problem. It is in this latter form that we extend the optimization problem to the CRS model, by seeking minimum time trajectories.

As opposed to RS, the CRS model lends itself directly to the use of optimal control techniques, because U is compact and convex. Using standard existence results from classical optimal control theory (e.g. those of [5] and [18]), we will conclude that given any two points p, q, there exists an optimal CRS path from p to q.

So now the difficulty due to Issue I1 has been overcome. Given any γ from p to q, we know *a priori* that γ can be shortened to an optimal (i.e time minimizing) $\hat{\gamma}$ from p to q. The price we have to pay is that

(#) we only know that $\hat{\gamma}$ is a CRS trajectory, and nothing guarantees that it is RS-admissible. (In fact, there are plenty of optimal trajectories of CRS that are not RS-admissible.

For instance, it is obvious that "turning left with maximum angular velocity while not moving" —i.e. choosing $u \equiv 0$, $v \equiv 1$— is time-optimal for CRS up to time π, but not RS-admissible.)

In spite of (#) we will apply the PMP to try to get information about $\hat{\gamma}$. However, when we do so we will find that we have to deal with I2, I3 and I4. For this reason, we will have to combine the PMP with the more novel tools of geometric control theory, and in particular with *the systematic use of Lie brackets of vector fields and the structural properties of the Lie algebras associated to a control system.* We will find that:

- in some cases, the situation described in I2 occurs: the PMP is too weak, and allows some nonoptimal trajectories. To overcome this difficulty, we will have to use some special methods. Most of the nonoptimal extremals can be dealt with by simple geometric considerations, but in one case —to exclude concatenations of five circles— we will have to use a geometric tool —the theory of envelopes— that was introduced in optimal control theory to study a totally different class of problems.

- I3 poses a problem as well. To write the conditions of the PMP in a more manageable form, we make systematic use of a technique that was also introduced elsewhere for the study of other problems. We translate the conditions of the PMP into a system of differential equations involving the switching functions, and get information about the optimal controls by analyzing the solutions. (The structure of this system of equations is closely related to the Lie algebraic relations of the vector fields of CRS.)

- Finally, I4 also causes trouble. The class of *all* optimal trajectories turns out to be too large. For instance, any trajectory corresponding to $v \equiv 1$, and of duration $\leq \pi$, is time-optimal. The control $t \to u(t)$ can be an *arbitrary* measurable function with values in $[-1, 1]$.) The subproblem of CRS obtained by restricting consideration to those trajectories for which $v \equiv 1$ will be called the "LTV problem," and what we have said implies that this problem is "degenerate," (A problem is *degenerate* if the cost is the same for all trajectories whose initial and terminal points agree, i.e. if for every pair p, q, of points, if γ, δ are two trajectories that go from p to q, then γ and δ have the same cost. A problem is *locally degenerate* if every point has a neighborhood on which the problem is degenerate.) Once again, it turns out that similar situations have been encountered in optimal control theory when studying other problems, and techniques used for these other problems also work here. Degenerate problems can be characterized in terms of the relation between the "accessibility Lie algebra" $L(\Sigma)$ and the "strong accessibility Lie algebra" $L_0(\Sigma)$ of a control system Σ, both of which were introduced in [48]. When degeneracy occurs, a technique introduced in [52] can be applied to prove

a "regularity with choice" theorem. In our case, this enables us to *replace* $\hat{\gamma}$ by a nice trajectory.

When all this work is complete, we end up having singled out a family $\mathcal{F}$ of trajectories of CRS which is sufficient for optimality and consists of trajectories of a very special form. In particular, *these trajectories happen to be admissible for RS*, so *our solution of the minimum time problem for the CRS model immediately yields a solution of the Reeds-Shepp problem as well.*

So, in fact, we will end up not having directly applied Step 3 of the E-E-R approach. Instead, we will have used a slight modification: rather than trying to prove regularity of *all* the solutions, we only seek to *choose*, from the set of all solutions, a sufficiently large subset where the desired regularity holds. (We refer to this as the "E-E-RC" approach. The "RC" stands for "regularity with choice.") It is a rather general fact that the need to deal with "regularity with choice" theorems for optimal trajectories, rather than just plain regularity results, is intimately related to the occurrence of degenerate problems.

Remark 2 Naturally, degenerate problems are not very interesting in themselves, since for such problems nothing special can be said about the optimal trajectories. However, it turns out that *degenerate problems often show up as subproblems of more interesting, nondegenerate ones*, as they do in the case of CRS and its LTV subproblem. It is for this reason that the degenerate case. has become important in recent work on optimal synthesis, e.g. in the series of papers [60], [61], [62]. ∎

3 "Classical" vs. "modern" optimal control theory

To understand why the "classical" tools such as the Maximum Principle need to be supplemented with Lie algebraic techniques and other geometric methods, we include a brief account of the evolution of optimal control theory, told from the —acknowledgedly biased— perspective of practitioners of the modern geometric approach.

Historically, the birth of optimal control theory can be dated to the publication of [39], where the *Pontryagin Maximum Principle* was stated and proved. (Cf. also [5], [34], and [55].) However, the conditions given by the Maximum Principle often require a considerable amount of work in order to yield concrete information about the optimal trajectories, e.g. that every optimal trajectory is necessarily of a particularly simple kind, such as a concatenation of four or five pieces of a special type. Such an analysis can be carried out, and was indeed carried out, for a number of specific problems that were widely studied in the 1960's.

However, if one wishes to go beyond the study of special problems and develop theories of *general classes* of systems, so as to be able to prove theorems of the form "for every system of

such-and-such kind the optimal trajectories are of such-and-such type," one needs, to begin with, a *language* with which to describe the properties of systems that may be relevant to the understanding of the structure of their optimal trajectories. We contend that one of the main virtues of the modern geometric approach is that it has provided such a language.

The following analogy may help understand the situation. Suppose we want to solve one particular system of linear equations. We can certainly do it using elementary high-school algebra recipes, such as adding to one equation a multiple of another equation, and so on. This can be done without ever using general concepts such as determinant, linear independence, or rank. If, however, one wishes to go beyond the study of one particular system at a time, and develop a *general theory* of the solutions of linear systems of equations, then the concepts of determinant, rank, etc. become necessary. For example, suppose we want to know if a particular $n \times n$ system S has a unique solution. This can be settled directly using elementary tools, without talking about determinants. Armed with these tools, one can thoroughly analyze the system, show that the solution is unique, and compute it. One could then believe that the problem is complete understood. However, most of us would prefer to look at it from the "higher" theoretical perspective offered by linear algebra, and would end up finding that the *real reason* why S has a unique solution is that the determinant of its matrix of coefficients is $\neq 0$. Whoever is solving S need not know this, and may even be legitimately puzzled and wonder whether this new knowledge about the determinant adds anything at all to an already quite thorough understanding of the particular system S. Since every particular system can always be analyzed by elementary tools, while maintaining a state of blissful ignorance of the concepts of determinant and rank, it might be hard to persuade our problem solver that these concepts do matter.

Similarly, one can study particular optimal control problems, and in some cases actually manage to describe their optimal trajectories completely, without mentioning a Lie bracket even once. From the "higher" perspective of Lie algebraic optimal control theory, we could say, for instance, that the *real reason* why there is a bang-bang theorem for linear time-optimal control is that for linear systems certain Lie brackets vanish, and these brackets are precisely the ones that, for more general systems, are responsible for the occurrence of non-bang-bang optimal controls. However, the linear system theorist already knows that for linear systems there is a bang-bang theorem, and may fail to be impressed by the new "conceptual explanation" proposed for this fact. Most nonlinear optimal control problems that were studied in the 1960's were studied as individual problems, and it is perhaps for this reason that the ubiquitousness of the Lie brackets went unnoticed. As will be illustrated in our analysis of the car problem, Lie brackets show up immediately whenever one differentiates a switching function, and differentiation of switching functions is the key step in the analysis of the structure of optimal trajectories. However, when one is studying one particular problem,

the switching functions and their derivatives are given by formulae, so one can compute these derivatives without ever noticing that they are expressible in terms of Lie brackets.

When one wants to prove general theorems and develop general techniques for general classes of nonlinear optimal control problems, the concept of a Lie bracket becomes an essential part of the vocabulary to be used. Not only do Lie brackets show up whenever one differentiates a switching function: they matter for many other profound reasons, e.g. because they determine the controllability and degeneracy properties of a system. (A lot of recent work on nonlinear controllability and optimal control —cf. [11], [25], [27], [28], [29], [42]-[44], [48], [46], [45], [51]-[65]— illustrates how Lie brackets play a crucial role in analyzing the controllability properties of nonlinear control systems, and the regularity properties of optimal trajectories.)

"Modern" optimal control theory, in our view, is a systematic effort to combine the use of the Maximum Principle and other tools from the 1960's with a number of other mathematical techniques, in order to prove *general* theorems about the structure of the solutions for *general* classes of problems. One example of this actually dates back to the 60's: the theory of *linear quadratic optimal control* is based on the combined use of the Maximum Principle and the Bellman equation, together with a substantial amount of linear algebra and the theory of the Ricatti equation. For most other classes of optimal control problems, little was done under the mid 70's because the tools that were really needed had not yet began to be exploited.

The situation changed when P. Brunovsky, in his two remarkable papers [16], [17], published in 1978 and 1980, proved general theorems on the existence of a regular synthesis, using the theory of *subanalytic sets*. Brunovsky's work showed that one can prove synthesis results provided one could produce a reasonably detailed description of the structure of the optimal trajectories. Moreover, in order to put together a synthesis one does not need to know all the optimal trajectories. A sufficient family, as defined above, will do. This provided the impetus for attempting to prove theorems on the structure of the optimal trajectories in a variety of cases, and trying to show the existence of sufficient families with a simple structure.

In the meanwhile, the tools that would eventually become important for the analysis of trajectories —that is, Lie brackets and Lie algebras of vector fields— were gradually being introduced in control theory, in the work of Hermann, Brockett, Lobry, Krener, Sussmann, Jurdjevic, and others (cf. [12], [13], [22], [23], [24], [27], [28], [35], [48]). Using these tools, a lot of work for several classes of problems was done in the 1980's, and several widely applicable techniques were developed.

In our study of the car problem, we will make systematic use of techniques that were developed for other classes of problems. We will use Lie brackets to analyze the switching functions, a selection technique for degenerate problems, and the theory of envelopes. For our particular problem, we could of course have carried out the calculations without mentioning

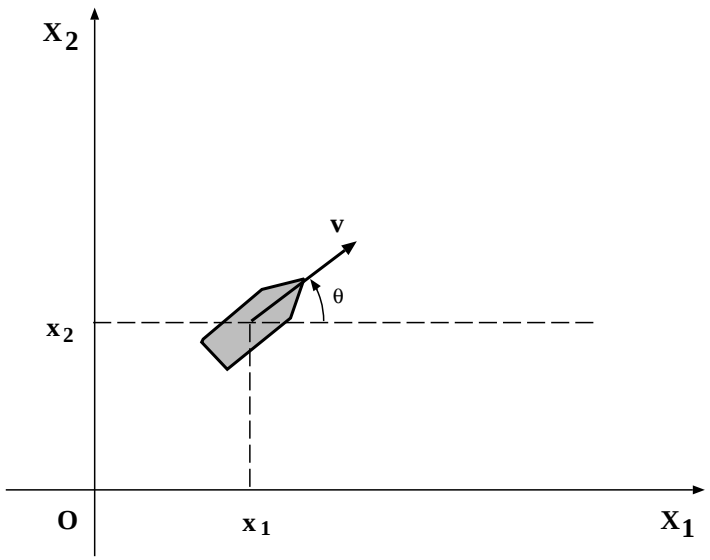

Figure 1: The coordinates.

Lie brackets, but this would fail to exhibit the basic underlying mathematical structure of the problem. Similarly, when we use in Section 14 the general technique of envelope theory to exclude certain extremals, our actual calculations will be self-contained and will not explicitly mention envelopes, but we will explain how envelope theory led us to them, for otherwise the calculations would appear unmotivated and mysterious. Our longer presentation will, we hope, show that behind the apparent madness of the various techniques used here there is a *unified method* that can be used to study many other problems as well.

4 The model

The car is modeled as a region $\mathcal{R}$ moving rigidly in the plane, to which a unit vector $\mathbf{v}$, the *orientation*, is rigidly attached (cf. Fig. 1).

Because the motion is rigid, the car has three degrees of freedom, namely, the coordinates x_1 and x_2 that give the position in the plane of some point of $\mathcal{R}$ (e.g. the center of mass), and the angle θ of the orientation vector $\mathbf{v}$ with the x_1 axis. With respect to these coordinates, the vector $\mathbf{v}$ is given by $\mathbf{v} = (\cos\theta, \sin\theta)$. We assume that the car can move forward or backward with a velocity vector equal to $u\mathbf{v}$, where the scalar u is required to satisfy $|u| \leq 1$. In addition, it can turn, i.e. change direction. The angular velocity $v = \dot{\theta}$ satisfies $|v| \leq 1$. The variables x_1 and x_2 are real numbers. On the other hand, θ is an angle, i.e. a real number modulo 2π or, equivalently, a member of S^1, the unit circle in the plane. We will refer to the point $(x_1, x_2) \in \mathbb{R}^2$ as the *position* of the car, to the angle θ as the *direction*, or *orientation*, and to the triple (x_1, x_2, θ) as the *state*. The *state space* M of the system, i.e. the space of the variables x_1, x_2, θ, is then the cylinder defined by $M = \mathbb{R}^2 \times \mathrm{S}^1$.

The full dynamical model for CRS is given by:

$$CRS: \quad \left\{ \begin{array}{rcl} \dot{x}_1 &=& u\cos\theta \ , \\ \dot{x}_2 &=& u\sin\theta \ , \\ \dot{\theta} &=& v \ , \end{array} \right. \tag{1}$$

where $(x_1, x_2, \theta) \in M$, $(u, v) \in U$, and $U = [-1, 1] \times [-1, 1]$.

The *Reeds-Shepp system* RS and the *Dubins system* DU are described by exactly the same equations (1), except that now the control vector (u, v) is required to belong to the sets $\tilde{U} = \{(u, v) \in U : |u| = 1\}$, $U_D = \{(u, v) \in U : u = 1\}$, respectively. We also want to consider the *LTV problem*, where the control is required to belong to the set $U_{LTV} = [-1, 1] \times \{1\}$, i.e. to be such that $v \equiv 1$, and the *lifted LTV* (LLTV) problem, which is exactly like LTV except that its state space is $\mathbb{R}^3$ rather than M, and θ is regarded as a true real number, rather than a number modulo 2π. (The letters "LTV" stand for "left turning velocity.")

We remark that, *for the systems CRS, DU, LTV and LLTV (but not for RS) the control set is compact and convex.*

5 Trajectories and optimal control

The equations for CRS can also be written in the form

$$CRS: \quad \dot{x} = uf(x) + vg(x), \quad (u, v) \in U, \ \ x = (x_1, x_2, \theta)^\dagger \in M \ , \tag{2}$$

where "$\dagger$" means "transpose," and the vector fields f and g are given by

$$f(x) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \quad g(x) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3}$$

An *admissible control* for CRS is a measurable function $\eta = (u(\cdot), v(\cdot)) : [a, b] \to U$, where $[a, b]$ is some compact interval. (In that case, the interval $[a, b]$ is the *domain* of η, and is denoted by $\mathrm{Dom}\,(\eta)$.) We use $\mathcal{U}$ to denote the class of all admissible controls for CRS. We identify admissible controls that agree almost everywhere. A *trajectory* of CRS for a control $\eta = (u(\cdot), v(\cdot))$ with domain $[a, b]$ is an absolutely continuous curve $\gamma : [a, b] \to M$ such that the equality

$$\dot{\gamma}(t) = u(t)f(\gamma(t)) + v(t)g(\gamma(t)) \tag{4}$$

holds for almost every $t \in [a, b]$. The set of all trajectories of CRS will be denoted by $\mathrm{Traj}(CRS)$. Since f and g are linearly independent at every point, it is clear that *a trajectory uniquely determines its corresponding control.*

Remark 3 For more general optimal control problems, where it is desired to minimize a cost functional $J = \int_a^b L(x(t), w(t))dt$, subject to a dynamical constraint $\dot{x} = F(x, w)$, it may happen that a trajectory $x(\cdot)$ can arise from more than one control, and then the cost J will depend on the specification of an *admissible pair* $(x(\cdot), w(\cdot))$ consisting of a trajectory and corresponding control. In our case, since the trajectory determines the control, there is no need to introduce the concept of an admissible pair. Also, for more general control systems, one has to be more careful about the precise technical conditions on the spaces M, W where the "state variable" x and the "control variable" w take values, the precise technical requirements on F and L, and the precise definition of "admissible control." At the time of this writing, the following conditions on M, W, F, and the class of admissible controls, seem best to us, in that they are the most general ones under which the theory (including the Pontryagin Maximum Principle) works: (1) M is a finite-dimensional, smooth (i.e. C^∞) manifold, (2) W is just a set, (3) $F(x, w)$ belongs, for each $x \in M$, $w \in W$, to the tangent space T_xM to M at x, (4) $F(x, w)$ is of class C^1 in x for each fixed w, (5) an "admissible control" is a W-valued function η, defined on an interval I, such that the time-varying vector field $(x, t) \to F(x, \eta(t))$ is Borel measurable and satisfies local Carathéodory conditions, i.e. for every compact subset K of the domain of a coordinate chart κ, and every compact subinterval J of I, there is a Lebesgue integrable function $\varphi : J \to [0, \infty)$ such that $||\kappa_*(F)(x, \eta(t))|| + ||\frac{\partial \kappa_*(F)}{\partial x}(x, \eta(t))|| \leq \varphi(t)$ for all $(x, t) \in \kappa(K) \times J$. (Here $\kappa_*(F)$ denotes, as usual, the image of F under κ, i.e. the expression of F with respect to the coordinates κ.) If, in addition, we are looking at an optimal control problem where a Lagrangian L occurs, then L should also be required to be C^1 in x, and admissible controls should be required to be such that the local Carathéodory conditions hold for L as well. However, the reader is warned that *most papers by other authors, and even some of our own earlier articles, use different definitions, and no universal consensus has been reached.* ∎

If $\gamma \in \text{Traj}(CRS)$, then $\text{Dom}(\gamma) = [a, b]$ for some $a, b \in \mathbb{R}$. We use $In(\gamma)$ to denote $\gamma(a)$, i.e. the initial point of γ, and $Term(\gamma)$ to denote the terminal point of γ. We use $T(\gamma)$ to denote $b - a$, i.e. the time along γ. If $\gamma \in \text{Traj}(CRS)$, then γ is said to be *time-optimal for CRS* if $T(\gamma) \leq T(\gamma')$ for all $\gamma' \in \text{Traj}(CRS)$ such that $In(\gamma') = In(\gamma)$ and $Term(\gamma') = Term(\gamma)$. The set of all time-optimal trajectories of CRS will be denoted by $\text{Opt}^1(CRS)$. (The superscript 1 refers to the fact that time-optimal trajectories minimize the integral $\int L(x(t), \eta(t))dt$ where the "Lagrangian" L is equal to 1.)

In a similar way, we define the classes $\tilde{\mathcal{U}}$, $\mathcal{U}_D$ of admissible controls for the Reeds-Shepp problem and for the Dubins problem. We can then define the corresponding classes $\text{Traj}(RS)$, $\text{Traj}(DU)$, $\text{Opt}^1(RS)$, $\text{Opt}^1(DU)$ of trajectories and optimal trajectories.

Notice that $\text{Traj}(DU) \subseteq \text{Traj}(RS) \subseteq \text{Traj}(CRS)$. Clearly, $\text{Opt}^1(CRS) \cap \text{Traj}(RS) \subseteq \text{Opt}^1(RS)$, i.e. a trajectory that is optimal for CRS and admissible for RS is optimal for RS.

Similarly, $\mathrm{Opt}^1(RS) \cap \mathrm{Traj}(DU) \subseteq \mathrm{Opt}^1(DU)$.

It is also true, but somewhat less obvious, that $\mathrm{Opt}^1(RS) \subseteq \mathrm{Opt}^1(CRS)$, i.e. every optimal trajectory of RS is also optimal for CRS. This follows easily from the stronger fact that $V \equiv \tilde{V}$, where $V : M \times M \to \mathbb{R}$ and $\tilde{V} : M \times M \to \mathbb{R}$ are, respectively, the optimal time functions for CRS and RS, that is,

$$\begin{array}{rcl} V(p,q) & = & \inf\{T(\gamma) : \gamma \in \mathrm{Traj}(CRS)\,,\, In(\gamma) = p\,,\, Term(\gamma) = q\}\ , \\ \tilde{V}(p,q) & = & \inf\{T(\gamma) : \gamma \in \mathrm{Traj}(RS)\,,\, In(\gamma) = p\,,\, Term(\gamma) = q\}\ . \end{array} \tag{5}$$

The equality $V \equiv \tilde{V}$ follows from standard results in optimal control theory, as will be explained below in Section 7.

6 The Lie algebra structure

As announced in Sections 2 and 3, the structure of the Lie algebra generated by the vector fields f and g will play a crucial role in the analysis of the optimal trajectories.

First let us recall that the *Lie bracket* of two vector fields $X, Y : \Omega \to \mathbb{R}^n$, of class C^1 on a open subset Ω of $\mathbb{R}^n$, is another vector field, denoted by $[X,Y]$, that can be defined in each of the following three equivalent ways:

1. Write the vectors $X(x)$ and $Y(x)$ as *columns*, with components $X_1(x), \ldots, X_n(x)$ and $Y_1(x), \ldots, Y_n(x)$. Then the Lie bracket $[X,Y]$ is the vector field Z given by

$$Z(x) = DY(x) \cdot X(x) - (DX)(x) \cdot Y(x)\ , \tag{6}$$

   where DX and DY are, respectively, the Jacobian matrices of X and Y.

2. Think of X and Y as *differential operators*, i.e. write $X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i}$, $Y = \sum_{i=1}^n Y_i \frac{\partial}{\partial x_i}$, where the functions $X_i(x)$, $Y_i(x)$ are the components of X, Y. Then X, Y act on functions $\varphi : \Omega \to \mathbb{R}$ via $X\varphi = \sum_i X_i \frac{\partial \varphi}{\partial x_i}$, $Y\varphi = \sum_i Y_i \frac{\partial \varphi}{\partial x_i}$. The Lie bracket $[X,Y]$ is then the differential operator $[X,Y] = XY - YX$. Of course, one has to verify that $[X,Y]\varphi = X(Y\varphi) - Y(X\varphi)$ is again of the form $\sum_i Z_i \frac{\partial \varphi}{\partial x_i}$, but this verification is immediate. Moreover, when one carries it out one gets an explicit formula for the Z_i, which shows that this second definition is equivalent to the previous one.

3. Use exponential notation to write the trajectories of X and Y so that, for instance $t \to p\, e^{tX}$ is the solution curve of $\dot{x}(t) = X(x(t))$, $x(0) = p$. Then

$$[X,Y](p) = \lim_{t \to 0+} \frac{p\, e^{\sqrt{t}X} e^{\sqrt{t}Y} e^{-\sqrt{t}X} e^{-\sqrt{t}Y} - p}{t}\ . \tag{7}$$

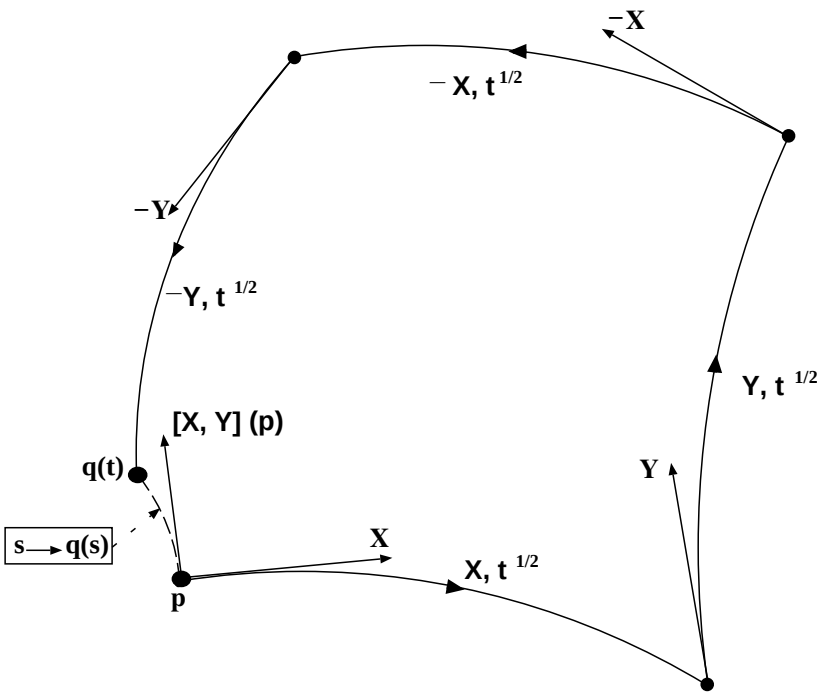

Figure 2: The Lie bracket.

The equivalence of Definition 3 with the other two is a simple exercise in Taylor expansions: one computes the Taylor expansion in powers of s of the expression $p\, e^{sX} e^{sY} e^{-sX} e^{-sY}$ to second order, and finds that it is given by $p + s^2 Z(p)$, where $Z(p)$ is precisely the vector given by Formula 6.

The importance of Definition 3 is that it gives the "geometric meaning of the Lie bracket," as the object that measures, at least infinitesimally, to what extent the "square" of Fig. 2 fails to close.

In other words, if $[X,Y](p) \neq 0$, then this is an indication that, if we first follow X and then Y, we do not get the same result as if we had first followed Y and then X.

A more precise result is the following

Theorem 1 *Let X and Y be vector fields of class C^1 on an open subset Ω of $\mathbb{R}^n$. Consider the three conditions: (i) for every $p \in \Omega$ there exists $\varepsilon > 0$ such that*

$$p\, e^{tX} e^{sY} = p\, e^{sY} e^{tX} \text{ whenever } |t| < \varepsilon,\ |s| < \varepsilon \tag{8}$$

(ii) $[X,Y] \equiv 0$, and (iii) the flows of X and Y "commute globally," i.e. $p\, e^{tX} e^{sY} = p\, e^{sY} e^{tX}$ for all $p \in \Omega$, $t, s \in \mathbb{R}$ such that both sides are defined. Then (i) is equivalent to (ii), and (iii) implies (i) and (ii). Moreover, if X and Y are complete then the converse holds, i.e. (i) (or (ii)) implies (iii).

We recall that a vector field X is *complete* if its trajectories are defined for all times. The fact that (i) implies (ii) is a trivial corollary of (7). The other implications are standard, but require some work. For the proof of these implications we refer the reader to [38] or [47]. ∎

Having defined the Lie bracket, we can define the concept of a Lie algebra of vector fields. Notice first that, if two vector fields X, Y are of class C^k, with $k \geq 1$, then $[X,Y]$ is of class C^{k-1}. In particular, the set of all smooth (i.e. C^∞) vector fields is closed under the operation of taking Lie brackets. We then define a *Lie algebra of vector fields* to be a linear space L of smooth vector fields such that, whenever $X \in L$, $Y \in L$, it follows that $[X,Y] \in L$. Given a set S of vector fields on Ω, we can define Lie(S), the *Lie algebra generated by* S, to be the smallest Lie algebra of vector fields containing S.

For our car problem, the state space is the manifold M, which is not quite an open subset of a Euclidean space $\mathbb{R}^n$. Naturally, the definitions of Lie bracket and Lie algebras of vector fields extend to manifolds, but here all we need to do is to proceed as if M was $\mathbb{R}^3$, and take care of identifying values of θ that differ by a multiple of 2π when necessary. We are interested in the set S that consists of the two vector fields f and g. First observe that *f and g do not commute*, i.e. $[f,g] \neq 0$. This is intuitively obvious, since f corresponds to moving straight forward, and g to turning left without moving. It is clear that moving forward first and then turning left produces a totally different outcome from turning left first and then moving. The Lie bracket $h = [g,f]$ is easily computed from the formulas for f and g together with (6), and we get

$$h(x) = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} . \tag{9}$$

Notice that $h(p)$ is a unit vector tangent to the x_1, x_2 plane and perpendicular to $f(p)$. Moreover, the pair $f(p), h(p)$ is oriented counterclockwise, so that $h(p)$ is perpendicular to the car and points to its left side. Motion along h trajectories —which is not directly possible using our controls u and v— is sideways motion of the car. One can, however, achieve the same result by combining permissible motions: to move sideways to the left, rotate 90 degrees, then move forward, and then rotate 90 degrees to the right (cf. Fig. 3). Mathematically, this is expressed by the formula

$$e^{th} = e^{\frac{\pi}{2}g} e^{tf} e^{-\frac{\pi}{2}g} . \tag{10}$$

Remark 4 The fact that our two controls suffice to connect any two points in M, even though M is three-dimensional, says that our model is an example of a *nonholonomic system*: although only two control directions are directly available, we can effectively move in all three directions by suitably combining permissible moves. (For details about nonholonomic systems and nonholonomic motion planning see [14], [15], [20], [21], [30], [31], [32], [33], [36], [37], [41], [49], and [50].) ∎

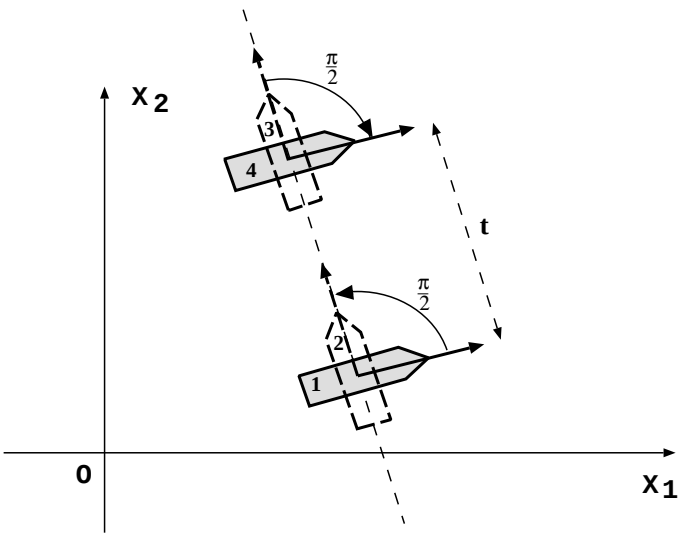

Figure 3: Moving sideways.

In general, a Lie algebra L of vector fields on an open subset Ω of $\mathbb{R}^n$ is said to satisfy the *full rank condition* if $L(p) = \mathbb{R}^n$ for all $p \in \Omega$. (Here we use $L(p) = \{X(p) : X \in L\}$.) A set S of vector fields is said to satisfy the *Lie algebra rank condition* (LARC) if Lie(S) satisfies the full rank condition.

For our car problem, we will use $L(f, g)$ to denote the Lie algebra generated by f and g. Then $h \in L(f, g)$. Moreover, it is clear that $f(p)$, $g(p)$ and $h(p)$ form a basis of $\mathbb{R}^3$ for each p, so the set $\{f, g\}$ satisfies the LARC.

Actually, we can completely determine the structure of $L(f, g)$. The Lie bracket $[g, h]$ can be easily computed, and turns out to be $-f$. The Lie bracket $[f, h]$ vanishes. (Again, this is geometrically obvious: forward motion and motion to the left commute, as long as no change in the direction of the car is involved.) So $L(f, g)$ is in fact three-dimensional, and it is spanned by f, g and h. Moreover, f, g and h satisfy the *commutation relations*

$$\boxed{[g, f] = h \ , \ [g, h] = -f \ , \ [f, h] = 0 \ .} \tag{11}$$

The commutation relations (11) will play a basic role in our analysis. We remark that higher order Lie brackets, such as $[[f, g], [g, [g, f]]]$, can be computed using (11) together with the skew-symmetry property, i.e. the identity $[X, Y] = -[Y, X]$. In particular, if we use ad_X to denote the operator $Y \to [X, Y]$ (so that, e.g., $\mathrm{ad}_X^3(Y) = [X, [X, [X, Y]]]$), it is easily shown by induction that

$$\mathrm{ad}_g^m(f) = \begin{cases} (-1)^k f & \text{if } m = 2k \\ (-1)^k h & \text{if } m = 2k + 1 \end{cases} \tag{12}$$

for all $k = 0, 1, \cdots$.

7 Elementary controllability properties

Now that we know that $\{f, g\}$ satisfies the LARC, we show that some significant information of control theoretic interest follows from this fact. In this section we will just give the simplest results. The really deep consequences of the Lie algebra structure will be presented later.

First let us study the *controllability* of our problem. In general, a control system is said to be *controllable* if, for every pair of points p and q in its state space, there is a trajectory that goes from p to q. A system is *small-time locally controllable* (STLC) from a point p if for every time $T > 0$ there is a full neighborhood $N(T, p)$ of p all whose points can be reached from p in time $\leq T$. For details regarding STLC we refer the reader to [26], [51], [54], [55], [57], and [59].

It is geometrically obvious that CRS is controllable, and the reader can easily persuade himself that CRS is SLTC from every point p. An analytical proof of these facts follows from general theorems in Lie-algebraic control theory, as we now explain.

To a general system $\Sigma : \dot{x} = F(x, w)$ with control set W, associate the set of vector fields $VF(\Sigma) = \{F(\cdot, w) : w \in W\}$, so that $VF(\Sigma)$ is the set of all vector fields that correspond to all possible constant values of the controls. We then define the *accessibility Lie algebra* (known sometimes as the *controllability Lie algebra*) $L(\Sigma)$ of Σ by $L(\Sigma) = \text{Lie}(VF(\Sigma))$, i.e. $L(\Sigma)$ is the smallest Lie algebra of vector fields that contains all the constant control vector fields $F(\cdot, w)$.

Call a system *symmetric* if every trajectory run backwards in time is also a trajectory. Then it is easy to verify that *CRS and RS are symmetric, but the Dubins system DU is not.* A system has the *accessibility property* (AP) from a point p if the set of all points that can be reached from p by trajectories of the system contains some interior point. (Notice that this interior point need not be p itself. That is, it may happen that no neighborhood N of p is such that all the points of N can be reached from p. For a trivial example of this, consider the problem $\dot{x} = w$ on the real line, with control values w constrained by $0 \leq w \leq 1$. Clearly, if $p \in \mathbb{R}$, then the set of points reachable from p is the half-line $[p, \infty)$, so the AP holds from p, but every neighborhood of p contains points that cannot be reached from p.)

With this terminology, the following theorem is true:

Theorem 2 *If the vector fields of a control system Σ have the LARC, then Σ has the accessibility property from every point. If in addition Σ is symmetric, then it is STLC from every point. In particular, if the state space of Σ is connected and Σ is symmetric, then it is controllable.* ∎

Remark 5 A completely precise statement of the theorem would require a detailed definition of admissible controls and trajectories for general systems, as we did for CRS. This is done in

detail in [55], [63], and [64], and we will not repeat it here. For our purposes, it suffices to know that the theorem holds true under the general technical conditions of Remark 3, provided that the vector fields $x \to F(x,w)$ are actually of class C^∞. (This property is needed to make sure that the Lie algebra $L(\Sigma)$ is well defined, i.e. that all possible iterated Lie brackets of all orders of the $F(\cdot,w)$ exist.) Moreover, the theorem does not require the use of all possible admissible controls, and is actually valid *for any class of admissible controls that contains all piecewise constant functions with values in the control set.* ∎

To apply this result to our problem, consider the vector fields $f+g$ and $f-g$. Their linear span is the same as that of f and g. So the Lie algebra $\mathrm{Lie}(f+g, f-g)$ is exactly $L(f,g)$. We have shown that $L(f,g)$ satisfies the full rank condition. So the Dubins system (whose associated vector fields include $f+g$ and $f-g$) has the LARC. This implies that the Dubins system has the AP from every point. Since CRS and RS are symmetric and have the LARC (because their Lie algebras are obviously equal to $L(f,g)$), we conclude that both CRS and RS are controllable, and STLC from every point.

Actually, the Dubins system is also controllable, as can be proved by the following simple argument. A trajectory of $f+g$ involves moving counterclockwise on a circle of radius one with angular velocity one. So in time 2π a car that follows such a trajectory will return to its original position and orientation. Using our exponential notation for flows, this says that $e^{2\pi(f+g)} = \mathbf{I}$, where $\mathbf{I} : M \to M$ is the identity map. Therefore, if $t < 0$, we can write $e^{t(f+g)} = e^{s(f+g)}$ for some $s > 0$ (e.g. $s = t + 2k\pi$, with k a large enough integer). This means that, although a trajectory of $f+g$ run backwards is not a trajectory of $f+g$, every point lying on such a trajectory can be reached from any other point by moving forward in time. A similar phenomenon occurs for $f-g$. Hence, if we consider the system $\hat{\Sigma}$ with four values of the control (u,v), namely $(\pm 1, \pm 1)$ and piecewise constant controls, and the smaller system $\check{\Sigma}$ with only two control values $(1, \pm 1)$, it follows that every point reachable from p by a trajectory of $\hat{\Sigma}$ can in fact be reached by a trajectory of $\check{\Sigma}$. Now, the Lie algebra of $\hat{\Sigma}$ is $L(f,g)$, so $\hat{\Sigma}$ has the LARC. Since $\hat{\Sigma}$ is symmetric, it is controllable. So $\check{\Sigma}$ is controllable as well. Since every trajectory of $\check{\Sigma}$ is a trajectory of DU, we conclude that DU is controllable as well.

To conclude, we show how to prove the equality $V \equiv \tilde{V}$ announced earlier. This is done by using (a) *approximation results* from control theory, together with (b) small-time local controllability. The approximation results are:

Theorem 3 *If $a, b \in \mathbb{R}$, $a < b$, K is a compact convex subset of $\mathbb{R}^m$, and K_0 is a subset of K whose closed convex hull is K, then the set* $\mathrm{Meas}([a,b], K_0)$ *of measurable K_0-valued functions on $[a,b]$ is weakly dense in the set* $\mathrm{Meas}([a,b], K)$ *of measurable K-valued functions.*

Theorem 4 *Assume that $a, b \in \mathbb{R}$, $a < b$, K is a compact convex subset of $\mathbb{R}^m$, $f_1, \ldots, f_m$ are locally Lipschitz vector fields on an open subset Ω of $\mathbb{R}^n$, and $\{\eta_j\}_{j=1}^{\infty}$ is a sequence of measurable K-valued functions on $[a,b]$ that converges weakly to $\eta = (\eta_1(\cdot), \ldots, \eta_m(\cdot)) : [a,b] \to K$. Assume that $\gamma : [a,b] \to \Omega$ is a trajectory of η for the system $\dot{x} = u_1 f_1(x) + \ldots + u_m f_m(x)$ (i.e. γ is absolutely continuous and $\dot{\gamma}(t) = \eta_1(t) f_1(\gamma(t)) + \ldots + \eta_m(t) f_m(\gamma(t))$ for almost all $t \in [a,b]$). Let $\bar{t} \in [a,b]$. Then, (a) for j sufficiently large, there exists a trajectory $\gamma_j : [a,b] \to \Omega$ of η_j that satisfies $\gamma_j(\bar{t}) = \gamma(\bar{t})$, (b) $\gamma_j \to \gamma$ uniformly as $j \to \infty$.*

In the above theorems, "$\eta_j \to \eta$ weakly" means that $\int_a^t \eta_j(s)ds \to \int_a^t \eta(s)ds$ as $j \to \infty$ for every t. (In this case, the convergence is actually uniform, because the functions η_j take values in a fixed compact set, and so the sequence of their indefinite integrals is equicontinuous.) For the proofs of these two theorems we refer the reader to [59] and [63].

We now use the theorems to prove

Theorem 5 *Let V, $\tilde{V}$ be the minimum time functions for CRS and RS, i.e. the functions defined by (5). Then $V \equiv \tilde{V}$.*

Proof. The inequality $V \leq \tilde{V}$ is trivial. To show the opposite inequality, let $T = V(p,q)$, and pick $\varepsilon > 0$. Let $\gamma : [a,b] \to M$ be a trajectory of CRS such that $\gamma(a) = p$, $\gamma(b) = q$, $b - a < T + \varepsilon$. Assume γ corresponds to the control $\eta : [a,b] \to U$. Let $K = U$, $K_0 = \tilde{U}$. Let $\bar{t} = b$. Let $\eta_j : [a,b] \to \tilde{U}$ converge weakly to η. Let $\gamma_j : [a,b] \to M$ be trajectories of η_j such that $\gamma_j(b) = q$. Then $\gamma_j \to \gamma$ uniformly. Using the STLC property of RS, let N be a neighborhood of p all whose points can be reached from p by trajectories of RS in time $< \varepsilon$. If j is large enough, then $\gamma_j(a) \in N$. So $\gamma_j(a)$ can be reached from p by a trajectory of RS in time $< \varepsilon$, and therefore q can be reached from p by a trajectory of RS in time $< T + 2\varepsilon$. So $\tilde{V}(p,q) < T + 2\varepsilon$. Since ε is arbitrary, we get $\tilde{V}(p,q) \leq T = V(p,q)$. So $\tilde{V} \leq V$, and the proof of the identity $V \equiv \tilde{V}$ is complete. ∎

8 Existence of Optimal Trajectories

In optimal control theory, there are various results giving existence of optimal trajectories, cf. e.g. [5] and [18]. One such theorem, that will suffice for our purposes, is the following.

Theorem 6 *Consider a control system of the form $\dot{x} = u_1 f_1(x) + \ldots + u_m f_m(x)$, where (i) the state variable takes values in a space M which is an open subset of $\mathbb{R}^n$ or, more generally, a differentiable manifold, (ii) the vector fields f_i are locally Lipschitz, (iii) the control vector $u = (u_1, \ldots, u_m)$ takes values in a compact convex subset K of $\mathbb{R}^m$, and (iv) the admissible controls are all K-valued measurable functions on compact subintervals of $\mathbb{R}$. Assume that the system is complete, in the sense that for every control function $\eta : [a,b] \to K$ and every initial*

condition $p \in M$, there exists a trajectory γ for η which is defined on the whole interval $[a,b]$ and satisfies $\gamma(a) = p$. Then, if two points p, q of M are such that there exists a trajectory from p to q, it follows that there exists a time-optimal trajectory from p to q.

We refer the reader to the optimal control literature for a detailed proof of this and many other, much more general, existence results, but we remark that the proof is actually quite simple: one picks a minimizing sequence $\eta_j : [0, T_j] \to K$ of controls giving rise to trajectories γ_j from p to q, restricts the η_j to $[0, T]$ (where T is the infimum of the times of all trajectories from p to q, so that $T_j \geq T$ and $T_j \to T$), uses the compactness of the space $\mathrm{Meas}([a,b], K)$ of Theorems 3 and 4 to extract a convergent subsequence from $\{\eta_j\}$, and then uses the completeness assumption to show that the limit η of this subsequence gives rise to a trajectory γ, and Theorem 4 to conclude that the γ_j converge to γ and so γ goes from p to q in time T. ∎

To apply Theorem 6 to our situation, we observe that the car problem satisfies the completeness condition, because f and g are bounded. Moreover, Problems CRS, RS and DU are controllable, so that given any p, q there is a trajectory from p to q. Finally, the control set is compact and convex for CRS and DU (but not for RS), so we get:

Theorem 7 *For Problem CRS, and for the Dubins problem DU, given any two points p, q of $\mathbb{R}^2 \times \mathrm{S}^1$, there exists a time-optimal trajectory from p to q.* ∎

For the Reeds-Shepp problem, existence of optimal trajectories does not follow immediately from general results of control theory, but we will establish existence below as a byproduct of our analysis of Problem CRS.

9 The Maximum Principle

Now that we know that every pair of points can be joined by an optimal trajectory of CRS, we use the Pontryagin Maximum Principle to study these trajectories. For a general statement of this result, we refer the reader to [5], [34], [39], and [55]. Here we shall just state it for the cases of interest to us, namely, for CRS and some other related problems.

For a general control system $\dot{x} = f(x,u)$, with $x \in \Omega$, Ω open in $\mathbb{R}^n$, and u in some set U, we can consider the *Hamiltonian* $H : \mathbb{R}^n_* \times \Omega \times \mathrm{U} \to \mathbb{R}$ given by $H(\lambda, x, u) = \langle \lambda, f(x,u) \rangle$. (Here $\mathbb{R}^n_*$ is the space of n-dimensional real row vectors, and $\langle \lambda, v \rangle$, when λ is a row vector and v a column vector, is just their ordinary matrix product, which is a scalar.) If $\eta : [a,b] \to U$ is a control, and $\gamma : [a,b] \to \Omega$ is a corresponding trajectory, an absolutely continuous $\mathbb{R}^n_*$-valued function λ on $[a,b]$ will be called an *adjoint vector* for (γ, η) if it satisfies the equation

$$\boxed{\dot{\lambda}(t) = -\tfrac{\partial H}{\partial x}(\lambda(t), \gamma(t), u(t))} \qquad (13)$$

(known as the "adjoint equation") for almost all t. Since H is linear in λ, (13) is linear in λ as well, so an adjoint vector λ such that $\lambda(t) = 0$ for some t in fact satisfies $\lambda(t) = 0$ for all t. An adjoint vector λ such that $\lambda(t) \neq 0$ for some (and hence every) t will be called *nontrivial*.

We will say that λ satisfies the *minimization condition* if there is a constant $\lambda_0 \geq 0$ such that, for almost all $t \in [a, b]$, the equality

$$-\lambda_0 = H(\lambda(t), \gamma(t), u(t)) = \min\{H(\lambda(t), \gamma(t), w) : w \in U\} \tag{14}$$

holds. A *minimizing adjoint vector* for (γ, η) is an adjoint vector that satisfies the minimization condition. If $\lambda_0 = 0$, then the adjoint vector is called *zero-minimizing*.

We are now ready to state the two versions of the Maximum Principle that will be needed here. They deal, respectively, with *time-optimal trajectories* and with *boundary trajectories*. (We already know what is meant by a time-optimal trajectory. A *boundary trajectory*, for a system defined on a state space M, is a trajectory $\gamma : [a, b] \to M$ such that $\gamma(b)$ belongs to the boundary of the set of all points that can be reached from $\gamma(a)$ by trajectories of the system.) In both cases, the Principle gives a *necessary* condition for a trajectory to have the corresponding property.

MP1 (The Maximum Principle for time-optimal control):

if a trajectory is time-optimal then it has a nontrivial minimizing adjoint vector.

MP2 (The Maximum Principle for boundary trajectories):

if γ is a boundary trajectory, then it has a zero-minimizing nontrivial adjoint vector.

Remark 6 Notice that the necessary conditions for optimality and for being a boundary trajectory are almost identical. The only difference is that in the time-optimal case the number λ_0 is only asserted to be nonnegative, whereas for the boundary case λ_0 actually has to vanish. This will be important to us later: it will turn out that, for the "lifted LTV problem":

1. MP1 gives no information whatsoever, and therefore we cannot use it to prove anything about optimal trajectories. (This is exactly as should have been expected, since for lifted LTV all trajectories are time-optimal.)

2. When the extra restriction $\lambda_0 = 0$ is imposed, i.e. when we use MP2, we can derive very strong conclusions about γ. ∎

We now translate the conditions of the Maximum Principle into formulas for the cases of interest to us. Precisely, we will write out explicitly:

1. necessary conditions for a trajectory of CRS to be time-optimal for CR,
2. necessary conditions for a trajectory of DU to be time-optimal for DU,
3. necessary conditions for a trajectory of the lifted LTV problem to be a boundary trajectory.

First consider Problem CRS. In this case, the Maximum Principle amounts to the following: if γ is a time-optimal trajectory defined on $[a,b]$, and $(u(\cdot),v(\cdot))$ is the corresponding control, then there exist a constant $\lambda_0 \geq 0$ and an absolutely continuous curve $\lambda(\cdot) : \mathrm{Dom}(\gamma) \to \mathbb{R}^3_*$, not identically zero, such that

$$\dot{\lambda}(t) = -\lambda(t)(u(t)Df(\gamma(t)) + v(t)Dg(\gamma(t))), \tag{15}$$

$$\langle \lambda(t), f(\gamma(t))\rangle u(t) + \langle \lambda(t), g(\gamma(t))\rangle v(t) = \min_{w\in U}(\langle \lambda(t), f(\gamma(t))\rangle w_1 + \langle \lambda(t), g(\gamma(t))\rangle w_2), \tag{16}$$

$$\langle \lambda(t), f(\gamma(t))\rangle u(t) + \langle \lambda(t), g(\gamma(t))\rangle v(t) + \lambda_0 = 0 \tag{17}$$

hold for almost every $t \in [a,b]$.

Next, let us study the optimal trajectories for Dubins' problem. In this case, everything is almost the same as before, except that $u(t) \equiv 1$, and the minimization is over U_D, i.e. over vectors $w \in U$ such that $w_1 = 1$. This means that in this case the minimization condition becomes

$$\langle \lambda(t), g(\gamma(t))\rangle v(t) = \min_{w\in[-1,1]}(\langle \lambda(t), g(\gamma(t))\rangle w)\,. \tag{18}$$

Finally, we use MP2 to get information about the boundary trajectories for the "lifted LTV problem." (Why this is needed will become clear later.) Recall that LTV is the subproblem in which $v \equiv 1$ (i.e. the car's velocity vector is always turning left), and the *lifted LTV problem* is the problem in $\mathbb{R}^3$ obtained from LTV by regarding θ as true real number, i.e. not identifying values of θ that coincide modulo 2π. In this case, the necessary conditions for γ to be a boundary trajectory are the same as for time-optimality in CRS, except that (i) $v(t) \equiv 1$, and the minimization is over $[-1,1] \times \{1\}$, i.e. over vectors $w \in U$ such that $w_2 = 1$, and (ii) $\lambda_0 = 0$. So in this case the minimization condition becomes

$$\langle \lambda(t), f(\gamma(t))\rangle u(t) = \min_{w\in[-1,1]}(\langle \lambda(t), f(\gamma(t))\rangle w)\ , \tag{19}$$

and we have in addition the supplementary condition $\lambda_0 = 0$.

If γ is a trajectory, $\eta = (u(\cdot), v(\cdot))$ is its corresponding control, $\lambda_0 \geq 0$ is a constant, $\lambda(\cdot)$ is a nontrivial solution of the system of linear time-varying differential equations (15), and Equations (16) and (17) hold for almost all $t \in [a, b]$, then the 4-tuple

$$\Lambda = (\gamma, (u(\cdot), v(\cdot)), \lambda(\cdot), \lambda_0) \tag{20}$$

will be called an *extremal lift* of γ, or of the pair (γ, η). (So the Maximum Principle says that every time-optimal trajectory has an extremal lift.)

10 The switching functions

We now try to bridge the gap between the somewhat complicated statement of the Maximum Principle and the analysis of the structure of the optimal trajectories. The main tool is the study of the *switching functions*.

First, let us introduce some terminology to describe different types of controls. A scalar control $\xi : [a, b] \to [-1, 1]$ is a *bang-bang control* if it only takes on the values 1 or -1 for almost every $t \in [a, b]$. If ξ in addition is actually a.e. constant on $[a, b]$, then we will call it a *bang* control. A *switching time* (or just "switching") of ξ is a time $t \in [a, b]$ such that ξ is not bang on any interval of the form $(t - \delta, t + \delta) \cap [a, b]$, $\delta > 0$. A control with finitely many switchings is called *regular bang-bang*. Equivalently, a regular bang-bang control is a control that is a finite concatenation of bang controls. One can then define a U-valued control to be *bang* (resp. *bang-bang, regular bang-bang*) if each of its two components is bang (resp. bang-bang, regular bang-bang). So η is regular bang-bang if and only if it is equal a.e. to a U^*-valued piecewise constant control, where U^* is the set of vertices of U, i.e. $U^* = \{(1, 1), (1, -1), (-1, 1), (-1, -1)\}$. If $\eta = (u(\cdot), v(\cdot))$, then a *u-switching* (resp. *v-switching*) is a switching time of $u(\cdot)$ (resp. $v(\cdot)$).

Now assume that Λ is an extremal lift of (γ, η), where $\eta = (u(\cdot), v(\cdot)))$, and γ, η have domain $[a, b]$. We define the *u-switching function* φ and the *v-switching function* ψ along Λ to be the functions $\varphi, \psi : [a, b] \to \mathbb{R}$ given by

$$\varphi(t) = \langle \lambda(t), f(\gamma(t)) \rangle \ , \qquad \psi(t) = \langle \lambda(t), g(\gamma(t)) \rangle \ . \tag{21}$$

These functions are important because the minimization condition (16) implies the following *switching properties*:

SP1: if $\varphi(t) > 0$ then $u(t) = -1$,
SP2: if $\varphi(t) < 0$ then $u(t) = 1$,
SP3: if $\psi(t) > 0$ then $v(t) = -1$,
SP4: if $\psi(t) < 0$ then $v(t) = 1$.

So the functions φ and ψ determine where the controls can switch. On any interval where $\varphi(t)$ is positive (resp. negative) the component $u(t)$ of the control has to be equal to -1 (resp. $+1$) almost everywhere. Similarly, on an interval where $\psi(t)$ is positive (resp. negative) the component $v(t)$ has to be a.e. equal to -1 (resp. $+1$). *A necessary condition for t to be a u-switching (resp. v-switching) is that $\varphi(t) = 0$ (resp. $\psi(t) = 0$).* Therefore, *on any interval where one of the switching functions has no zeros (resp. has finitely many zeros) the corresponding control is bang (resp. regular bang-bang).*

At the other extreme of the bang-bang controls, we find the so-called *singular controls.* (Cf. [8]-[10], [28], [43], [44], [53], and [60]-[62].) The reader is warned that not all authors define "singular control" in the same way. Our definition is as follows: we say that Λ is *u-singular* (resp. *v-singular*) on an interval $[c, d]$ if $c < d$ and the switching function φ (resp. ψ) vanishes identically on $[c, d]$. We call extremals that are both u-singular and v-singular arcs *doubly singular extremals.* (It will be shown below that for our car problem there do not exist optimal doubly singular extremals.)

Notice that singularity is not a property of a control, or even of a trajectory, but of an extremal, since it is defined in terms of the switching functions. (This should be contrasted with the bang, bang-bang and regular bang-bang properties, that are expressed in terms of the control itself.) However, there are cases where the control itself is singular, in the sense that any extremal lift of any trajectory corresponding to such a control will be singular. Such a control will be called "essentially singular". More precisely, let us call a (scalar or vector) control *essentially singular* on a interval $[c, d]$ with $c < d$ if every interior point of $[c, d]$ is a switching. (For instance, a scalar control $u : [a, b] \to [-1, 1]$ all whose values $u(t)$, for $t \in [c, d]$, satisfy $|u(t)| < 1$, is essentially singular.)

It should now be clear that, in order to understand the structure of the optimal controls, we have to investigate the zeros of the switching functions φ, ψ. Moreover, the *derivatives* of the switching functions will undoubtedly have to play a role. (For instance, if we show that $\dot{\varphi}$ never vanishes on $[a, b]$, we can conclude that u is bang-bang with at most one switching.)

It is at this point that the Lie bracket enters the scene again, via the following elementary observation:

Lemma 1 *Let Z be a smooth vector field in $\mathbb{R}^2 \times \mathrm{S}^1$, and let Λ be an extremal lift of (γ, η), where $\eta = (u(\cdot), v(\cdot)))$, and γ, η have domain $[a, b]$. Then the derivative with respect to t of the function $t \to \varphi_Z(t) = \langle \lambda(t), Z(\gamma(t)) \rangle$ is given by*

$$\dot{\varphi}_Z(t) = u(t)\langle \lambda(t), [f, Z](\gamma(t)) \rangle + v(t)\langle \lambda(t), [g, Z](\gamma(t)) \rangle \ . \tag{22}$$

Proof. $\dot{\varphi}_Z = \langle \lambda, \frac{\partial Z}{\partial x}\dot{x} \rangle - \langle \dot{\lambda}, Z \rangle$. But $\dot{\lambda} = -\frac{\partial H}{\partial x} = -u\Big\langle \lambda, \frac{\partial f}{\partial x} \Big\rangle - v\Big\langle \lambda, \frac{\partial g}{\partial x} \Big\rangle$, and $\dot{x} = uf + vg$, so

$$\dot{\varphi}_Z = u\Big\langle \lambda, \frac{\partial Z}{\partial x} f \Big\rangle + v\Big\langle \lambda, \frac{\partial Z}{\partial x} g \Big\rangle - u\Big\langle \lambda, \frac{\partial f}{\partial x} Z \Big\rangle - v\Big\langle \lambda, \frac{\partial g}{\partial x} Z \Big\rangle \ ,$$

and therefore, using the definition of the Lie bracket, we get

$$\dot{\varphi}_Z = u\langle\lambda,[f,Z]\rangle + v\langle\lambda,[g,Z]\rangle \quad , \tag{23}$$

which is our desired conclusion. ∎

Formula (23) can be rewritten as

$$\dot{\varphi}_Z = u\varphi_{[f,Z]} + v\varphi_{[g,Z]} \quad . \tag{24}$$

The identity (24) is the key fact that will enable us to compute derivatives of switching functions and relate them to the Lie brackets. Let us apply it to compute $\dot{\varphi}$ and $\dot{\psi}$. Since $\varphi = \langle\lambda, f\rangle$, we get $\dot{\varphi} = u\langle\lambda,[f,f]\rangle + v\langle\lambda,[g,f]\rangle$. Using $[f,f] = 0$ and $[g,f] = h$ we get $\dot{\varphi} = v\langle\lambda,h\rangle$. Similarly, if we use (22) together with $\psi = \langle\lambda,g\rangle$ and $[g,g] = 0$, $[f,g] = -[g,f] = -h$, we get $\dot{\psi} = -u\langle\lambda,h\rangle$. These two formulas show that the function $\langle\lambda,h\rangle$ is also important, so we give it a name. We define $\chi : \mathrm{Dom}\,(\Lambda) \to \mathbb{R}$ as follows

$$\chi(t) = \langle\lambda(t), h(\gamma(t))\rangle \ , \tag{25}$$

so that our formulas simply say that $\dot{\varphi} = v\chi$, and $\dot{\psi} = -u\chi$. Finally, we apply (22) again to compute $\dot{\chi}$ and get $\dot{\chi} = u\langle\lambda,[f,h]\rangle + v\langle\lambda,[g,h]\rangle$. Using $[f,h] = 0$, $[g,h] = -f$, we end up with the equation $\dot{\chi} = -v\varphi$. So we have obtained the system of equations

$$\boxed{\dot{\varphi} = v\chi \ , \quad \dot{\psi} = -u\chi \ , \quad \dot{\chi} = -v\varphi} \tag{26}$$

(i.e. $\dot{\varphi}(t) = v(t)\chi(t)$, $\dot{\psi}(t) = -u(t)\chi(t)$, $\dot{\chi}(t) = -v(t)\varphi(t)$ for almost every t). This is almost a system of ordinary differential equations for the vector (φ,ψ,χ), so that in some sense they determine the time evolution of this vector once its value at some initial time is given. They do not quite do so, however, because *the controls u and v are themselves determined by φ and ψ but are not necessarily locally Lipschitz or single-valued.* Indeed, the minimization condition (16) implies that

$$\boxed{u(t) = -\mathrm{sign}\,\varphi(t) \ , \ v(t) = -\mathrm{sign}\,\psi(t) \ ,} \tag{27}$$

where "sign s" means "1 if $s > 0$, -1 if $s < 0$, any number between -1 and 1 if $s = 0$." (So we should actually think of "sign" as a set-valued function, and write $y \in \mathrm{sign}\,s$ rather than $y = \mathrm{sign}\,s$.) If we plug these values for $u(t)$ and $v(t)$ into (26), then we get a system of differential equations with a discontinuous (and, in fact, multivalued) right-hand side. Initial value problems for such a system need not have solutions, nor do the solutions have to be

unique when they exist. However, as we will show, in many cases the solutions of such systems can be studied, and used to derive conclusions about the properties of the optimal trajectories. Next we link the switching functions and λ_0. This is done by observing that (17) implies

$$\boxed{-|\varphi(t)| - |\psi(t)| + \lambda_0 = 0 \ .} \tag{28}$$

Finally, we recall that the Maximum Principle also tells us that $\lambda(t)$ does not vanish. So we can conclude that

$$\boxed{|\varphi(t)| + |\psi(t)| + |\chi(t)| \neq 0 \ .} \tag{29}$$

Formulas (26), (27), (28), (29) will be called the *Switching Structure Equations*. As we now show, they determine the switching properties of the optimal trajectories.

11 Elementary properties of optimal trajectories

Let us start with the simplest properties of the extremals. In the statement of the Maximum Principle there is a number λ_0 which is required to be ≥ 0 but could be 0. Extremals for which $\lambda_0 = 0$ are called "abnormal." They occur in many optimal control problems and cause great difficulty. However, for our car problem we can rule them out. Let us refer to a trajectory that goes from a point p to itself in zero time as "trivial." We then have:

Lemma 2 *Nontrivial optimal abnormal extremals do not exist.*

Proof. Along an abnormal extremal, Equation (28) implies that $\varphi \equiv \psi \equiv 0$. So $\chi(t) \neq 0$ by (29). But the Switching Structure Equations (26) imply that $u(t)\chi(t) = v(t)\chi(t) = 0$. So $u(t) \equiv v(t) \equiv 0$. This means that along this trajectory the car is not moving at all. This can only be time-optimal if the duration of our trajectory is zero, i.e. if our trajectory is trivial. ∎

Lemma 3 *For a nontrivial optimal extremal, the switching functions φ, ψ cannot have a common zero.*

Proof. If at some time t both φ and ψ were equal to zero, then it would follow from (28) that $\lambda_0 = 0$. So our trajectory we would be abnormal, contradicting the previous lemma. ∎

Lemma 4 *If an optimal extremal is v-singular, then $v \equiv 0$.*

Proof. Since the switching function ψ vanishes identically, we conclude that $\varphi(t) \neq 0$ for every t, and then $|u(t)| = 1$. From (26), we get $0 = \dot{\psi}(t) = -u(t)\chi(t)$. So $\chi(t) \equiv 0$ on $\mathrm{Dom}(\gamma)$. So $\dot{\chi}(t) = 0$. But then (26) implies $\dot{\chi}(t) = -v(t)\varphi(t)$. Since $\varphi(t) \neq 0$, we get $v(t) = 0$. ∎

Next, we observe that the equations $\dot{\varphi} = v\chi$ and $\dot{\chi} = -v\varphi$ imply that the number

$$\boxed{\kappa = \varphi(t)^2 + \chi(t)^2} \tag{30}$$

is a constant. It will be important to distinguish the cases $\kappa > 0$ and $\kappa = 0$.

Lemma 5 *Along an extremal of CRS, $\kappa = 0$ if and only if $\varphi \equiv 0$.*

Proof. It is clear that $\kappa = 0$ implies $\varphi \equiv 0$. To prove the converse, assume that $\varphi \equiv 0$ but $\kappa \neq 0$. Then ψ never vanishes, by Lemma 3. So v is $\neq 0$ almost everywhere. But then the equations (26) imply that $\chi \equiv 0$, since $\dot{\varphi} \equiv 0$. ∎

Moreover, we also have

Lemma 6 *Either $\varphi \equiv 0$ or all the zeros of φ are isolated. Moreover, at an isolated zero of φ, $\dot{\varphi}$ exists and is $\neq 0$.*

Proof. Assume that $\varphi \not\equiv 0$. Then $\kappa > 0$. Let $t_0 \in \mathrm{Int}(\mathrm{Dom}\,\gamma)$ be such that $\varphi(t_0) = 0$. Then, by Lemma 3, $\psi(t_0) \neq 0$. Since ψ is continuous, there is an subinterval I of the domain of Λ, containing t_0 in its relative interior, such that ψ never vanishes —and therefore has constant sign— on I. This implies that either $v = 1$ a.e. or $v = -1$ a.e. on I. In either case, since the equation $\dot{\varphi}(t) = -v(t)\chi(t)$ holds a.e. on I, φ is absolutely continuous, and χ is continuous, we conclude that φ is continuously differentiable on I and $\dot{\varphi}$ is given, at *every* $t \in I$, by $\dot{\varphi}(t) = \pm\chi(t)$, where the sign is $+$ everywhere if $v = -1$ a.e., and $-$ everywhere if $v = 1$ a.e. On the other hand, since $\kappa = \varphi(t_0)^2 + \chi(t_0)^2 > 0$ and $\varphi(t_0) = 0$, it follows that $\chi(t_0) \neq 0$. Hence $\dot{\varphi}(t_0) = \pm\chi(t_0) \neq 0$. So t_0 is an isolated zero, and $\dot{\varphi}(t_0) \neq 0$, as stated. ∎

Remark 7 If we use λ_1, λ_2, λ_3 to denote the components of the vector λ, we have $\varphi = \lambda_1 \cos\theta + \lambda_2 \sin\theta$, $\psi = \lambda_3$, and $\chi = -\lambda_1 \sin\theta + \lambda_2 \cos\theta$. Therefore κ is also equal to $\lambda_1^2 + \lambda_2^2$. The derivatives of the λ_i can be computed from (16), and turn out to be

$$\dot{\lambda}_1(t) = 0, \quad \dot{\lambda}_2(t) = 0, \quad \dot{\lambda}_3(t) = (\lambda_1(t)\sin\theta(t) - \lambda_2(t)\cos\theta(t))u(t)\ . \tag{31}$$

These equations can be regarded as a different way of describing the evolution of the adjoint vector λ, by giving the derivatives of its components relative to the canonical basis of $\mathbb{R}^3$ rather than with respect to the basis (f, g, h). When one uses this representation, the role of the Lie brackets is less evident, so we will work as much as possible with the φ, ψ, χ representation. ∎

At this point, we have learned the following:

Every time-optimal trajectory is of one of two (not mutually exclusive) types, namely:

T1: *a trajectory for which u only has finitely many switchings,*

T2: *a trajectory for which $\varphi \equiv 0$.*

To study Type T1 trajectories we will first analyze the pieces between u-switchings, i.e. optimal trajectories where $u \equiv 1$ or $u \equiv -1$. Since both cases are equivalent under the change of coordinates $y_j = -x_j$, it suffices to study the case $u \equiv 1$. This is precisely the property that characterizes the Dubins trajectories, so we will start by studying the trajectories of DU that are optimal for CRS.

Next we will study the Type T2 trajectories. Since $\varphi \equiv 0$, ψ can never vanish because of Lemma 3. So $v \equiv 1$ or $v \equiv -1$. Once again, both cases are equivalent in an obvious way, so it will suffice to study the case $v \equiv 1$. This is precisely what we have called the "LTV subproblem."

Finally, we will put together the information obtained in our analysis of both subproblems and determine a family of trajectories of CRS which is sufficient for time-optimality.

12 Notations for trajectory types

At this point, it will be convenient to introduce some general notational conventions to describe different types of trajectories. The letters L, R, S refer to the direction in which the velocity vector of the car is turning: they mean, respectively, left (i.e. $v = 1$), straight (i.e. $v = 0$) and right (i.e. $v = -1$). The superscript $+$ or $-$ indicates whether the motion is forward (i.e. $u = 1$) or backwards ($u = -1$). So an L^- trajectory is a trajectory for which $u = -1$ and $v = 1$. (Notice that for an L^- trajectory the car is actually *turning to the right*. The L refers to the direction of rotation of the *velocity vector*.) The six trajectory types are shown in Figure 4.

We will often simply use the symbols $L^\pm$, $R^\pm$, $S^\pm$ as alternative names for the vector fields themselves, so that $L^+ = f + g$, $L^- = -f + g$, $R^+ = f - g$, $R^- = -f - g$, $S^+ = f$, $S^- = -f$. Also, we will sometimes use subscripts to indicate the length of the time-interval of a trajectory, so that, for instance, an L_t^+ trajectory is just a trajectory corresponding to $u = 1$, $v = 1$, defined on an interval of length t.

Concatenations of trajectories of various types will be labeled by their symbols, written in order *from left to right*, so that, for instance, an L^+L^- trajectory consists of an L^+ piece followed by an L^-.

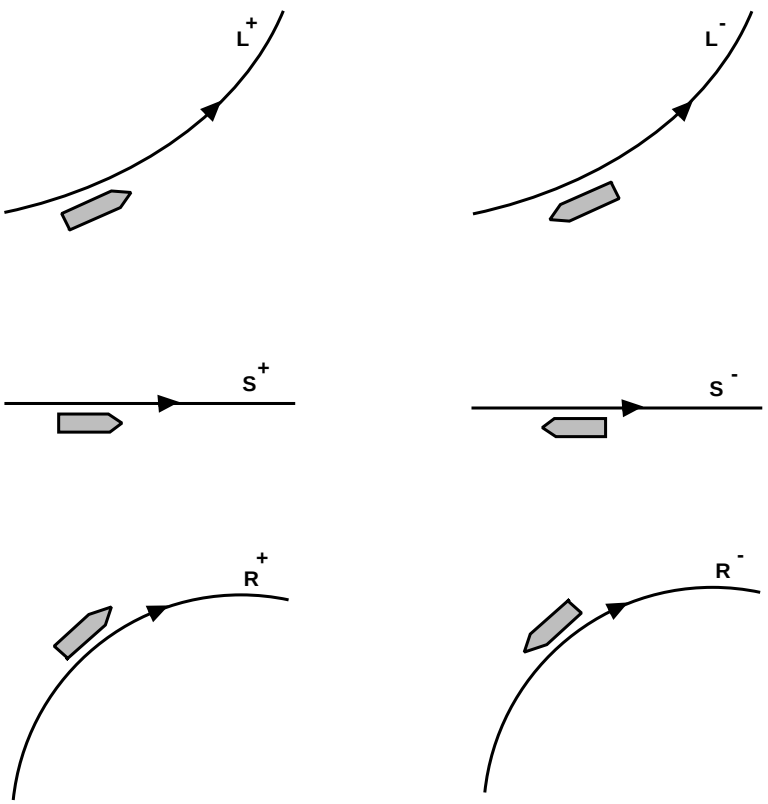

Figure 4: The six basic trajectory types.

Finally, the letter B (for "bang") will refer, indistinctly, to L or R. Whenever we write a symbol such as LSR, or BSB, or BSR, it is understood that one of the pieces may have zero duration, i.e. may actually be missing. (So, for instance, an LS trajectory is LSR. On the other hand, if the durations are specified, e.g. if we talk about an $L_aS_bR_c$ trajectory with, say, $c > 0$, then of course the R piece is there.) Whenever we want to talk about a combination where all the pieces actually have nonzero duration, we will add the word *strict* so that, for example, an LS trajectory is BSB but is not strict BSB.

On some occasions, we will want to make it clear that, at a particular point, a certain control does change value. We will do this by writing there the name of the control. So, for instance, if $a > 0$, $b > 0$, a B_avB_b trajectory γ can either be L_aR_b or R_aL_b, but not L_aL_b or R_aR_b. (If γ is optimal, then we know that u cannot switch at the same time as v, so the only possibilities for B_avB_b are $L_a^+R_b^+$, $L_a^-R_b^-$, $R_a^+L_b^+$ and $R_a^-L_b^-$.)

13 CRS-optimal Dubins trajectories

We now completely determine the structure of the CRS-optimal trajectories that satisfy $u \equiv 1$, i.e. are such that the car is only moving forwards. These are exactly the trajectories of the Dubins system that are optimal for CRS. We show that such trajectories have to be of the form BSB, i.e. concatenations of at most three pieces, of which the middle one is singular and the first and third one are bang (cf. Fig. 5).
First we make the following trivial observation:

Lemma 7 *If an optimal trajectory γ of CRS is B_α, then $\alpha \leq \pi$.*

Proof. It is clear that any of the 4 B-controls returns the car to its starting state (i.e. position and orientation) in time 2π, so α must be $< 2\pi$. The path of γ in the plane is a circle of radius one. Clearly, any state that can be attained in time α from an initial state p can also be attained in time $2\pi - \alpha$ by moving along the same circle in the opposite direction. (Analytically, $pe^{2\pi X} = p$ if X is any of the four B vector fields, so $pe^{\alpha X} = pe^{(2\pi-\alpha)Y}$, if we let $Y = -X$.) So, if $\alpha > \pi$, then we can replace γ by another trajectory that goes to the same point in the smaller time $2\pi - \alpha$. ∎

Now assume that γ is a CRS-optimal Dubins trajectory, defined on an interval $[a, b]$. If $\varphi \equiv 0$ then ψ never vanishes, so $v \equiv 1$ a.e. or $v \equiv -1$ a.e., and therefore γ is actually a B-trajectory. Then the previous lemma tells us that γ is B_α with $\alpha \leq \pi$.

Now assume that φ is not $\equiv 0$. Then we know from Lemma 6 that all the zeros of φ are isolated, and if any of these zeros is an interior point of $[a, b]$, then φ must change sign there, so that u cannot be $\equiv 1$ a.e. on $[a, b]$. Therefore φ has no zeros on (a, b).

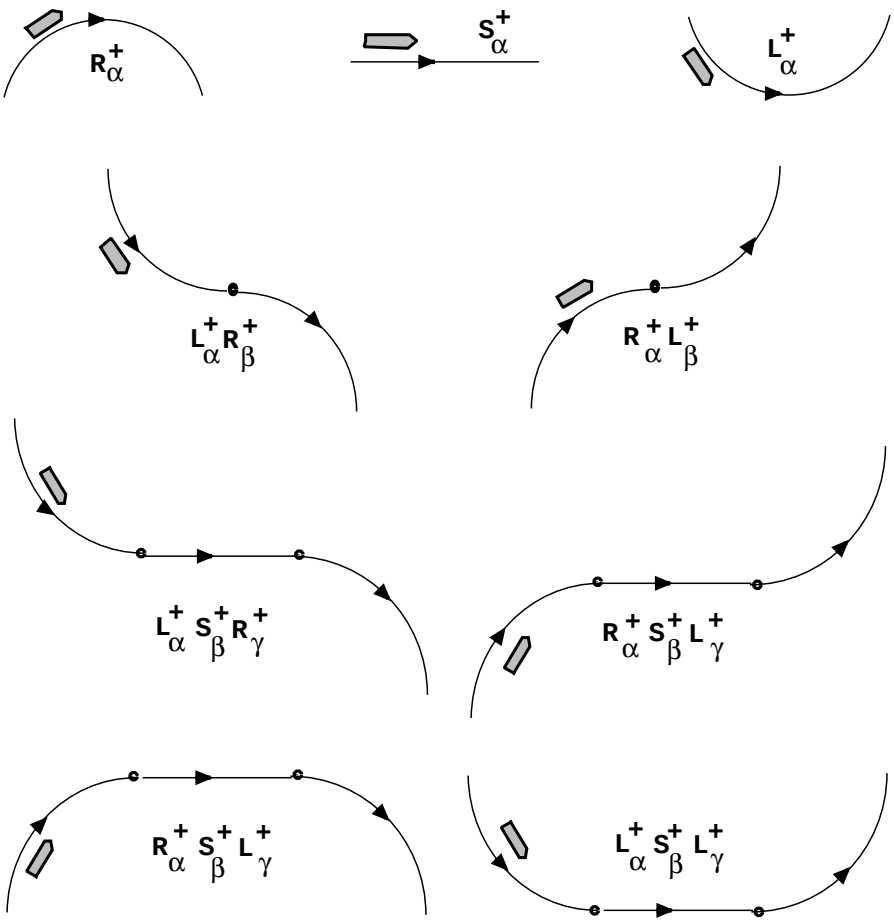

Figure 5: CRS-optimal Dubins trajectories.

The Switching Structure Equations give $\dot{\psi} = -\chi$ and $\dot{\chi} = -v\varphi$. So ψ is actually continuously differentiable and $\ddot{\psi} = v\varphi$. On the other hand, φ must be ≤ 0, for otherwise u would be equal to -1. So in fact $\varphi < 0$ on (a, b). In addition, we have $v = -\text{sign}\,\psi$. So we have $\ddot{\psi} = |\varphi|\text{sign}\,\psi$. This means that ψ *is convex on any interval where* $\psi > 0$, *and concave on any interval where* $\psi < 0$. This implies:

Lemma 8 *An optimal trajectory* $\gamma : [a, b] \to M$ *for CRS which is of the Dubins type is necessarily of the form BSB. More precisely,* γ *is of one of the following types:*

D1 B_α *with* $0 \leq \alpha \leq \pi$,

D2 $B_\alpha v B_\beta$ *with* $0 < \alpha \leq \frac{\pi}{2}$, $0 < \beta \leq \frac{\pi}{2}$,

D3 $B_\alpha v S_\sigma v B_\beta$ *with* $0 < \sigma$ *and* $0 \leq \alpha \leq \frac{\pi}{2}$, $0 \leq \beta \leq \frac{\pi}{2}$.

Proof. If v has no switchings then γ is B, and Lemma 7 yields the desired conclusion.

Now assume that v does have switchings. We know that ψ is convex when it is positive, and concave when it is negative. Let I be the subset of $[a, b]$ that consists of all t such that $\psi(t) \neq 0$. Then I is relatively open in $[a, b]$. Let $\mathcal{I}$ denote the set of connected components of I, so that each element of $\mathcal{I}$ is a relatively open subinterval of $[a, b]$. Suppose that $J \in \mathcal{I}$. If $J = (c, d)$, then $a \leq c < d \leq b$, and $\psi(c) = \psi(d) = 0$. Since ψ is either negative and concave or positive and convex on J, it follows that $\psi \equiv 0$ on J, which is a contradiction. Therefore no $J \in \mathcal{I}$ can be an open interval. This shows that $\mathcal{I}$ consists of at most two intervals, and that each of these intervals contains one of the endpoints of $[a, b]$. So $[a, b]$ is partitioned into at most three intervals I_1, I_2, I_3, such that ψ never vanishes on $I_1 \cup I_3$, and $\psi \equiv 0$ on I_2 (cf. Fig. 6).

Moreover, on each interval $J \in \mathcal{F}$ the control v is constant, and equal to 1 or -1. The equations $\dot{\chi} = -v\varphi$ and $\dot{\varphi} = v\chi$ give $\ddot{\chi} + \chi = 0$.

We have shown (assuming that v has switchings) that the set of zeros of ψ is itself a nonvoid interval $[c, d]$. If $a < c$, then ψ is convex and positive (or concave and negative) on $[a, c]$, and $\psi(c) = 0$. Therefore both χ and $\dot{\chi}$ have constant sign on (a, c). (For example, if $\psi > 0$ on $[a, c)$, then $\dot{\chi} = -v\varphi$ and $v = -1$, so $\dot{\chi} = \varphi$ and then $\dot{\chi}$ does not have zeros. Also, $\chi = -\dot{\psi}$, so χ has no zeros either, because a zero of the derivative of a convex function is a minimum of the function.) This implies that $c - a \leq \frac{\pi}{2}$. A similar reasoning applies to the interval $[d, b]$. So, if we let $\alpha = c - a$, $\beta = b - d$, $\sigma = d - c$, then: (i) if $\sigma > 0$, we are in Case [D3]; (ii) if $\sigma = 0$ and both α and β are > 0, we are in Case [D2] if ψ changes sign at c, and in Case [D1] if it does not. ∎

Remark 8 The previous results do not solve the Dubins problem. All we have done is determine the structure of the Dubins trajectories that are CRS-optimal.

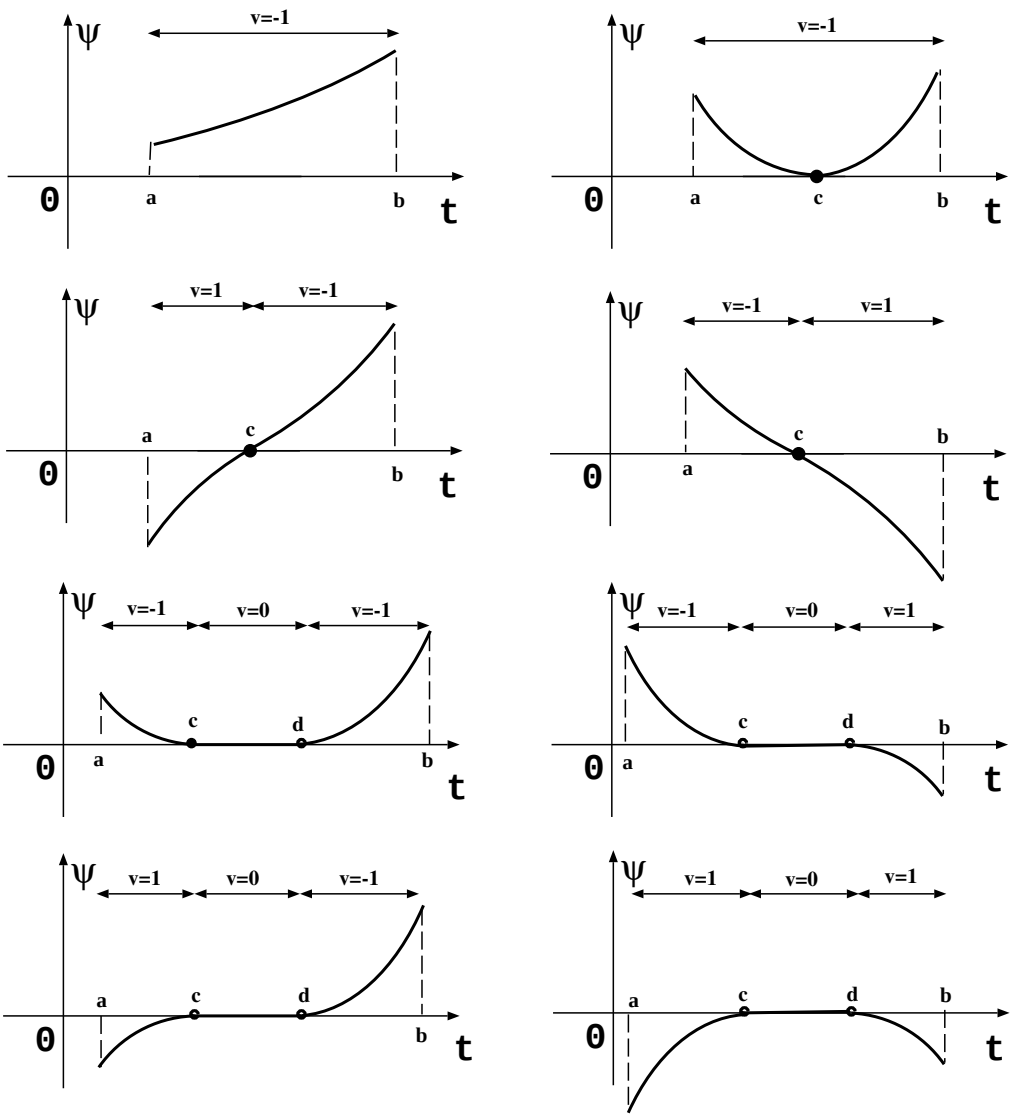

Figure 6: The v-switching function ψ for CRS-optimal Dubins trajectories.

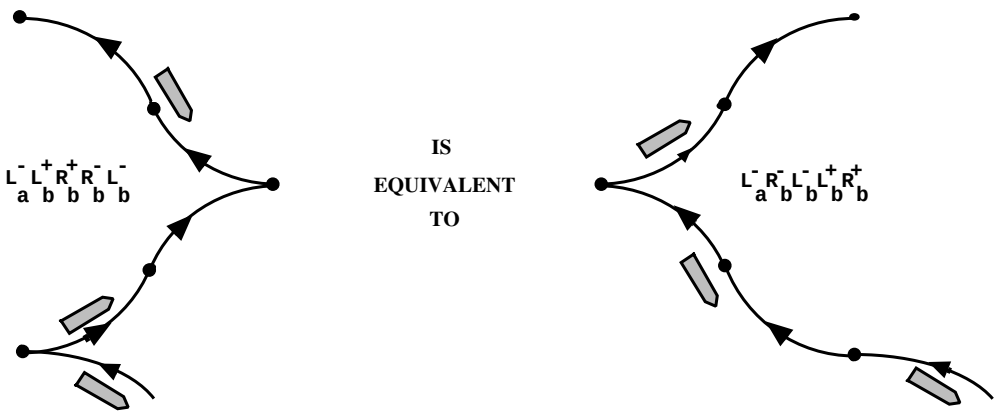

Figure 7: A $B_a^- B_b^+ v B_b^+ B_b^- B_b^-$ trajectory is not optimal.

While it is clear that every optimal CRS trajectory which is of the Dubins type is indeed optimal for the Dubins problem DU, the converse is not true. A Dubins trajectory γ may be optimal for DU but there may be CRS trajectories that are not Dubins and do better than γ. (For example, the replacement used in Lemma 7 would not be permissible if we only considered Dubins trajectories.) Indeed, although we have shown that all CRS-optimal Dubins trajectories are BSB, the corollary of Theorem 1 in [19] indicates that there exist time-optimal trajectories for DU that are of the form BBB. This implies that the family of CRS-optimal Dubins trajectories is not sufficient for solving the Dubins's problem. On the other hand, our methods can be applied to solve the Dubins problem as well, and in fact we can actually obtain stronger results than those of [19]. This is done in the Appendix. ∎

The Dubins trajectories are those where $u \equiv 1$, i.e. where the car is always moving forwards. Let us call a trajectory γ "Dubins-like" if either γ or γ with the car orientation reversed is Dubins. Then it is clear that

Lemma 9 *The conclusions of Lemma 8 hold for Dubins-like trajectories.* ∎

We now prove a few simple corollaries:

Lemma 10 *Let $a > 0$, $b > 0$. Then a $B_a u B_b v B_b u B_b v B_b$ trajectory cannot be optimal.*

Proof. Without loss of generality, we may assume that γ is $L_a^- L_b^+ R_b^+ R_b^- L_b^-$. (The other three cases are similar.) Then it is clear from Figure 7 that γ can be replaced by an $L_a^- R_b^- L_b^- L_b^+ R_b^+$ trajectory γ_{new} without changing the initial and terminal states or the time. If γ was optimal, then γ_{new} would be optimal as well. But the $L_a^- R_b^- L_b^-$ piece $\hat{\gamma}$ of γ_{new} is a Dubins-like trajectory with three bang pieces. So $\hat{\gamma}$ is not optimal, and therefore γ_{new} is not optimal, and *a fortiori* γ is not optimal either. ∎

Lemma 11 *Let $a > 0$. Then a $B_a u B_\pi$ trajectory cannot be optimal.*

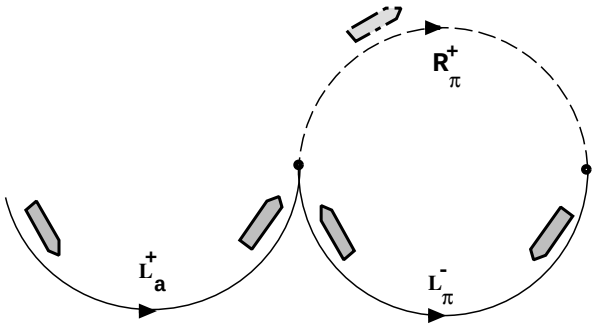

Figure 8: An $L_a^+ L_\pi^-$-trajectory cannot be optimal.

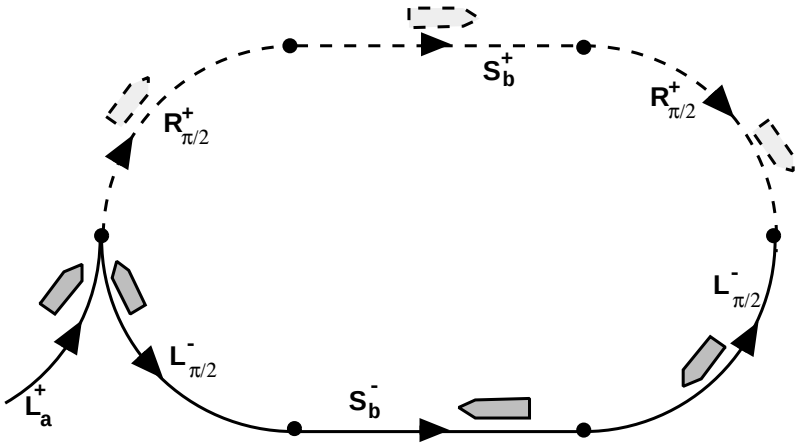

Figure 9: An $L_a^+ L_{\frac{\pi}{2}}^- S_b^- L_{\frac{\pi}{2}}^-$-trajectory cannot be optimal.

Proof. Let γ be $B_a u B_\pi$. The control v never switches, so we may as well assume it is always equal to one, i.e that both B's are actually L's. We may also assume that the first piece corresponds to $u = 1$, i.e. is an L^+. (The case when the first piece is an L^-, and the other two cases that correspond to $v \equiv -1$, are similar.) Then γ is actually $L_a^+ L_\pi^-$. It is clear from Figure 8 that if we replace the L_π^- by an R_π^+ piece then we get a trajectory γ_{new} with the same initial and terminal points and the same duration. So γ_{new} is also optimal. But γ_{new} is Dubins and $B_a v B_\pi$, so we know from Lemma 8 that it cannot be optimal, because the length of an optimal Dubins BB-trajectory cannot exceed π.

This contradiction proves our conclusion. ∎

Lemma 12 *Let $a > 0$, $b \geq 0$. Then an $L_a^+ L_{\frac{\pi}{2}}^- S_b^- L_{\frac{\pi}{2}}^-$- or $R_a^+ R_{\frac{\pi}{2}}^- S_b^- R_{\frac{\pi}{2}}^-$-trajectory cannot be optimal.*

Proof. First assume that $b > 0$. Then, as shown in Figure 9, an $L_a^+ L_{\frac{\pi}{2}}^- S_b^- L_{\frac{\pi}{2}}^-$-trajectory can be replaced by an $L_a^+ R_{\frac{\pi}{2}}^+ S_b^+ R_{\frac{\pi}{2}}^+$-trajectory changing neither the initial and terminal states nor the time. The latter trajectory is Dubins and strict $BBSB$, so it is not optimal. So γ is not

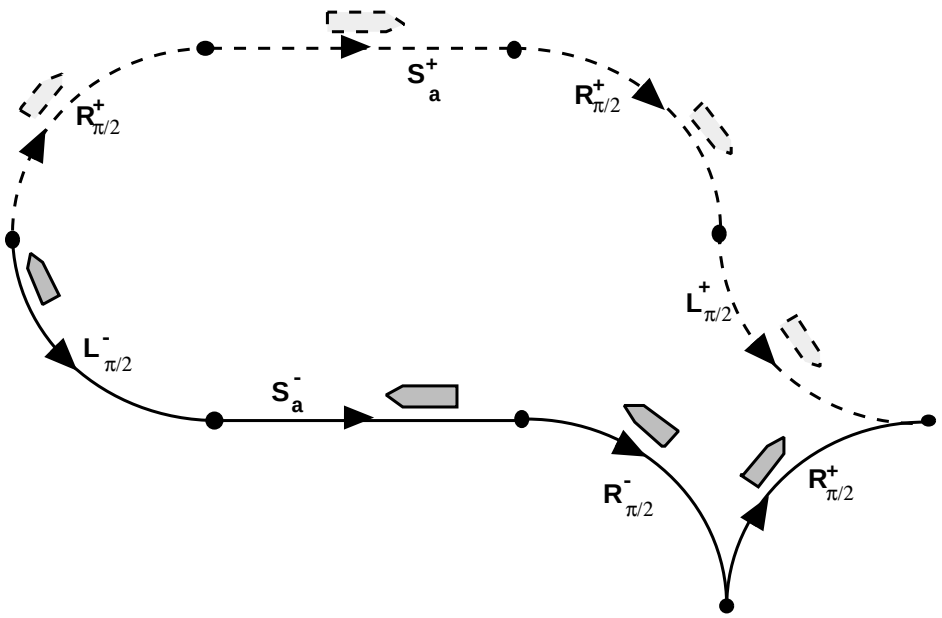

Figure 10: An $L^-_{\frac{\pi}{2}} S^-_a R^-_{\frac{\pi}{2}} R^+_{\frac{\pi}{2}}$-trajectory cannot be optimal.

optimal either. If $b = 0$ then γ is a $B_a u B_\pi$-trajectory, so it is not optimal by Lemma 11. The proof for the $R^+_a R^-_{\frac{\pi}{2}} S^-_b R^-_{\frac{\pi}{2}}$ case is similar. ∎

Lemma 13 *Let $a \geq 0$. Then an $L^-_{\frac{\pi}{2}} S^-_a R^-_{\frac{\pi}{2}} R^+_{\frac{\pi}{2}}$-trajectory cannot be optimal.*

Proof. First assume that $a > 0$. Then as shown in Figure 10, an $L^-_{\frac{\pi}{2}} S^-_a R^-_{\frac{\pi}{2}} R^+_{\frac{\pi}{2}}$-trajectory can be replaced by an $R^+_{\frac{\pi}{2}} S^+_a R^+_{\frac{\pi}{2}} L^+_{\frac{\pi}{2}}$-trajectory changing neither the initial and terminal states nor the time. The latter trajectory is Dubins and strict $BSBvB$, so it is not optimal. So γ is not optimal either.

If $a = 0$, then γ is replaced by an $R^+_\pi L^+_{\frac{\pi}{2}}$-trajectory, which is BB Dubins of length $> \pi$, and therefore not optimal by Lemma 8. ∎

14 Envelopes

The purpose of this section is to illustrate the use of the theory of envelopes by proving the following result (cf. Figure 11) that slightly strengthens Lemma 10:

Lemma 14 *Let $\alpha > 0$, $\beta > 0$. Then a $B^-_\alpha B^+_\beta v B^+_\beta B^-_\beta$ trajectory cannot be optimal.*

The reader who is only interested in the result need not read the rest of this section. Moreover, the only use of this result is to narrow down even further the sufficient class of the main theorem and eliminate bang-bang trajectories with four switchings. The reader who is

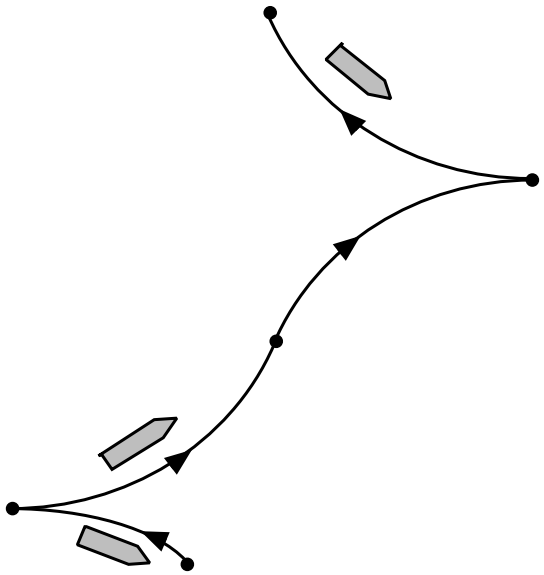

Figure 11: A $B_\alpha^- B_\beta^+ v B_\beta^+ B_\beta^-$ trajectory.

satisfied with a slightly larger class where five switchings are excluded but four are permitted can rely on the much easier Lemma 10.

As explained in Section 3, the proof of Lemma 14 can be done by means of a totally self-contained computation, without using any results from the theory of envelopes, and without even mentioning envelopes, and this is in fact how we will eventually present it. However, such a computation is likely to appear unmotivated and mysterious, and the fact that it works can only be understood by referring to the general theory, so we begin with a brief review of this theory.

Consider first problems in the classical Calculus of Variations, where the objective is to find a curve $[a,b] \ni t \to x(t) \in \mathbb{R}^n$ that minimizes an integral $\int_a^b L(x(t),\dot{x}(t))\,dt$, where L —the "Lagrangian"— is a given C^1 function on $\mathbb{R}^n \times \mathbb{R}^n$. (Here "curve" means "absolutely continuous function.") Let us call a curve $x(\cdot)$ a *minimizer* if it minimizes the above integral among all curves $\tilde{x}$ defined on the same interval $[a,b]$ that satisfy $\tilde{x}(a) = x(a)$, $\tilde{x}(b) = x(b)$. Then it is well known that, under suitable regularity and growth conditions on L, a necessary condition for x to be a minimizer is that it satisfy the *Euler-Lagrange equations.* Call any curve that satisfies these equations an *extremal.* Then extremality is necessary but not sufficient for optimality, so one wants to have extra conditions for optimality that will enable us to detect nonoptimal extremals. The *theory of envelopes* is a particularly elegant way of doing that. It works as follows: assume that an extremal γ can be embedded in a *one-parameter family of extremals*, i.e. a family $\{\gamma_s : s \in I\}$ of extremals, each one defined on $[a,b]$, such that $\gamma_0 = \gamma$, $\gamma_s(0) = \gamma(a)$ for each s, and $\gamma_s(t)$ is sufficiently smooth as a function of s and t. (Here I is an interval of the form $[-\varepsilon, 0]$, with $\varepsilon > 0$.)

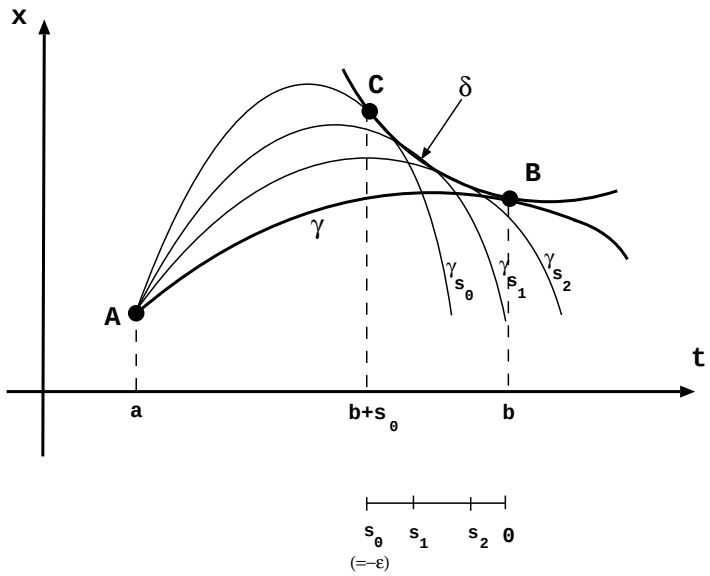

Figure 12: A classical envelope.

Let δ be the curve $[b-\varepsilon, b] \ni \sigma \to \delta(\sigma) \in \mathbb{R}^n$ defined by $\delta(b+s) = \gamma_s(b+s)$. Then the graphs of δ and γ_s intersect at $p(s) = (\delta(b+s), b+s)$. If these graphs are tangent at $p(s)$ for each s, then δ is said to be an *envelope* of the family $\{\gamma_s\}$ (cf. Figure 12).

It then turns out that the following identity is true: let $\tilde{\gamma}$ be the curve obtained by following $\gamma_{-\varepsilon}$ from a up to time $b-\varepsilon$, and then δ from $b-\varepsilon$ to b (so that $\tilde{\gamma}$ is the curve going from A to C and then to B in Figure 12). Then $\tilde{\gamma}$ and γ have exactly the same cost.

This rather remarkable fact can then be used to prove nonoptimality of γ if some extra conditions hold. For instance, suppose that the Lagrangian L is nondegenerate, in the sense that the Hessian matrix of L with respect to $\dot{x}$ is nonsingular at every point. Then γ cannot be optimal for the following reason. If γ was optimal, then $\tilde{\gamma}$ would be optimal as well, because it has the same initial and terminal points, and the same cost, as γ. So $\tilde{\gamma}$ would have to be an extremal. Since $\tilde{\gamma}$ and γ go through $\gamma(b)$ at time b and are tangent there, it would follow that through $\gamma(b)$ there pass two different extremals with the same tangent vector. But the nondegeneracy of L implies that this is a contradiction. (Indeed, the Euler-Lagrange equations $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$ can be rewritten as

$$\frac{\partial^2 L}{\partial \dot{x}^2}\ddot{x} + \frac{\partial^2 L}{\partial \dot{x} \partial x}\dot{x} = \frac{\partial L}{\partial x}$$

and this can be solved for $\ddot{x}$ as a function of x and $\dot{x}$ if L is nondegenerate, so the initial value problem given by these equations plus the specification of $x(b)$ and $\dot{x}(b)$ has uniqueness of solutions.)

It turns out that, properly generalized, the theory of envelopes extends to control theory as well. For a general control problem $\Sigma : \dot{x} = F(x,w)$, $w \in U$ on a state space M, (and for time optimal control, for simplicity) assume that $\gamma : [a,b] \to M$ is a trajectory corresponding

to a control $\eta : [a, b] \to U$ such that the pair (γ, η) is an extremal, i.e. that (γ, η) satisfies the conditions of the Pontryagin Maximum Principle. Assume that (γ, η) can be embedded in a *one-parameter family of extremals* $\Gamma = \{(\gamma_s, \eta_s)\}_{s \in [-\varepsilon, 0]}$, where each (γ_s, η_s) is defined on an interval $[a, b_s]$, that may depend on s, and satisfies $\gamma_s(a) = \gamma(a)$. Assume that this family is smooth, in some suitable sense, and that the corresponding minimizing adjoint vectors λ_s are also smooth. (The precise definition of smoothness is a somewhat delicate matter, because the trajectories involved usually have switchings. But this need not concern us here, since in any case the proof we will give will not use the general result.) In this case, we would call the curve $\delta : s \to \gamma_s(b_s)$ an *envelope* of the family Γ if two conditions hold. First, δ should be a trajectory of our control system, corresponding to some control $\zeta : [-\varepsilon, 0] \to U$. Second, the identity

$$H(\lambda_s(b_s), \gamma_s(b_s), \eta_s(b_s)) = H(\lambda_s(b_s), \gamma_s(b_s), \zeta_s(b_s))$$

should hold. (Notice that the first condition is trivial in the classical Calculus of Variations situation, because in that case every curve is a trajectory. And the second condition holds trivially, because in that case the control is just $\dot{x}$, so the tangency condition says precisely that $\eta_s(b_s) = \zeta_s(b_s)$.) With these definitions, the same conclusion as for the classical case holds: the cost (i.e., in this case, the time) along the curve $\tilde{\gamma}$ obtained by first following $\gamma_{-\varepsilon}$ from $\gamma(a)$ to $\gamma_{-\varepsilon}(b_{-\varepsilon})$ and then δ from $\gamma_{-\varepsilon}(b_{-\varepsilon})$ to $\gamma(b)$ is exactly the same as the cost along γ. In other words: $b_{-\varepsilon}$ is necessarily equal to $b - \varepsilon$.

To apply this to our situation, we will proceed as follows (cf. Figure 13).

Consider, say, an $L_\alpha^- L_\beta^+ R_\beta^+ R_\beta^-$ trajectory. Then this trajectory happens to be an extremal, if $\alpha < \beta$. (We will not actually use this fact in the proof either, but we point it out here because it is one of the reasons why the proof works.) By allowing α and β to vary in a neighborhood of the values $\bar{\alpha}$, $\bar{\beta}$ of our trajectory, while keeping the starting point fixed, we get a *two-parameter family of extremals* $\gamma_{\alpha,\beta}$. We will verify by a direct computation that the terminal points $q(\alpha, \beta)$ of these extremals describe a two-dimensional surface S. To get a one-parameter family of extremals that satisfies the envelope conditions, we need to find a trajectory of our control system that is actually contained in S. A basis for the tangent vectors to S is given by A and B, where $A = \frac{\partial q}{\partial \alpha}$ and $B = \frac{\partial q}{\partial \beta}$. So a vector field X on S is specified by giving a linear combination of A and B, with variable coefficients. If this X happens to be of the form $\theta_1 f + \theta_2 g$, where the functions θ_1, θ_2 have values in $[-1, 1]$, then the integral curves of X will be trajectories of our system as well. Notice that so far, since for every point $q \in S$ the span of $A(q)$ and $B(q)$ is two dimensional, as is that of $f(p)$ and $g(p)$, we can expect that the intersection of these two spans will be a one-dimensional space, which means that X will be uniquely determined up to multiplication by a scalar function. We now try to satisfy the second envelope condition. The Hamiltonian along γ_s at time b_s is $\langle \lambda_s(b_s), -f(\gamma_s(b_s)) - g(\gamma_s(b_s)) \rangle$, whereas that along δ is $\langle \lambda_s(b_s), X(\gamma_s(b_s)) \rangle$. On the other

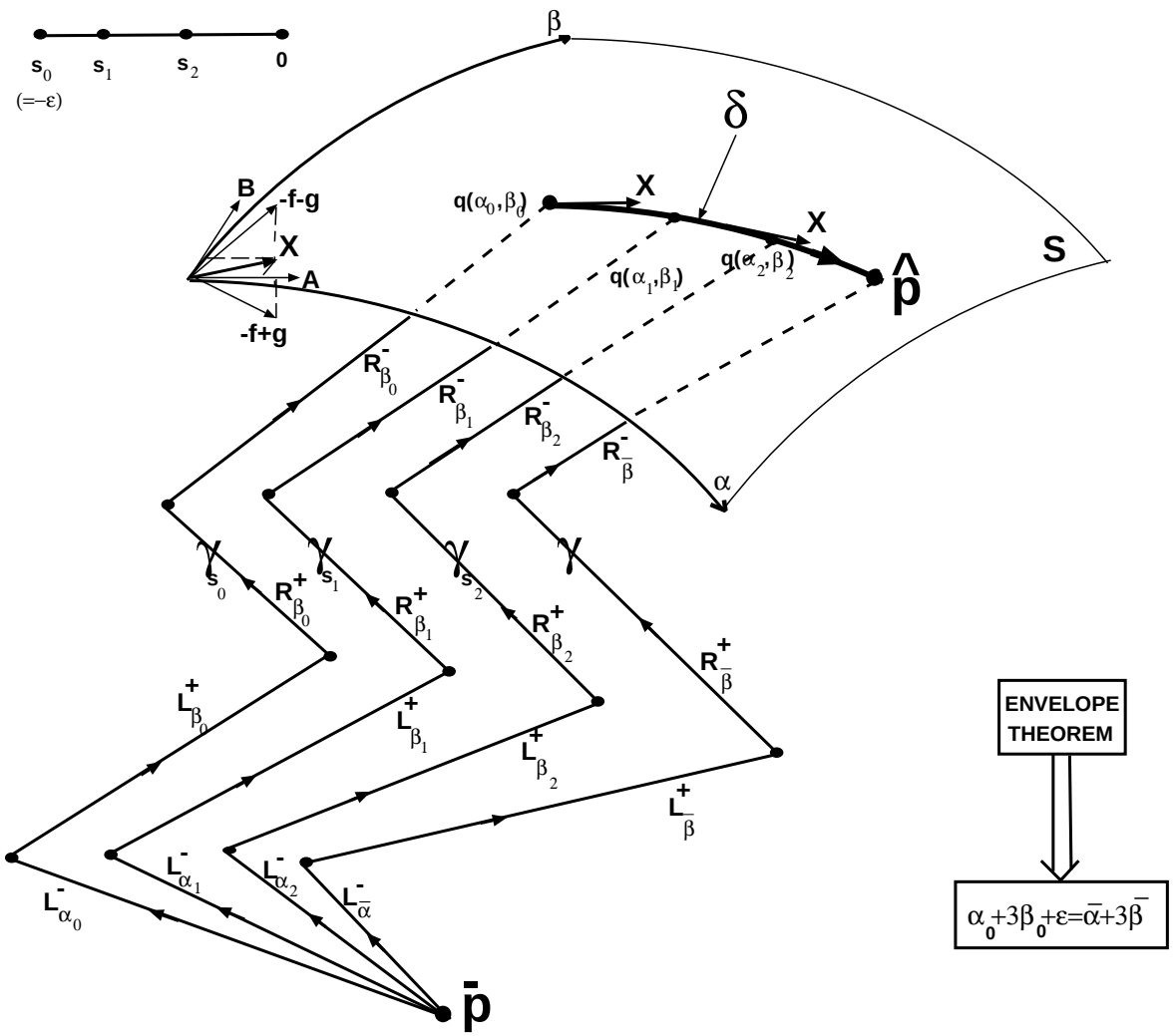

Figure 13: A control envelope.

hand, $\langle \lambda_s(b_s), g(\gamma_s(b_s))\rangle = 0$. (Indeed, not only are curves $L_\alpha^- L_\beta^+ R_\beta^+ R_\beta^-$ extremals, but in fact an even longer curve $L_\alpha^- L_\beta^+ R_\beta^+ R_\beta^- L_\beta^-$ is also an extremal, which means that, for γ_s, the switching function for g vanishes at the terminal point. So the second envelope condition will hold if X happens to be of the form $-f + \sigma g$ for some function σ. Once this is achieved, the general envelope theorem already guarantees that the time along the trajectory obtained by first following $\gamma_{-\varepsilon}$ and then δ is the same as that along γ. (But in our case we will verify this directly, so the general theorem will not be used.) It follows that δ is optimal. This can then be proved impossible in a number of ways. For instance, the control v has to be constant along δ, because it is equal to 1, 0 or -1, and continuous. We will verify by a direct computation that the derivative of σ is $\neq 0$, and this will complete the proof.

We now give the details.

Proof of Lemma 14. Let γ be a $B_{\bar\alpha}^- B_{\bar\beta}^+ v B_{\bar\beta}^+ B_{\bar\beta}^-$ trajectory. Then γ is either $L_{\bar\alpha}^- L_{\bar\beta}^+ R_{\bar\beta}^+ R_{\bar\beta}^-$ or $R_{\bar\alpha}^- R_{\bar\beta}^+ L_{\bar\beta}^+ L_{\bar\beta}^-$. We will assume that γ is $L_{\bar\alpha}^- L_{\bar\beta}^+ R_{\bar\beta}^+ R_{\bar\beta}^-$. (The $R_{\bar\alpha}^- R_{\bar\beta}^+ L_{\bar\beta}^+ L_{\bar\beta}^-$ case is similar.)

Since a portion of an optimal trajectory is optimal, we may assume that $\bar\alpha < \bar\beta$. Also, we know that an $L_{\bar\beta}$ trajectory with $\bar\beta > \pi$ is not optimal, so we may assume $\bar\beta \leq \pi$.

By changing cooordinates, if necessary, we may assume that the initial point $\bar p$ of γ is $(0,0,0)$. Let $\hat p = (\hat x_1, \hat x_2, \hat\theta)$ be the terminal point of γ. Then

$$\hat p = \bar p e^{\bar\alpha L^-} e^{\bar\beta L^+} e^{\bar\beta R^+} e^{\bar\beta R^-} \quad . \tag{32}$$

One can then easily compute $\hat p$ explicitly, and the result is

$$\hat p = \begin{pmatrix} -4\sin\bar\alpha + 2\sin(\bar\alpha+\bar\beta) - \sin(\bar\beta-\bar\alpha) \\ -1 + 4\cos\bar\alpha - 2\cos(\bar\alpha+\bar\beta) - \cos(\bar\beta-\bar\alpha) \\ \bar\alpha - \bar\beta \end{pmatrix} . \tag{33}$$

Let us make a "variation" of γ by considering, for (α,β) in some neighborhood of $(\bar\alpha,\bar\beta)$, the trajectory $\gamma_{\alpha,\beta}$ that starts at $\hat p$ and is of type $L_\alpha^- L_\beta^+ R_\beta^+ R_\beta^-$. Let $q(\alpha,\beta)$ be the terminal point of this trajectory, so

$$q(\alpha,\beta) = \begin{pmatrix} -4\sin\alpha + 2\sin(\alpha+\beta) - \sin(\beta-\alpha) \\ -1 + 4\cos\alpha - 2\cos(\alpha+\beta) - \cos(\beta-\alpha) \\ \alpha - \beta \end{pmatrix} \tag{34}$$

and $q(\bar\alpha,\bar\beta) = \hat p$. We now compute the derivatives A, B of q with respect to α and β. The result is

$$A(\alpha,\beta) = \frac{\partial q}{\partial \alpha} = \begin{pmatrix} -4\cos\alpha + 2\cos(\alpha+\beta) + \cos(\beta-\alpha) \\ -4\sin\alpha + 2\sin(\alpha+\beta) - \sin(\beta-\alpha) \\ 1 \end{pmatrix} , \tag{35}$$

$$B(\alpha,\beta) = \frac{\partial q}{\partial \beta} = \begin{pmatrix} 2\cos(\alpha+\beta) - \cos(\beta-\alpha) \\ 2\sin(\alpha+\beta) + \sin(\beta-\alpha) \\ -1 \end{pmatrix} . \tag{36}$$

Let $\hat{A} = A(\bar{\alpha}, \bar{\beta})$, $\hat{B} = B(\bar{\alpha}, \bar{\beta})$. Then it is easy to verify that $\hat{A}$ and $\hat{B}$ are linearly independent. (If $\mu\hat{A} + \nu\hat{B} = 0$, then by looking at the third component we get $\mu = \nu$, so $\mu(\hat{A} + \hat{B}) = 0$, and linear independence follows if $\hat{A} + \hat{B} \neq 0$. But

$$\hat{A} + \hat{B} = \begin{pmatrix} 4\cos(\bar{\alpha} + \bar{\beta}) - 4\cos\bar{\alpha} \\ 4\sin(\bar{\alpha} + \bar{\beta}) - 4\sin\bar{\alpha} \\ 0 \end{pmatrix} , \tag{37}$$

so $\hat{A} + \hat{B}$ can only equal 0 if $\bar{\beta} = 0$ modulo 2π. Since $0 < \bar{\beta} \leq \pi$, this is impossible.)

It follows that, for (α, β) in some small neighborhood N of $(\bar{\alpha}, \bar{\beta})$, the points $q(\alpha, \beta)$ describe a surface S. Our goal now is to find a trajectory of our control system that goes through the point $\hat{p}$ and is entirely contained in S. For this purpose, it suffices to find smooth functions $\mu(\alpha, \beta)$, $\nu(\alpha, \beta)$ such that the linear combination

$$X(\alpha, \beta) = \mu(\alpha, \beta)A(\alpha, \beta) + \nu(\alpha, \beta)B(\alpha, \beta)$$

belongs to the linear span of $f(q(\alpha, \beta))$ and $g(q(\alpha, \beta))$. (Indeed, once such functions are found, the integral curve of the vector field X through $\hat{p}$ will have the desired properties.)

Clearly, both vectors $A(\alpha, \beta)$ and $B(\alpha, \beta)$ can be expressed as linear combinations of $f(q(\alpha, \beta))$, $h(q(\alpha, \beta))$ and $g(q(\alpha, \beta))$, since f, g and h form a basis at every point. So what we need is to find a linear combination of A and B such that the coefficient of h vanishes. Computing the inner products of A and B with h, we get

$$\langle A(\alpha, \beta), h(q(\alpha, \beta))\rangle = -4\sin\beta + 2\sin 2\beta \tag{38}$$

and

$$\langle B(\alpha, \beta), h(q(\alpha, \beta))\rangle = 2\sin 2\beta \ . \tag{39}$$

To satisfy $\langle \mu A + \nu B, h\rangle = 0$, we need $\mu(\alpha, \beta)(2\sin 2\beta - 4\sin\beta) + \nu(\alpha, \beta)2\sin 2\beta = 0$. Writing $\sin 2\beta = 2\sin\beta\cos\beta$ and factoring $4\sin\beta$, we get the equation $(\mu + \nu)\cos\beta - \mu = 0$. An obvious solution is given by $\mu = \cos\beta$, $\mu + \nu = 1$, which yields $\nu = 1 - \cos\beta$. So the vector field Y, defined on S by $Y(q(\alpha, \beta)) = \cos\beta\, A(\alpha, \beta) + (1 - \cos\beta)B(\alpha, \beta)$, is a linear combination of f and g. Let us compute the coefficient of f, i.e. the inner product $\langle Y, f\rangle$. We have

$$\langle A(\alpha, \beta), f(q(\alpha, \beta))\rangle = -4\cos\beta + 2\cos 2\beta + 1 \ ,$$

and

$$\langle B(\alpha, \beta), f(q(\alpha, \beta))\rangle = 2\cos 2\beta - 1 \ .$$

Then

$$\langle Y, f\rangle = -4\cos^2\beta + 2\cos\beta\cos 2\beta + \cos\beta + 2\cos 2\beta - 1 - 2\cos\beta\cos 2\beta + \cos\beta \ ,$$

so that

$$\langle Y(\alpha,\beta), f(q(\alpha,\beta))\rangle = -3 + 2\cos\beta \ . \tag{40}$$

So, if instead of μ and ν we use the functions

$$\eta(\alpha,\beta) = \frac{\cos\beta}{3-2\cos\beta} \ , \quad \zeta(\alpha,\beta) = \frac{1-\cos\beta}{3-2\cos\beta} \ , \tag{41}$$

and define $X = \eta A + \zeta B$, then the f-component of X is exactly equal to -1. The h-component is of course 0, and the g-component is clearly $\eta - \zeta$, so we get, if $q \in S$, the identity $X(q) = -f(q) + \sigma(q)g(q)$, where

$$\sigma(q(\alpha,\beta)) = \frac{2\cos\beta - 1}{3-2\cos\beta} \ . \tag{42}$$

We remark that $\sigma = -1 + \frac{2}{3-2\cos\beta}$. Since $3-2\cos\beta \geq 1$, the function σ satisfies $|\sigma(q)| \leq 1$. Hence every integral curve of X is in fact a trajectory of CRS.

Now pick a small $\varepsilon > 0$, let I denote the interval $[-\varepsilon, 0]$, and let $\delta : I \to S$ be the integral curve of X such that $\delta(0) = \hat{p}$. Since the curve δ is entirely contained in S, we can write $\delta(s) = q(\alpha(s),\beta(s))$ for $-\varepsilon \leq s \leq 0$. For each $s \in I$, let $\hat{\gamma}_s$ be the curve obtained as follows: first follow the $L^-_{\alpha(s)}L^+_{\beta(s)}R^+_{\beta(s)}R^-_{\beta(s)}$ curve $\gamma_{\alpha(s),\beta(s)}$, which goes from $\bar{p}$ to $q(\alpha(s),\beta(s))$, and then follow δ from time s to time 0. Then $\hat{\gamma}_s$ goes from $\bar{p}$ to $\hat{p}$ in time $\tau(s) = \alpha(s) + 3\beta(s) - s$. (Recall that $s < 0$.)

On the other hand, using prime to denote differentiation with respect to s, we have

$$\delta'(s) = X(\delta(s)) = \eta(\alpha(s),\beta(s))A(\alpha(s),\beta(s)) + \zeta(\alpha(s),\beta(s))B(\alpha(s),\beta(s)) \ , \tag{43}$$

whereas, using $\delta(s) = q(\alpha(s),\beta(s))$, the chain rule, and the definition of A and B, we get

$$\delta'(s) = \alpha'(s)A(\alpha(s),\beta(s)) + \beta'(s)B(\alpha(s),\beta(s)) \ . \tag{44}$$

Equating coefficients, we find $\alpha'(s) = \eta(\alpha(s),\beta(s))$ and $\beta'(s) = \zeta(\alpha(s),\beta(s))$. So

$$\tau'(s) = \eta(\alpha(s),\beta(s)) + 3\zeta(\alpha(s),\beta(s)) - 1 \ . \tag{45}$$

On the other hand, it follows directly from the formulas for η and ζ that

$$\eta + 3\zeta = 1 \ . \tag{46}$$

This implies that $\tau'(s) \equiv 0$, so the function $s \to \tau(s)$ is in fact a constant.

In particular, the trajectory $\hat{\gamma}_{-\varepsilon}$ goes from $\bar{p}$ to $\hat{p}$ in *exactly the same time* as γ does. Therefore $\hat{\gamma}_{-\varepsilon}$ is optimal if γ is. Since δ is a part of $\hat{\gamma}_{-\varepsilon}$, we conclude in particular that δ is optimal as well. On the other hand, $\delta'(s) = -f(\delta(s)) + v(s)g(\delta(s))$, where $v(s) = \sigma(\delta(s))$. We know that along any optimal trajectory of CRS v can only take the values 1, -1 and

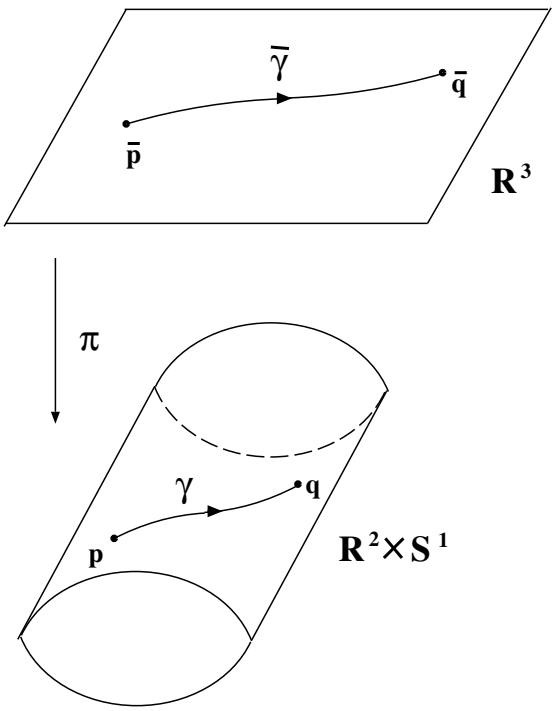

Figure 14: The LTV and LLTV problems.

0. Since σ is continuous, the only possibility is that $v(s)$ is constant, and equal to 1, -1, or 0. Since $\sigma = \frac{2\cos\beta - 1}{3 - 2\cos\beta}$, it is clear that $\sigma(\delta(s))$ cannot be constant unless $\beta(s)$ is constant. But $\beta'(s) = \zeta(\delta(s))$, so ζ has to vanish along δ. Since $\zeta = \frac{1-\cos\beta}{3-2\cos\beta}$, this can only happen if $\cos\beta = 1$ along δ, so in particular $\bar{\beta}$ has to be an integer multiple of 2π. Since $0 < \bar{\beta} \leq \pi$, this is impossible, and we have reached a contradiction. ∎

15 The lifted LTV problem and CRS-optimal LTV trajectories

We now study the regularity properties of CRS-optimal LTV trajectories, i.e. time-optimal trajectories of CRS for which $v \equiv 1$. Recall that the LTV problem is the subproblem of CRS obtained by setting $v \equiv 1$, i.e. assuming that the velocity vector of the car is only turning left.

In order to study the LTV problem, we will also want to consider the *lifted LTV problem* (abbreviated LLTV) in $\mathbb{R}^3$ obtained from LTV by regarding θ as true real number, i.e. not identifying values of θ that differ by a multiple of 2π (cf. Fig. 14). The precise reason why the lifting is necessary is explained in Remark 12.

The dynamical law for LLTV is given by

$$LLTV: \qquad \dot{x} = g(x) + uf(x), \qquad x \in \mathbb{R}^3, \qquad |u| \leq 1\ . \tag{47}$$

(We write $x = (x_1, x_2, x_3)$, $\pi(x) = (x_1, x_2, \theta)$, where θ is the class of x_3 modulo 2π. We allow ourselves a slight abuse of language and use the same notation for f, g and h regarded as functions of x or of $\pi(x)$.)

Notice that for LTV (resp. LLTV) one of the equations of motion is $\dot{\theta} = 1$ (resp. $\dot{x}_3 = 1$). For LLTV, this implies that for any trajectory γ from a point p to another point q, the time is exactly $x_3(q) - x_3(p)$. In particular, this time is the same for all trajectories from p to q, so *Problem LLTV is degenerate*, that is

D1 every trajectory of LLTV is time-optimal.

The situation is almost the same for LTV trajectories, except that now p and q only determine the time modulo 2π, so we can only conclude that

D2 every trajectory of LTV whose duration is $\leq \pi$ is time-optimal.

In particular, *the LTV problem is locally degenerate.*

As stated in Section 2, the degenerate situation can be well characterized by use of the *strong accessibility Lie algebra* $L_0(\Sigma)$ of a control problem Σ, introduced in [48]. We explain this in detail here for the particular case of the LTV and LLTV problems.

We let L_0 be the ideal generated by f of the Lie algebra $L = L(f, g)$. (In other words, L_0 is the smallest linear subspace S of $L(f, g)$ that contains f and is such that $X \in S$, $Y \in L(f, g)$ imply $[X, Y] \in S$. It follows from the Commutation Relations (11) that $L_0 = L(f, h) = \text{span}(f, h)$, i.e. L_0 is in fact the Lie subalgebra generated by f and h, which is the same as the linear span of f and h. (Indeed, since $[f, h] = 0$, the span S of f and h is a Lie subalgebra. Since $[g, f] = h$ and $[g, h] = -f$, it is clear that $[X, Y] \in S$ whenever $X \in L(f, g)$ and $Y \in S$, so S is indeed an ideal. Finally, since $[g, f] = h$, any ideal that contains f must contain h as well, so S is indeed the smallest one.)

Remark 9 We really should be using a notation such as $L_0(LTV)$, to indicate that this is the "strong accessibility Lie algebra" L_0 that corresponds to the LTV problem. (The general definition of L_0 for an arbitrary control problem is briefly explained in Remark 13 below.) And, moreover, we should actually distinguish between $L_0(LTV)$ and $L_0(LLTV)$. Abusing notation once again, we will not distinguish between the two, and we will only write L_0, since in this paper we will not have the occasion to use L_0 for any other problem. ∎

Next we write $L_0(p)$, for a point p, to denote the set of all vectors $X(p)$ for $X \in L_0$, that is, $L_0(p) = \text{span}(f(p), h(p))$. Finally, following the terminology of [52], we call a trajectory of LLTV a *strong extremal* if it has a minimizing adjoint vector λ such that $\lambda(t)$ is nontrivial on $L_0(\gamma(t))$ for each t.

Remark 10 Notice that the full tangent space at p to the state space of our model is just $L(p)$, where $L = L(f,g)$. So the statement that $\lambda(t)$ is nontrivial could be reformulated by saying that "$\lambda(t)$ is nontrivial on $L(\gamma(t))$ for each t." We thus see that strong extremality is just a more restricted form of extremality, in which L is replaced by L_0. ∎

The concept of a strong extremal is important because boundary trajectories of LLTV are strong extremals. To explain this, let us first introduce notations for reachable sets. For any $p \in \mathbb{R}^3$, let us write $\mathcal{R}_T(p)$ to denote the *time T reachable set from p*, i.e. the set of all points q that can be reached from p in time T by an LLTV trajectory. Let $\mathcal{R}(p) = \cup_{t\geq 0}\mathcal{R}_t(p)$, so $\mathcal{R}(p)$ is the *reachable set from p*.

We will use the following well known fact from control theory, whose elementary proof we include for completeness:

Lemma 15 *The reachable set $\mathcal{R}(p)$ is closed.*

Proof. Let $\{q_j\}$ be a sequence of points in $\mathcal{R}(p)$ that converges to q as $j \to \infty$. Then each q_j can be reached from p by means of a control $\eta_j : [0, T_j] \to U$. Then $T_j = x_3(q_j) - x_3(p) \to x_3(q) - x_3(p) = T$. So, if we pick $\hat{T} > x_3(q) - x_3(p)$ and take j large enough, we may assume that $T_j \leq \hat{T}$ for all j. Extend η_j to the whole interval $[0, \hat{T}]$ by setting it equal to 0 for $t > T_j$. The sequence of $\mathbb{R}^2$-valued functions $F_j(t) = \int_0^t \eta_j(s)\, ds$ is uniformly bounded and equicontinuous, so by the Ascoli-Arzelà theorem it has a subsequence $\{F_{j(k)}\}$ that converges uniformly to a continuous function F. Write $F_j = (F_j^1, F_j^2)$, $F = (F^1, F^2)$. Then it is easy to see that all the F_j^i are in fact Lipschitz with Lipschitz constant 1, so the F^i are also Lipschitz with constant 1. So F is absolutely continuous and its derivative η is U-valued. Let γ_j (resp. γ) be the trajectory of η_j (resp. η) from p. By Theorem 4, the γ_j converge uniformly to γ. So $q_j = \gamma_j(T_j) \to \gamma(T)$. But then $\gamma(T) = q$, which shows that q is reachable from p. ∎

The *boundary* $\partial\mathcal{R}(p)$ of $\mathcal{R}(p)$ is the set $\mathcal{R}(p) - \operatorname{Int}\mathcal{R}(p)$. A *boundary trajectory* of LLTV is a trajectory $\gamma : [a,b] \to \mathbb{R}^3$ such that $\gamma(b) \in \partial\mathcal{R}(\gamma(a))$.

Lemma 16 *A boundary trajectory of LLTV is a strong extremal.*

Proof. Let $\gamma : [a,b] \to \mathbb{R}^3$ be a boundary trajectory. Let $p = \gamma(a)$, $q = \gamma(b)$. Then q belongs to the boundary of $\mathcal{R}(p)$, so we know from Section 9 that γ must satisfy the conditions of the Maximum Principle for boundary trajectories. This means that there must be a nontrivial minimizing adjoint vector such that the value of the Hamiltonian is zero. In terms of switching functions, this says that $-|\varphi| + \psi = 0$. On the other hand, the switching function equations $\dot{\varphi}(t) = v(t)\chi(t)$ and $\dot{\chi}(t) = -v(t)\varphi(t)$ (cf. equations (26)) imply, as we saw in Section 11, that the number $\kappa = \varphi(t)^2 + \chi(t)^2$ is a constant. If $\kappa = 0$, it would follow that φ and χ vanish identically and then the identity $-|\varphi| + \psi = 0$ would imply that $\psi \equiv 0$ as well, so λ would

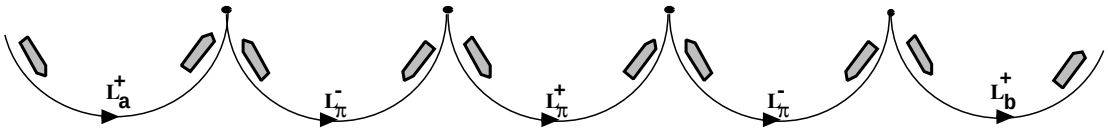

Figure 15: A boundary trajectory of LLTV.

be trivial, which is a contradiction. Therefore $\kappa \neq 0$. But this says precisely that $\langle \lambda, f \rangle$ and $\langle \lambda, h \rangle$ do not both vanish, i.e. that γ is a strong extremal. ∎

Remark 11 In the proof of Lemma 16 we used the Maximum Principle in the version labeled MP2 in Section 9. This version appears to say that "a boundary trajectory is an abnormal extremal," whereas Lemma 2 says that "optimal abnormal extremals do not exist." The explanation for the apparent contradiction is as follows: abnormal extremals do not exist for the CRS problem, because for that problem the Hamiltonian minimization condition gives $|\varphi| + |\psi| = \lambda_0$, since $u\varphi + v\psi + \lambda_0$ has to be minimized *with respect to both u and v for* $|u| \in [-1, 1]$, $|v| \in [-1, 1]$. (As indicated in the proof of Lemma 2, if $\lambda_0 = 0$ this would imply that $\varphi \equiv \psi \equiv 0$, and then nontriviality implies that $\chi(t) \neq 0$, whereas the equations (26) imply that $u(t)\chi(t) = v(t)\chi(t) = 0$. So $u(t) \equiv v(t) \equiv 0$, and γ is not optimal.) Notice, however, that for LTV or LLTV the Hamiltonian minimization condition only gives $|\varphi| - \psi = \lambda_0$, because now it is $u\varphi + \psi + \lambda_0$ that has to be minimized *with respect to u only.* Now $\lambda_0 = 0$ no longer implies that φ and ψ must vanish, since ψ can be < 0. ∎

Using Lemma 16, we can determine the structure of the boundary trajectories of LLTV.

Lemma 17 *For a boundary trajectory γ of LLTV, the control u has finitely many switchings. The interval between consecutive switchings is exactly π, and every interval of length $> \pi$ contains a switching.*

In other words: every boundary trajectory of LLTV is of the form $L_a^{\sigma_0} L_\pi^{\sigma_1} L_\pi^{\sigma_2} \dots L_\pi^{\sigma_k} L_b^{\sigma_{k+1}}$, where $0 \leq a < \pi$, $0 \leq b < \pi$, k is a nonnegative integer, and the signs $\sigma_0, \sigma_1, \dots, \sigma_{k+1} \in \{-, +\}$ alternate (cf. Figure 15).

Proof of Lemma 17. Let γ be a boundary trajectory. We know that γ is a strong extremal. So there is a minimizing adjoint vector such that the switching functions φ and χ do not both vanish identically, i.e. $\kappa > 0$. But then the equations $\dot{\varphi}(t) = v(t)\chi(t)$ and $\dot{\chi}(t) = -v(t)\varphi(t)$, together with $v \equiv 1$, imply that φ is a nontrivial solution of $\ddot{\varphi} + \varphi = 0$. So the intervals between consecutive zeros of φ are exactly π, and φ changes sign at each zero. Then the switching properties **SP1** and **SP2** of Section 10 imply our conclusion. ∎

Now that we have a simple description of the boundary trajectories of LLTV, we want to study more general trajectories. The key point here is to use a geometric argument, illustrated

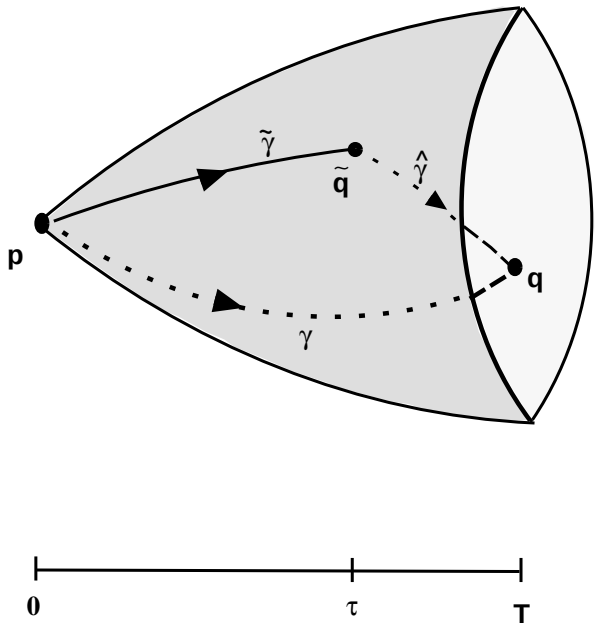

Figure 16: The proof of the Lemma on LLTV trajectories.

in Figure 16, to be able to replace an arbitrary trajectory by a concatenation of two trajectories of simple types, namely, boundary and bang.

Lemma 18 *For every trajectory γ of LLTV there exists another trajectory γ' that steers $In(\gamma)$ to $Term(\gamma)$ in time $T(\gamma)$ and is a concatenation of a boundary trajectory and a bang arc of LLTV for the control $u \equiv 1$.*

Proof. Because of the equation $\dot{x_3} = 1$, if $q \in \mathcal{R}_T(p)$ for some T, then this T is unique, and is none other than $x_3(q) - x_3(p)$.

Let γ be defined on $[0, T]$, $\gamma(0) = p$, $\gamma(T) = q$. If q belongs to the boundary of $\mathcal{R}(p)$, then γ is a boundary trajectory and our conclusion follows.

Now suppose that $q \in \text{Int}(\mathcal{R}(p))$. We then look at the LLTV-trajectory $\hat{\gamma} : \mathbb{R} \to \mathbb{R}^3$ that corresponds to the control $u \equiv 1$ and goes through q at time T (cf. Figure 16).

Since $\mathcal{R}(p)$ is closed (cf. Lemma 15), the set $\{s : 0 \leq s \leq T \text{ and } \hat{\gamma}(s) \in \mathcal{R}(p)\}$ has a smallest element τ. Let $\tilde{q} = \hat{\gamma}(\tau)$. Then q is reachable from $\tilde{q}$ in time $T - \tau$, so $x_3(\tilde{q}) = x_3(q) - (T - \tau)$. On the other hand, $\tilde{q}$ is reachable from p in some time τ'. But then, as remarked earlier, τ' has to be $x_3(\tilde{q}) - x_3(p)$. Since $T = x_3(q) - x_3(p)$, we conclude that $\tau = \tau'$.

If $\tau = 0$, then $\tilde{q}$ is reachable from p in time 0, and this is only possible if $\tilde{q} = p$, in which case $\hat{\gamma}$ is a trajectory from p to q for the control $u \equiv 1$.

Now suppose that $\tau > 0$. Then it is clear that $\hat{\gamma}(\tau) \in \partial\mathcal{R}(p)$, because the points $\hat{\gamma}(t)$ for $t < \tau$ are not in $\mathcal{R}(p)$. In that case, let $\tilde{\gamma} : [0, \tau] \to \mathbb{R}^3$ be a trajectory from p to $\tilde{q}$. Then $\tilde{\gamma}$ is a boundary trajectory, so our conclusion holds in this case as well. ∎

We now return to the LTV problem. Suppose $\gamma : [0, T] \to M$ is an LTV trajectory which is time-optimal for CRS. Then $\gamma = \pi \circ \gamma^*$, where $\gamma^* : [0, T] \to \mathbb{R}^3$ is a trajectory of LLTV. We can then replace γ^* by a concatenation γ^*_{new} of a boundary trajectory $\tilde{\gamma}^*$ and a bang trajectory $\hat{\gamma}^*$ with $u \equiv 1$, and then project these down to trajectories γ_{new}, $\tilde{\gamma}$, $\hat{\gamma}$ in M. Then $\tilde{\gamma}$ is of the form $L_a^{\sigma_0} L_\pi^{\sigma_1} L_\pi^{\sigma_2} \dots L_\pi^{\sigma_k} L_b^{\sigma_{k+1}}$. But then Lemma 11 implies that $k = 0$, so actually $\tilde{\gamma}$ is $L_a^+ L_b^-$ or $L_a^- L_b^+$, so that γ_{new} is of the form $L_a^+ L_b^- L_c^+$ or $L_a^- L_b^+ L_c^+$. In the latter case, γ_{new} is $L_\alpha^- L_\beta^+$, if we let $\beta = b + c$. But then necessarily $\beta \leq \pi$ and so, if we now relabel again and let $a = 0$, $b = \alpha$, $c = \beta$, we get again an $L_a^+ L_b^- L_c^+$ trajectory. So we have shown:

Lemma 19 *If $\gamma : [0, T] \to M$ is an LTV trajectory which is optimal for CRS, then there exists another trajectory $\gamma_{new} : [0, T] \to M$ such that $\gamma_{new}(0) = \gamma(0)$, $\gamma_{new}(T) = \gamma(T)$, and γ_{new} is of the form $L_a^+ L_b^- L_c^+$ with $a \leq \pi$, $b \leq \pi$, $c \leq \pi$.* ∎

Naturally, the same analysis is valid for the "RTV problem," in which $v \equiv -1$, except that now all the L's have to be replaced by R's. Combining the two results, we can conclude that

Lemma 20 *If $\gamma : [0, T] \to M$ is a time-optimal CRS-trajectory along which $v \equiv 1$ or $v \equiv -1$, then there exists another trajectory $\gamma_{new} : [0, T] \to M$ such that $\gamma_{new}(0) = \gamma(0)$, $\gamma_{new}(T) = \gamma(T)$, and γ_{new} is of the form $L_a^+ L_b^- L_c^+$ or $R_a^+ R_b^- R_c^+$, with $a \leq \pi$, $b \leq \pi$, $c \leq \pi$.* ∎

Remark 12 The preceding discussion would not have been possible if we had only worked with LTV, without lifting it to LLTV. The reason is that for LTV we cannot produce a boundary trajectory by the procedure we used for LLTV. (Actually, it is not hard to prove that LTV is completely controllable, and this means that the boundary of the reachable sets is empty. The proof of complete controllability can be carried out as for the Dubins problem, using the fact that the vector fields $f + g$ and $-f + g$ are periodic.) The precise step where our proof breaks down for LTV is the following: the time τ of the proof of Lemma 18 can be 0, and for LTV this does not imply that $\hat{\gamma}(\tau)$ is reachable from p in time 0, i.e. that $\hat{\gamma}(\tau) = p$, but only that $\hat{\gamma}(\tau)$ is reachable from p in a time which is $\equiv 0$ modulo 2π. ∎

Remark 13 For a general control system $\Sigma : \dot{x} = F(x, w)$, $w \in W$, the *strong accessibility Lie algebra* $L_0(\Sigma)$ is defined as follows: we let $L(\Sigma)$ be, as before, the Lie algebra of vector fields generated by the $F(\cdot, w)$, for all possible values $w \in W$. We then define $L_0(\Sigma)$ to be the smallest ideal of $L(\Sigma)$ that contains all the differences $F(\cdot, w_1) - F(\cdot, w_2)$ for all possible pairs w_1, w_2 of control values. It is a general theorem that a real analytic minimum time optimal control problem Σ that satisfies the LARC and has a connected state space M is

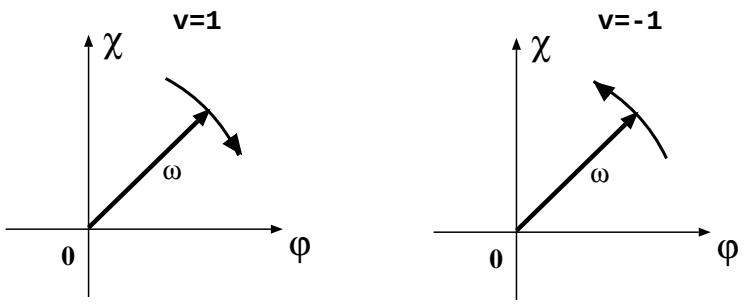

Figure 17: The vector ω.

locally degenerate if and only if $L_0(\Sigma)(x) \neq T_xM$ for all $x \in M$. (Here M is a C^ω manifold, and T_xM is the tangent space at x.) ∎

16 The analysis of general CRS-optimal trajectories

We now study the structure of general time-optimal trajectories for the CRS problem. Any such trajectory is an extremal, so there is a nontrivial minimizing adjoint vector $\lambda(\cdot)$. Then we know that there are two possibilities, namely, Type T1 and T2 (cf. Section 11).

Recall that (γ, λ) is of Type T2 if the corresponding u-switching function φ is $\equiv 0$. In that case Lemma 3 tells us that the v-switching function ψ never vanishes. So v is identically 1 or -1. By Lemma 20, γ can be replaced by a CRS-optimal LTV trajectory γ_{new} from $In(\gamma)$ to $Term(\gamma)$ which is bang-bang with at most two u-switchings and no v-switching. A similar conclusion clearly holds when $v \equiv -1$. This completely disposes of Type T2 extremals.

We now turn our attention to the Type T1 case. Let $\gamma : [a, b] \to M$ be an optimal trajectory of CRS such that, for some nontrivial minimizing adjoint vector, (γ, λ) is of Type T1. Then $\kappa \neq 0$, so the switching structure equations $\dot{\varphi}(t) = v(t)\chi(t)$, $\dot{\chi}(t) = -v(t)\varphi(t)$, say that the *nonzero* vector $\omega = (\varphi, \chi)$ —which is of constant length $\sqrt{\kappa}$— is rotating in the plane with angular velocity v. In particular:

The vector ω rotates with angular velocity ± 1 if $\lvert v\rvert \equiv 1$, and the rotation is clockwise as long as $v = 1$ and counterclockwise when $v = -1$ (cf. Figure 17).

Moreover, we know from Lemma 6 that the zeros of φ are isolated. In between any two zeros, $u \equiv 1$ or $u \equiv -1$. So between any two zeros of φ we have a *Dubins-like trajectory*, whose structure we completely know thanks to Lemma 9.

We now have to study how these Dubins-like pieces fit together. For this purpose, let us first remark that the identity $|\varphi| + |\psi| = \lambda_0$ implies that

$\lvert\psi(t)\rvert = \lambda_0$ at every u-switching time t.

Now let us study a *maximal Dubins-like interval* (henceforth abbreviated MDLI), i.e. a maximal subinterval $I = [t_1, t_2]$ of $[a, b]$ whose interior contains no u-switchings. (That is, t_1 and t_2 may be consecutive u switchings, or $t_1 = a$ and t_2 is the first u-switching, or $t_2 = b$ and t_1 is the last u-switching, or $t_1 = a$, $t_2 = b$, and $[a, b]$ contains no u-switchings.)

Lemma 9 implies that we can distinguish four cases:

C1: the switching function ψ never vanishes on $[t_1, t_2]$,

C2: ψ has exactly one zero s between t_1 and t_2 and changes sign there,

C3: the zero set of ψ is a nontrivial interval $[s_1, s_2]$,

C4: ψ has exactly one zero s between t_1 and t_2 but does not change sign there.

In Case C1 u and v are constant and equal to 1 or -1. So the equations $\dot{\varphi} = v\chi$ and $\dot{\chi} = -v\varphi$ yield $\ddot{\varphi} + \varphi = 0$, and therefore $\varphi(t) = \sqrt{\kappa}\sin(t - t_0)$ and $\chi(t) = \sqrt{\kappa}\cos(t - t_0)$ for some t_0. In particular, the fact that φ has no zeros on (t_1, t_2) implies that $t_2 - t_1 \leq \pi$, with equality holding if and only if $\varphi(t_1) = \varphi(t_2) = 0$. Moreover, since $|\psi| = \lambda_0 - |\varphi|$, we have $\psi(t) = \pm\lambda_0 \pm \varphi(t)$, i.e. $\psi(t) = \pm\lambda_0 \pm \sqrt{\kappa}\sin(t - t_0)$. If $\varphi(t_1) = \varphi(t_2) = 0$ then necessarily $|\psi(t_1)| = |\psi(t_2)| = \lambda_0$. Finally we show that, *if φ vanishes at one of the endpoints of $[t_1, t_2]$, then (i) $\lambda_0 > \sqrt{\kappa}\sin(t_2 - t_1)$, and (ii) if $t_2 - t_1 > \frac{\pi}{2}$ then $\lambda_0 > \sqrt{\kappa}$.* (Proof: assume $\psi(t) > 0$ on $[t_1, t_2]$, and let $\varphi(t_1) = 0$. Then $\psi(t_1) = \lambda_0$. Also, $\varphi(t_1 + s) = \pm\sqrt{\kappa}\sin s$ for $0 \leq s \leq t_2 - t_1$, so $\psi(t_1 + s) = \lambda_0 \pm \sqrt{\kappa}\sin s$. Since $|\psi| + |\varphi| = \lambda_0$ so that $|\psi| \leq \lambda_0$, the sign must actually be a minus. Since ψ does not vanish on $[t_1, t_2]$, the equation $\sqrt{\kappa}\sin s = \lambda_0$ cannot have a solution in $[0, t_2 - t_1]$. This implies that $\lambda_0 > \sqrt{\kappa}\sin(t_2 - t_1)$, and also that $\lambda_0 > \sqrt{\kappa}$ if $t_2 - t_1 > \frac{\pi}{2}$. The proofs when $\psi < 0$, and for the endpoint t_2, are similar.)

Next consider Case C2. We first show that *ω cannot go through the φ axis, except possibly at time s.* To see this, recall that $\dot{\psi} = -u\chi$ so, if our assertion was not true, the function $\dot{\psi}$ would have a zero in $(t_1, s) \cup (s, t_2)$. If $\dot{\psi}$ has a zero on (t_1, s) and ψ is positive there, then ψ has to be convex on (t_1, s) and so this zero would be a minimum of ψ on (t_1, s), which would make it impossible for $\psi(s)$ to vanish. The other three possibilities —i.e. the case when $\psi < 0$ on (t_1, s), and the two cases when the zero of $\dot{\psi}$ occurs on (s, t_2)— are excluded by similar considerations. Since there are no u-switchings on $[t_1, t_2]$ except possibly at the endpoints, we conclude that that ω does not hit the χ axis on (t_1, t_2). In particular, in each of the intervals (t_1, s), (s, t_2), ω rotates with angular velocity one without changing the sense of rotation, and does not meet any of the two axes. Therefore these two intervals have length $\leq \frac{\pi}{2}$. If $\varphi(t_1) = \varphi(t_2) = 0$, then ω starts at time t_1 on the χ axis, rotates clockwise (if $v = 1$) or counterclockwise (if $v = -1$) until time s, and then reverses the sense of rotation until it hits the χ axis again at time t_2. This means that $s - t_1 = t_2 - s$. Let us use ρ to denote this

common value. Then we know that $\rho \leq \frac{\pi}{2}$. Moreover, it is clear that ρ is the angle of $\omega(s)$ with the χ axis, so $|\chi(s)| = \sqrt{\kappa}\cos\rho$ and $|\varphi(s)| = \sqrt{\kappa}\sin\rho$. Finally, the equations $\dot{\psi} = -u\chi$ and $\dot{\varphi} = v\chi$ hold on each of the intervals $[t_1, s]$, $[s, t_2]$. Since u and v are constant on this interval, and $\varphi(t_1) = \varphi(t_2) = \psi(s) = 0$, it follows that $|\psi(t_1)| = |\int_{t_1}^{s} \chi(r)dr| = |\varphi(s)|$, and similarly $|\psi(t_2)| = |\varphi(s)|$. So $|\psi(t_1)| = |\psi(t_2)| = \sqrt{\kappa}\sin\rho = \lambda_0$.

Now consider the remaining possibilities for case C2, that is, assume it is not true that $\varphi(t_1) = \varphi(t_2) = 0$. If, say, $\varphi(t_1) = 0$ but $\varphi(t_2) \neq 0$ (so that $t_2 = b$), then it is still true that ω starts at t_1 on the χ axis, rotates an angle $\rho = s - t_1$, and then rotates back. Since we are now assuming that φ has no zeros in $[s, t_2]$, this means that ω does not get back to the χ axis, so $t_2 - s < \rho$. Moreover, the identity $\lambda_0 = |\psi(t_1)| = \sqrt{\kappa}\sin\rho$ still holds. Similarly, if $\varphi(t_2) = 0$ but $\varphi(t_1) \neq 0$ (so that $t_1 = a$), then we can define ρ by $\rho = t_2 - s$, and conclude that $s - t_1 < \rho$ and $\lambda_0 = |\psi(t_2)| = \sqrt{\kappa}\sin\rho$.

Next we consider Case C3. Now the vector ω does not move at all for $s_1 \leq t \leq s_2$, and the equation $\dot{\psi} = -u\chi$, together with $u \neq 0$, implies that $\chi = 0$ on $[s_1, s_2]$. This means that $\omega(s)$ lies on the φ axis for $s \in [s_1, s_2]$. Since ω rotates without changing the sense of rotation and without meeting the χ axis on each of the intervals (t_1, s_1) (s_2, t_2), it follows that the length of each of these intervals is $\leq \frac{\pi}{2}$, and that (i) $\varphi(t_1) = 0$ if and only if $s_1 - t_1 = \frac{\pi}{2}$, and in that case $\lambda_0 = |\psi(t_1)| = \sqrt{\kappa}\sin\rho$, where $\rho = \frac{\pi}{2}$ (so that in fact $\lambda_0 = \sqrt{\kappa}$), (ii) $\varphi(t_2) = 0$ if and only if $t_2 - s_2 = \frac{\pi}{2}$, and in that case $\lambda_0 = |\psi(t_2)| = \sqrt{\kappa}\sin\rho$, where $\rho = \frac{\pi}{2}$ (so that $\lambda_0 = \sqrt{\kappa}$).

Finally, we consider Case C4. The point here is that, since ψ has a local extremum at s, $\dot{\psi}$ is continuous and equal to $-u\chi$, and u is a nonzero constant, it follows that $\chi(s) = 0$. Using this, the rest of the analysis is exactly as in Case C3, and we conclude that the intervals $[t_1, s]$ and $[s, t_2]$ have length $\leq \frac{\pi}{2}$, and (i) and (ii) of C3 hold.

We summarize these conclusions in the following

Lemma 21 *Let $(\gamma, \lambda) : [a, b] \to M \times \mathbb{R}^3$ be a Type T1 extremal of the CRS problem, and let φ, ψ, χ be the corresponding switching functions. Let $I = [t_1, t_2] \subseteq [a, b]$ be an MDLI, and use γ^* to denote the restriction of γ to I. Then one of the following possibilities occurs:*

1. *The B case:*

   (a) *γ^* is B_α with $0 < \alpha \leq \pi$,*

   (b) *$\alpha = \pi$ if and only if $\varphi(t_1) = \varphi(t_2) = 0$,*

   (c) *if τ is an endpoint of $[t_1, t_2]$ such that $\varphi(\tau) = 0$, then $|\psi(\tau)| = \lambda_0 > \sqrt{\kappa}\sin\tilde{\alpha}$, where $\tilde{\alpha} = \min(\alpha, \frac{\pi}{2})$.*

2. *The BB case;*

   (a) *γ^* is $B_\alpha v B_\beta$,*

(b) $0 < \alpha \leq \frac{\pi}{2}$ *and* $0 < \beta \leq \frac{\pi}{2}$,

(c) if $\varphi(t_1) = 0$ *then* $\alpha \geq \beta$ *and* $|\psi(t_1)| = \lambda_0 = \sqrt{\kappa}\sin\alpha$,

(d) if $\varphi(t_2) = 0$ *then* $\alpha \leq \beta$ *and* $|\psi(t_2)| = \lambda_0 = \sqrt{\kappa}\sin\beta$,

3. The BSB case:

(a) γ^* *is* $B_\alpha v S_\sigma v B_\beta$ *with* $0 \leq \alpha \leq \frac{\pi}{2}$, $0 \leq \beta \leq \frac{\pi}{2}$, *and* $0 \leq \sigma$,

(b) $\varphi(t_1) = 0$ *if and only if* $\alpha = \frac{\pi}{2}$, *and in that case* $|\psi(t_1)| = \lambda_0 = \sqrt{\kappa} = \sqrt{\kappa}\sin\alpha$,

(c) $\varphi(t_2) = 0$ *if and only if* $\beta = \frac{\pi}{2}$, *and in that case* $|\psi(t_2)| = \lambda_0 = \sqrt{\kappa} = \sqrt{\kappa}\sin\beta$. ∎

Remark 14 Notice that we have subsumed Case C4 into the third possibility, allowing σ to be equal to zero. ∎

Next we study how different Dubins-like intervals are put together. That is, we study a CRS-optimal Type T1 extremal (γ, λ), defined on an interval $[a, b]$, that has at least one u-switching. (If it has no u-switchings, then the structure of γ is completely described by Lemma 21.)

First recall that the function $|\psi|$ has the value λ_0 at all the u-switching times. Lemma 21 implies that, if an interior MDLI is in the B case, then the length of this interval must be equal to π, and then $\lambda_0 > \sqrt{\kappa}$. But then our trajectory cannot contain any BB or BSB MDLI's. (Otherwise, one of the endpoints of at least one of these intervals would have to be a u-switching, which would imply $\lambda_0 \leq \sqrt{\kappa}$.) Therefore

if one interior MDLI is in the B case, then $\lambda_0 > \sqrt{\kappa}$ and all the MDLI's are in the B case.

Now suppose that $\lambda_0 > \sqrt{\kappa}$. Then all the MDLI's are in the B case, and all the interior MDLI's have length π. So γ is of the form $B_\alpha u B_\pi u B_\pi \ldots B_\pi u B_\beta$, with $0 \leq \alpha < \pi$ and $0 \leq \beta < \pi$. If at least two pieces occur, and one of them is a B_π, then Lemma 11 applies, and we conclude that γ is not optimal. So γ is $B_\alpha u B_\beta$ with $0 \leq \alpha < \pi$ and $0 \leq \beta < \pi$.

Next we consider the case when $\lambda_0 < \sqrt{\kappa}$. In that case, define β to be the unique number in $(0, \frac{\pi}{2})$ that satisfies $\sqrt{\kappa}\sin\beta = \lambda_0$, so that $|\psi| = \sqrt{\kappa}\sin\beta$ at each of the switching points. Then all the interior MDLI's are in the BB case, and on each of them γ is strict $B_\beta v B_\beta$. If t_1, t_m are, respectively, the first and last u-switchings, then $[a, t_1]$ and $[t_m, b]$ could be in the B or BB cases. If $[a, t_1]$ is in the BB case, then the second B piece must also have length β, and the first B piece will have length $\alpha \leq \beta$. If $[a, t_1]$ is in the B case, then $t_1 - a < \frac{\pi}{2}$, for otherwise Lemma 21 would imply that $|\psi(t_1)| > \sqrt{\kappa}$. But then Lemma 21 also implies that $\lambda_0 = |\psi(t_1)| > \sqrt{\kappa}\sin(t_1 - a)$, so that $t_1 - a < \beta$. So the interval contributes a piece of length $< \beta$. A similar reasoning applies to the other endpoint. So we end up with a trajectory of the

form $B_\alpha(vB_\beta)uB_\beta vB_\beta uB_\beta \dots uB_\beta vB_\beta u(B_\beta v)B_\delta$, where $0 \le \alpha \le \beta$ and $0 \le \delta \le \beta$, and the parentheses indicate that the corresponding pieces may be missing. (But, if the initial vB_β is missing, then $\alpha > 0$, and similarly for the final $B_\beta v$ part.) On the other hand, Lemma 14 implies that, as long as the trajectory contains a piece of the form $B_t uB_\beta vB_\beta uB_\beta$ with $t > 0$, then it is not optimal. This only leaves the possibilities:

1. $B_\alpha uB_\beta vB_\beta uB_\delta$ with $0 < \alpha \le \beta$, $0 < \delta < \beta$;

2. $B_\alpha vB_\beta uB_\beta vB_\beta uB_\delta$ with $0 \le \alpha \le \beta$, $0 < \delta < \beta$.

The second possibility is ruled out because in that case, if we run γ backwards, we get a $B_\delta uB_\beta vB_\beta uB_\beta vB_\alpha$ piece, which is not optimal because of Lemma 14. A similar argument rules out the first possibility if $\alpha = \beta$, so we are only left with the case $B_\alpha uB_\beta vB_\beta uB_\delta$ with $0 < \alpha < \beta$, $0 < \delta < \beta$.

The above analysis of the $\lambda_0 < \sqrt{\kappa}$ case is valid as long as there is an interior MDLI. We still have to take care of the possibility that γ has exactly one u-switching $\bar{t}$. In that case, the part to the left of $\bar{t}$ can be B_α or $B_\alpha vB_\beta$, and the one to the right can be B_δ or $B_\beta vB_\delta$, with $0 < \beta < \frac{\pi}{2}$, $0 \le \alpha \le \beta$, $0 \le \delta \le \beta$. This gives a total of four possibilities for γ, of which the first one (i.e. $B_\alpha B_\delta$) will be omitted because it has already been encountered before. The remaining three possibilities are: $B_\alpha vB_\beta uB_\delta$, $B_\alpha uB_\beta vB_\delta$, $B_\alpha vB_\beta uB_\beta vB_\delta$, always with $0 < \beta < \frac{\pi}{2}$, $0 \le \alpha \le \beta$, $0 \le \delta \le \beta$.

Finally, we look at the case when $\lambda_0 = \sqrt{\kappa}$. In this case the number β is equal to $\frac{\pi}{2}$. Each of the interior MDLI's I_j contributes a $B_{\frac{\pi}{2}} vS_{\sigma_j} vB_{\frac{\pi}{2}}$ piece, where σ_j is in principle arbitrary, and is allowed to vanish (i.e. the singular piece need not be there, and in that case v may or may not jump). If there are two consecutive MDLI's, then they contribute a $B_{\frac{\pi}{2}} vS_{\sigma_1} vB_{\frac{\pi}{2}} uB_{\frac{\pi}{2}} vS_{\sigma_2} vB_{\frac{\pi}{2}}$ piece $\hat{\gamma}$. This possibility is excluded as follows. Assume without loss of generality that the first B is a B^+, so $\hat{\gamma}$ is in fact $B^+_{\frac{\pi}{2}} vS^+_{\sigma_1} vB^+_{\frac{\pi}{2}} uB^-_{\frac{\pi}{2}} vS^-_{\sigma_2} vB^-_{\frac{\pi}{2}}$. If the last two B's are L's, then $\hat{\gamma}$ is $B^+_{\frac{\pi}{2}} vS^+_{\sigma_1} vL^+_{\frac{\pi}{2}} uL^-_{\frac{\pi}{2}} vS^-_{\sigma_2} vL^-_{\frac{\pi}{2}}$, so it contains an $L^+_t uL^-_{\frac{\pi}{2}} vS^-_{\sigma_2} vL^-_{\frac{\pi}{2}}$ piece with $t > 0$, and therefore it is not optimal by Lemma 9. The case when the last two B's are R's is excluded in a similar way. If the last B is an R but the previous one is an L, then $\hat{\gamma}$ is $B^+_{\frac{\pi}{2}} vS^+_{\sigma_1} vL^+_{\frac{\pi}{2}} uL^-_{\frac{\pi}{2}} vS^-_{\sigma_2} vR^-_{\frac{\pi}{2}}$, so it contains an $L^+_{\frac{\pi}{2}} uL^-_{\frac{\pi}{2}} vS^-_{\sigma_2} vR^-_{\frac{\pi}{2}}$ piece $\check{\gamma}$. Then $\check{\gamma}$ run backwards is $L^+_{\frac{\pi}{2}} S^+_{\sigma_2} R^+_{\frac{\pi}{2}} R^-_{\frac{\pi}{2}}$, and this is not optimal by Lemma 10.

So we cannot have more than one interior MDLI. Suppose that one such interval I exists, and let $I = [t_1, t_2]$. Then $[a, t_1]$ is an interval where γ is either B or BB or BSB. The former case either corresponds to what we called "the B case," or to the BSB case with S absent. In the B case we know that the length $t_1 - a$ has to be $< \frac{\pi}{2}$, for otherwise the inequality $\lambda_0 > \sqrt{\kappa} \min(\frac{\pi}{2}, t_1 - a)$ would imply $\lambda_0 > \sqrt{\kappa}$. If γ is B on $[a, t_1]$ because it falls in the "BSB case with S absent," then in particular there is a $B_{\frac{\pi}{2}}$ piece immediately before the switching. The same happens if γ is BB or BSB. So we see that either there is a $B_{\frac{\pi}{2}}$ piece before

the switching, or the part before the switching is B_α with $\alpha < \frac{\pi}{2}$. In the former case, the argument we used to exclude two MDLI's works as well, since that argument only depended on the fact that there was a whole $B_{\frac{\pi}{2}}$ piece before the switching, and not on what came before. So in fact the only possibility left for $[a, t_1]$ is B_α with $\alpha < \frac{\pi}{2}$. Similarly, the only possibility left for the piece after the last switching is B_β with $\beta < \frac{\pi}{2}$. So we end up with a structure $B_\alpha u B_{\frac{\pi}{2}} S_\sigma B_{\frac{\pi}{2}} B_\beta$, with $0 \leq \alpha < \frac{\pi}{2}$, $0 \leq \beta < \frac{\pi}{2}$, $0 \leq \sigma$. Moreover, in this case we can still exclude one possibility: if the middle B's are both L's or both R's, and α or β is $\neq 0$, then γ cannot be optimal either. (Proof: assume without loss of generality that the middle B's are L^-'s. Then γ is $L^+_\alpha L^-_{\frac{\pi}{2}} S^-_\sigma L^-_{\frac{\pi}{2}} L^+_\beta$. If $\alpha > 0$ then the $L^+_\alpha L^-_{\frac{\pi}{2}} S_\sigma L^-_{\frac{\pi}{2}}$ piece is not optimal by Lemma 9. If $\beta > 0$ then γ run backwards is $R^+_\beta R^-_{\frac{\pi}{2}} S^-_\sigma R^-_{\frac{\pi}{2}} R^+_\alpha$, and Lemma 9 also applies to exclude optimality.)

Finally, we have to consider the case when there is exactly one u- switching. In this case, the part before the switching may be a B_α, or a $B_\alpha S_\sigma B_{\frac{\pi}{2}}$ with $0 \leq \sigma$, $0 \leq \alpha \leq \frac{\pi}{2}$. Similarly, there are two possibilities for the part after the switching. So we get a total of four possibilities, of which the first one ($B_\alpha u B_\beta$ with $0 \leq \alpha \leq \frac{\pi}{2}$, $0 \leq \beta \leq \frac{\pi}{2}$) is not new, since it has already been encountered in our analysis of the $\lambda_0 < \sqrt{\kappa}$ case. The remaining three possibilities are: $B_\alpha u B_{\frac{\pi}{2}} S \sigma B_\beta$, $B_\alpha S_\sigma B_{\frac{\pi}{2}} u B_\beta$, $B_\alpha S_{\sigma_1} B_{\frac{\pi}{2}} u B_{\frac{\pi}{2}} S \sigma_2 B_\beta$ with $0 \leq \alpha \leq \frac{\pi}{2}$, $0 \leq \beta \leq \frac{\pi}{2}$, $0 \leq \sigma$, $0 \leq \sigma_1$, $0 \leq \sigma_2$.

We have now completed our analysis of the possible structures of optimal Type T1 extremals of CRS, and have proved the following

Theorem 8 *Let γ be a time-optimal trajectory of the CRS problem that, for some choice of a nontrivial minimizing adjoint vector, is a Type 1 extremal. Then one of the following seven possibilities holds:*

1. *γ is $B_\alpha u B_\beta$ where $0 \leq \alpha \leq \pi$, $0 \leq \beta \leq \pi$, $\alpha = 0$ if $\beta = \pi$, $\beta = 0$ if $\alpha = \pi$,*

2. *γ is $B_\alpha v S_\sigma v B_\beta$ where $0 \leq \alpha \leq \frac{\pi}{2}$, $0 \leq \beta \leq \frac{\pi}{2}$, and $0 \leq \sigma$,*

3. *γ is $B_\alpha B_\beta B_\delta$, where $0 \leq \alpha \leq \beta$, $0 \leq \delta \leq \beta$, $\beta \leq \frac{\pi}{2}$, one of the switchings is a v and the other one is a u,*

4. *γ is $B_\alpha B_\beta B_\beta B_\delta$, where $0 \leq \alpha < \beta$, $0 \leq \delta < \beta$, $\beta \leq \frac{\pi}{2}$, and the switchings alternate between u and v,*

5. *γ is $B_\alpha u B_{\frac{\pi}{2}} S_\sigma B_{\frac{\pi}{2}} u B_\beta$, where $0 \leq \alpha < \frac{\pi}{2}$, $0 \leq \beta < \frac{\pi}{2}$, $0 \leq \sigma$, and the two middle B's are different, i.e. one is an L and the other one is an R,*

6. *γ is $B_\alpha u B_{\frac{\pi}{2}} S \sigma B_\beta$ or $B_\alpha S_\sigma B_{\frac{\pi}{2}} u B_\beta$, where $0 \leq \alpha \leq \frac{\pi}{2}$, $0 \leq \beta \leq \frac{\pi}{2}$, $0 \leq \sigma$, $\alpha + \beta < \pi$, and, if the two B's adjacent to the singular piece are the same (i.e. both L's or both R's), and are both of length $\frac{\pi}{2}$, then the remaining B is not present),*

7. γ *is* $B_\alpha S_{\sigma_1} B_{\frac{\pi}{2}} u B_{\frac{\pi}{2}} S \sigma_2 B_\beta$ *, where* $0 \leq \alpha < \frac{\pi}{2}$, $0 \leq \beta < \frac{\pi}{2}$, $0 \leq \sigma$, $0 \leq \sigma_1$, $0 \leq \sigma_2$.

Most of the information in the above result is contained in the following more compact statement:

Theorem 9 *Every Type 1 optimal extremal is a concatenation of at most six B or S pieces, of which at most four are B's and at most two are S's. The possibilities with one S are S, BS, SB, BBS, SBB, BBBS, BBSB, BSBB, SBBB, BBBSB, BBSBB, BSBBB (that is, all possibilities except for BBBBS and SBBBB). The possibilities with two S's are SBBS, SBBSB, BSBBS and BSBBSB. In all the cases containing one or two S pieces, all the B's except possibly the extreme ones have length* $\frac{\pi}{2}$*, and all the other B's have length* $\leq \frac{\pi}{2}$*, except that the first B of a BBSB and the last of a BSBB are allowed to have length* $\leq \pi$*. When no S's are present, all the B parts have length* $\leq \pi$*, and the length is in fact always* $\leq \frac{\pi}{2}$ *in all cases except B or BuB. In all cases, the jumps alternate between u and v, except that, of course, the jumps on both sides of an S are v's. In the BBBB case the lengths of the two middle B's are equal. In the BBB and BBBB cases, the length* β *of the inner B's (which is the same for both of them in the BBBB case) is* $\leq \frac{\pi}{2}$ *and the lengths of the other two B's are* $\leq \beta$*.* ∎

Remark 15 It follows in particular from the above result that *every optimal Type T1 extremal of CRS has at most two u-switchings.* (Two u-switchings can occur in the $BBBB$ case, where we can have a $B_\alpha u B_\beta v B_\beta u B_\delta$ trajectory.) ∎

17 Sufficient families for the RS problem.

The concept of a *sufficient family for optimality* was defined in Section 2. We now exhibit two sufficient families for the Reedds-Shepp problem.

We let $\hat{\mathcal{F}}_1$ be the family of all trajectories that satisfy one of the seven conditions of Theorem 8.

Also, we let $\mathcal{F}_2$ be the family of all trajectories of the form $BuBuB$, with each B part having length $\leq \pi$. Recall that we have shown that every time-optimal Type T2 trajectory can be replaced by a trajectory in $\mathcal{F}_2$ without changing the time or the initial and terminal points.

If we then let $\hat{\mathcal{F}} = \hat{\mathcal{F}}_1 \cup \mathcal{F}_2$, then Theorem 8 together with the previous observation imply:

Theorem 10 *The family* $\hat{\mathcal{F}}$ *is sufficient for time optimality for the CRS problem.* ∎

Actually, Theorem 10 can be improved by making one extra choice. Indeed, while almost all the families of trajectories listed in Theorem 8 depend on three parameters, the last one

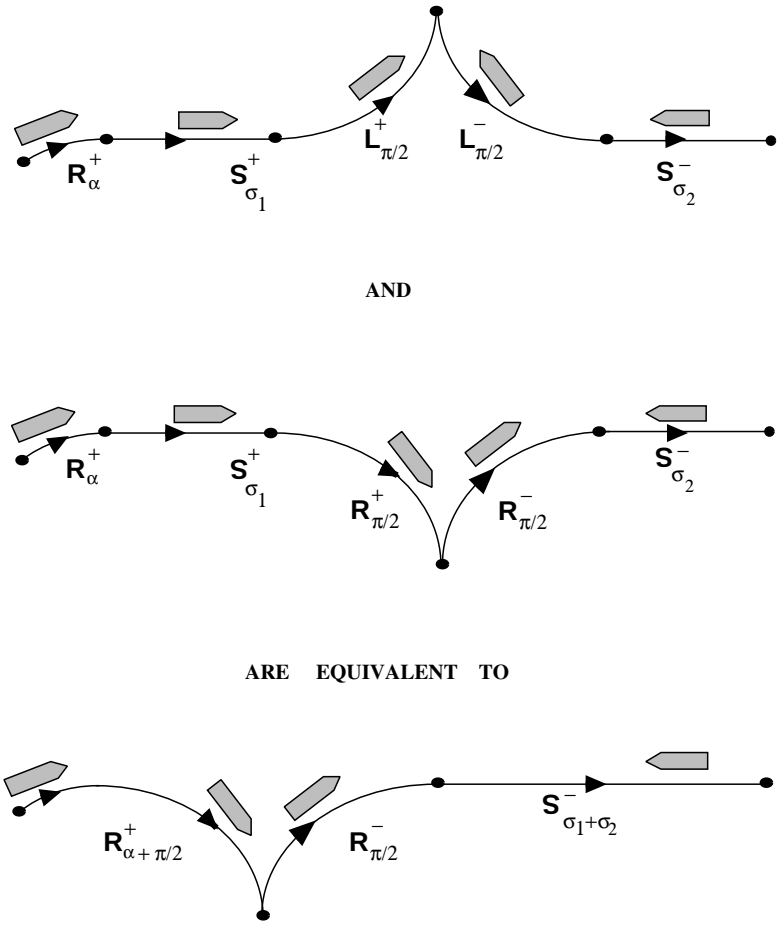

Figure 18: Two singular arcs can always be replaced by one.

constitutes an exception, and involves four parameters. We can eliminate one parameter, and be left with only one S piece, by observing that a piece of the form $S_{\sigma_1} B_{\frac{\pi}{2}} u B_{\frac{\pi}{2}} S_{\sigma_2}$ is equivalent to a piece $B_{\frac{\pi}{2}} u B_{\frac{\pi}{2}} S_{s_1+\sigma_2}$. (This is geometrically obvious, as shown in Figure 18, and can easily be verified analytically.) So in fact Item 7 in the list of Theorem 8 can be replaced by $B_\alpha B_{\frac{\pi}{2}} u B_{\frac{\pi}{2}} S_\sigma B_\beta$. Then the two middle B's have to be the same (i.e both L's or both R's), and we can *choose* them to be either L's or R's. So in particular we can choose them to match the B_α piece, and then we end up with a $BBSB$ trajectory or, more precisely, a $B_{\alpha'} u B_{\frac{\pi}{2}} S_\sigma B_\beta$ trajectory, where $\alpha' = \alpha + \frac{\pi}{2}$. Such a trajectory is almost in Item 6 of our list, the only difference being that in Item 6 the α of $B_\alpha u B_{\frac{\pi}{2}} S \sigma B_\beta$ and the β of $B_\alpha S_\sigma B_{\frac{\pi}{2}} u B_\beta$ are required to be $\leq \frac{\pi}{2}$, whereas now we need them to be $\geq \frac{\pi}{2}$ but $< \pi$. (The case when they are equal to π is easily excluded using Lemma 11.)

So, let us now define a family $\mathcal{F}_1$ that consists of all the trajectories γ that satisfy one of the first five conditions in the list of Theorem 8, or the following

6' γ is $B_\alpha u B_{\frac{\pi}{2}} S_\sigma B_\beta$ with $0 \leq \alpha \leq \pi$, $0 \leq \beta \leq \frac{\pi}{2}$, $0 \leq \sigma$, or $B_\alpha S_\sigma B_{\frac{\pi}{2}} u B_\beta$, where $0 \leq \alpha \leq \frac{\pi}{2}$,

$0 \leq \beta \leq \pi$, $0 \leq \sigma$.

We then define $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$. Then

Theorem 11 *The family $\mathcal{F}$ is sufficient for time-optimality for the CRS problem. In particular, for every pair of points p, q of M, there exists a time-optimal trajectory γ from p to q which is a concatenation of at most five pieces, each of which is either B or S, with at most four B's and at most one S.* ∎

If follows by direct inspection that all the trajectories in $\mathcal{F}$ are actually admissible for the Reeds-Shepp problem. So in fact, we have proved:

Theorem 12 *For the Reeds-Shepp problem, given any two points p, q of M, there exists a time-optimal trajectory from p to q that belongs to $\mathcal{F}$.* ∎

Remark 16 The various trajectory types that define $\mathcal{F}$ involve three continuous parameters. (The exception is the one in Item 1 of Theorem 8, which involves only two. However, this set of trajectories is already contained in $\mathcal{F}_2$, so we will not consider it separately.) Each type, however, depends as well on a discrete parameter, namely, the choice of which B's are L's or R's, whether the car is going forwards or backwards, and which controls switch. For example, BBB gives rise to three possible switching sequences, namely, uu, uv, and vu. (We do not include vv because it does not occur in the list of Theorem 8. We do include uu because it appears in $\mathcal{F}_2$.) Also, the first B can be L or R, and then the remaining B's are determined by the switching structure. (Example: if the first B is an L and the switching structure is uv, then the trajectory has to be LLR.) Finally, the car can start going forwards or backwards. (In the uv LLR case, this would give the possibilities $L^+L^-R^-$ and $L^-L^+R^+$.) So we have a total of $3 \times 2 \times 2 = 12$ choices. Similar counting is possible for the other cases. (For the singular cases, there are extra choices. For instance, BSB can be LSL or LSR or RSL or RSR.) The final result of the counting is that $\mathcal{F}$ gives rise to 48 three-parameter families of trajectories. This is exactly the number of families obtained by Reeds and Shepp. ∎

Remark 17 The number 48 of the previous remark can actually be lowered to 46. Indeed, four of the 48 trajectory types come from the $BuBuB$ combination, which occurs in $\mathcal{F}_2$, i.e. as replacements for general Type T2 trajectories. However, the result we showed in Lemma 20 is actually stronger, and says that a general trajectory along which $v \equiv 1$ or $v \equiv -1$ can always be replaced by a $BuBuB$ trajectory of the special form $L^+L^-L^+$ or $R^+R^-R^+$, so that the combinations $L^-L^+L^-$ and $R^-R^+R^-$ are actually not needed. ∎

Appendix

A The structure of time-optimal trajectories for the Dubins problem

We now study the structure of time-optimal trajectories for the Dubins' problem DU. Recall that the dynamical law for DU is given by

$$DU: \quad \dot{x} = f(x) + vg(x), \quad x \in M, \ \ v \in [-1, 1]. \tag{48}$$

In Theorem 7 we established the existence of time-optimal trajectories joining any two given points in M. The following theorem shows that the time-optimal trajectories of DU are of very special form, and slightly improves upon the result obtained by Dubins in [19].

Theorem 13 *Every time-optimal trajectory for DU is of the form BBB or BSB. If it is of the form BBB, and a, b, c are the times along the first, second, and third bang arc respectively, then necessarily the following three conditions hold: (i) $\pi < b < 2\pi$, (ii) $\min\{a, c\} < b - \pi$, and (iii) $\max\{a, c\} < b$.*

We split the proof of Theorem 13 into the following five lemmas.

Lemma 22 *An optimal trajectory of DU is either of the form BSB or regular bang-bang. In the latter case the times along all the interior bang arcs are equal and in $[\pi, 2\pi)$.*

Proof. Let γ be an optimal trajectory of DU with $\mathrm{Dom}\,(\gamma) = [a, b]$. Let $v(\cdot) : [a, b] \to [-1, 1]$ be the corresponding control. The optimality of γ implies that it has a nontrivial minimizing adjoint vector $\lambda(\cdot)$, i.e. there exists a constant $\lambda_0 \geq 0$ such that

$$\begin{aligned} -\lambda_0 &= \langle \lambda(t), f(\gamma(t)) \rangle + v(t) \langle \lambda(t), g(\gamma(t)) \rangle \\ &= \min\{\langle \lambda(t), f(\gamma(t)) \rangle + w \langle \lambda(t), g(\gamma(t)) \rangle : w \in [-1, 1]\} \end{aligned} \tag{49}$$

holds for almost all $t \in [a, b]$. Let φ, ψ, and χ be defined as in (21) and (25). Then the Switching Structure Equations (26) yield

$$\dot{\varphi} = v\chi, \quad \dot{\psi} = -\chi, \quad \text{and} \quad \dot{\chi} = -v\varphi.$$

This implies that $\dot{\psi}$ is continuous on $[a, b]$.

Let $I_+ = \{t \in [a, b] : \psi(t) > 0\}$ and $I_- = \{t \in [a, b] : \psi(t) < 0\}$. Then I_+ and I_- are relatively open in $[a, b]$. Let $\mathcal{F}_+$ (resp. $\mathcal{F}_-$) denote the set of connected components of I_+ (resp. I_-), so that each element of $\mathcal{F}_+$ (resp. $\mathcal{F}_-$) is a relatively open subinterval of $[a, b]$. For any $J \in \mathcal{F}_+$, if $J = (c, d)$ with $a < c < d < b$, then $\psi(c) = \psi(d) = 0$, and $\psi(t) > 0$ for all $t \in J$. So $v(t) = -1$ on $[c, d]$. The equations $\dot{\varphi} = v\chi$ and $\dot{\psi} = -\chi$, together with $v = -1$ on $[c, d]$, give $\dot{\psi}(t) = \dot{\varphi}(t)$. Hence $\varphi(d) - \varphi(c) = \psi(d) - \psi(c) = 0$. If $\varphi \equiv 0$ on

$[c,d]$, since the minimization condition (49) gives $\varphi = -v\psi - \lambda_0$, it follows that $\psi = \lambda_0$ on $[c,d]$. But $\psi(c) = 0$. So $\psi \equiv 0$ on $[c,d]$, contradicting $\psi(t) > 0$ on (c,d). So $\varphi \not\equiv 0$, and this implies that $\kappa = \varphi(t)^2 + \chi(t)^2$ —which is a constant— is positive. On the other hand, as proved in Section 16, the nonzero vector $\omega = (\varphi, \chi)$ —which is of constant length $\sqrt{\kappa}$— is rotating in the (φ,χ)-plane with angular velocity v. Moreover, the rotation is clockwise if $v = 1$ and counterclockwise if $v = -1$. Hence the equality $\varphi(c) = \varphi(d)$ implies that two points $(\varphi(c), \chi(c))$ and $(\varphi(d), \chi(d))$ are symmetric with respect to the φ-axis. The equality $\varphi(t) = \psi(t) - \lambda_0$ together with $\psi(c) = 0$ implies that $\varphi(c) = -\lambda_0 \leq 0$. Since ψ is continuously differentiable on $[a,b]$, $\psi(c) = 0$, and $\psi(t) > 0$ for all $t \in (c,d)$, it follows that $\dot{\psi}(c) \geq 0$. Hence $\chi(c) = -\dot{\psi}(c) \leq 0$. By the symmetry of $(\varphi(c), \chi(c))$ and $(\varphi(d), \chi(d))$ with respect to the φ-axis, we know $\chi(d) \geq 0$. This implies that $d - c \geq \pi$, since the rotation of ω is counterclockwise from time c to time d.

Assume now that one of the endpoints of $[c,d]$, say c, is such that $\dot{\psi}(c) = 0$. Then $\chi(c) = 0$. Hence $(\varphi(c), \chi(c)) = (\varphi(d), \chi(d)) = (-\lambda_0, 0)$. (So $\lambda_0 = \sqrt{\kappa}$.) This means that ω starts at the point $(-\lambda_0, 0)$ at time c, and rotates counterclockwise until it reaches the point $(-\lambda_0, 0)$ again at time d. Thus $d - c = 2\pi$. So γ contains a bang arc of length 2π. Since the vector fields $f + g$ and $f - g$ are periodic with period 2π, we conclude that γ is not time-optimal, contradicting our assumption. So we have proved that (i) $\mathcal{F}_+$ consists of only finitely many intervals, (ii) each interval of $\mathcal{F}_+$ either contains one of the endpoints of $[a,b]$ or has length $\geq \pi$, and (iii) if $(c,d) \in \mathcal{F}_+$, then $\dot{\psi}$ does not vanish at c and d. With the same argument, we can prove that the same conclusions hold for $\mathcal{F}_-$.

Let $\mathcal{F} = \mathcal{F}_+ \cup \mathcal{F}_-$. Then $\mathcal{F}$ is finite. It is clear that, if $\mathcal{F}$ does not contain any interior subinterval of $[a,b]$, then γ is of the form BSB. If $\mathcal{F}$ does contain an interior subinterval of $[a,b]$, then γ does not contain any singular arc. Otherwise γ would contain one of the following strict subarcs: SBB, BBS, or SBS. Since $\dot{\psi}$ vanishes at a switching which is the junction of bang and singular arcs, it follows that $\mathcal{F}$ contains an interior subinterval (t_1, t_2) of $[a,b]$ such that $\dot{\psi}(t_1)\dot{\psi}(t_2) = 0$, contradicting (iii). The contradiction proves our assertion. Let $I = \cup\{J : J \in \mathcal{F}\}$. If $\mathcal{F}$ contains an interior subinterval of $[a,b]$, then $[a,b] - I$ —which is finite since $\mathcal{F}$ is— is the set of the switchings of γ. Hence γ is regular bang-bang if $\mathcal{F}$ contains an interior subinterval of $[a,b]$.

Now we assume that γ is regular bang-bang. It remains to prove that the times along all the interior bang arcs are equal. Assume that $J_1 = (t_1, t_2)$ and $J_2 = (t_2, t_3)$ are two adjacent interior subintervals of $[a,b]$. Since $\psi(t) \neq 0$ on $(t_1, t_2) \cup (t_2, t_3)$, and $\dot{\psi}(t_2) \neq 0$, it follows that $v(s)v(t) = -1$ for almost all $s \in (t_1, t_2)$ and $t \in (t_2, t_3)$. For simplicity, we assume that $v = 1$ on $[t_1, t_2)$ and $v = -1$ on $[t_2, t_3)$. (The proof for the other case is similar.) Then $\psi(t_i) = 0$ for $i = 1, 2, 3$ imply that $\varphi(t_1) = \varphi(t_2) = \varphi(t_3) = -\lambda_0$. This means that ω starts at the point $(\varphi(t_1), \chi(t_1))$ at time t_1, rotates clockwise until time t_2, and then reverses the sense of rotation

until it reaches the point $(\varphi(t_1), \chi(t_1))$ again at time t_3. We then conclude that $t_3 - t_2 = t_2 - t_1$. The proof of the lemma is now complete. ∎

With the same argument as in the proof of Lemma 4, one can show that $v \equiv 0$ for any optimal singular extremal $\Lambda = (\gamma, v(\cdot), \lambda(\cdot), \lambda_0)$ for DU. Thus any optimal singular extremal trajectory of DU is in fact an integral curve of the vector field $S^+ = f$. As before, L^+ and R^+ denote the vector fields $f + g$ and $f - g$. Since we are now studying the Dubins problem, so that $u \equiv 1$, and there are no S^-, L^- or R^- trajectories, we will simply write S, L and R instead of S^+, L^+ or R^+.

The following lemma is an extension of the sublemma stated without proof in [19].

Lemma 23 *Let γ be a strict BBB-trajectory for DU, and let a, b and c be the times along the first, second, and third bang arc, respectively. Assume that γ is time-optimal. Then $b > \pi$ and* $\min\{a, c\} < b - \pi$.

Proof. Assume that γ is $R_a L_b R_c$. (The case when γ is $L_a R_b L_c$ is similar.) It follows from Lemma 22 that γ cannot be time-optimal if $b < \pi$. So $b \geq \pi$. Assume $b = \pi$. Since a part of an optimal trajectory is optimal, we may replace both R parts by an R_α part, with $0 < \alpha \leq \min(a, c)$. But then it follows from Figure 19 that γ can be replaced by a trajectory of the form $SL_{\pi-2\alpha}S$, which is obviously shorter. (Its length is in fact $\pi - 2\alpha + 4\tan\frac{\alpha}{2}$, whereas the length of γ is $\pi + 2\alpha$. Since $\tan\frac{\alpha}{2} < \alpha$ for $0 < \alpha < \frac{\pi}{2}$, the new curve is indeed shorter.)

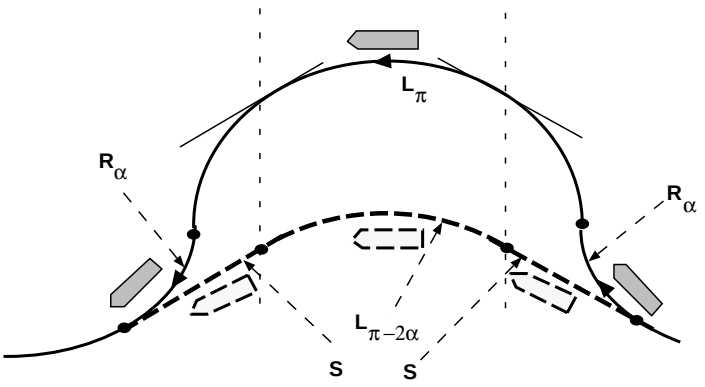

Figure 19: An $R_\alpha L_\pi R_\alpha$ trajectory cannot be optimal.

Now suppose that $b > \pi$ and both a and c are $\geq b - \pi$. Let $b_1 = b - \pi$. Then γ contains an $R_{b_1} L_b R_{b_1}$ piece, and Figure 20 shows that this piece can be replaced by a clearly shorter L_{b_2} piece, where $b_2 = b - 2b_1$. So γ is not optimal. ∎

We now apply the envelope technique of Section 14 to prove the following lemma.

Lemma 24 *Let $a > 0$, $b > 0$. Then a $B_a B_b B_b$-trajectory of DU is not time-optimal.*

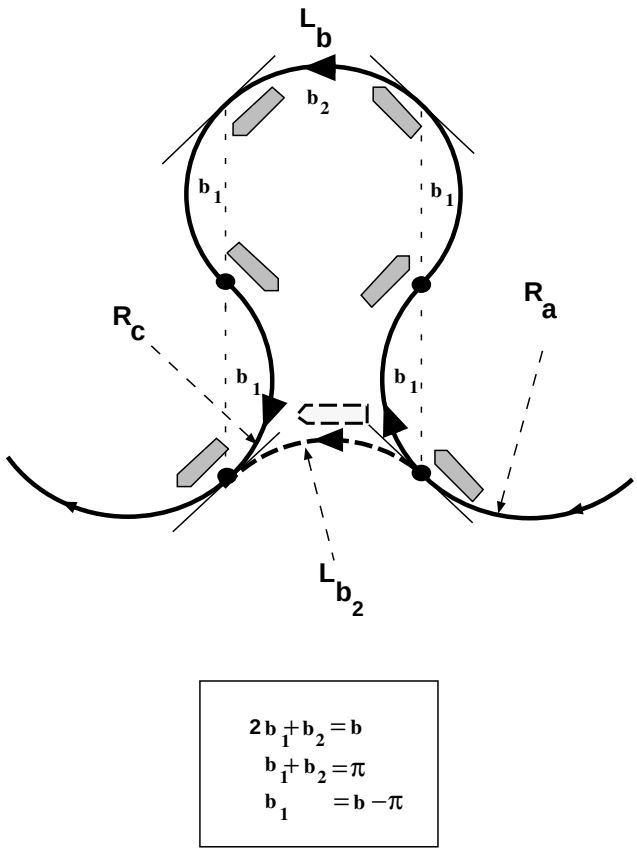

Figure 20: An $R_aL_bR_c$ trajectory with $\min\{a, c\} < b - \pi$ cannot be optimal.

Proof. Let γ be a $B_aB_bB_b$-trajectory. Then γ is either an $L_aR_bL_b$-trajectory or an $R_aL_bR_b$-trajectory. We will assume that γ is $R_aL_bR_b$. (The $L_aR_bL_b$ case is similar.) It follows from Lemma 23 that, if $b \leq \pi$, $b \geq 2\pi$, or $\pi < b < 2\pi$ and $a \geq b - \pi$, then γ is not optimal. So we may assume that $\pi < b < 2\pi$ and $a < b - \pi$.

By changing coordinates if necessary, we may assume that the initial point $\bar{p}$ of γ is $(0, 0, 0)$. Let $\hat{p} = (\hat{x}_1, \hat{x}_2, \hat{\theta})$ be the terminal point of γ. Then

$$\hat{p} = \bar{p}e^{aR}e^{bL}e^{bR},$$

and the right-hand side can be computed explicitly, yielding

$$\hat{p} = \begin{pmatrix} 3\sin a + 2\sin(b-a) \\ -1 + 3\cos a - 2\cos(b-a) \\ -a \end{pmatrix}. \tag{50}$$

We now allow (α, β) to vary in some neighborhood N of (a, b), chosen so that $\pi < \beta < 2\pi$ and $\alpha < \beta - \pi$ for $(\alpha, \beta) \in N$. To each $(\alpha, \beta) \in N$ we associate the trajectory $\gamma_{\alpha,\beta}$ that starts

at $\bar{p}$ and is of type $R_\alpha L_\beta R_\beta$. We let $q(\alpha,\beta)$ be the terminal point of this trajectory, so that

$$q(\alpha,\beta) = \begin{pmatrix} 3\sin\alpha + 2\sin(\beta-\alpha) \\ -1 + 3\cos\alpha - 2\cos(\beta-\alpha) \\ -\alpha \end{pmatrix}, \tag{51}$$

and $q(a,b) = \hat{p}$. We define $A = \frac{\partial q}{\partial \alpha}$ and $B = \frac{\partial q}{\partial \beta}$. Then

$$A(\alpha,\beta) = \begin{pmatrix} 3\cos\alpha - 2\cos(\beta-\alpha) \\ -3\sin\alpha - 2\sin(\beta-\alpha) \\ -1 \end{pmatrix} \tag{52}$$

and

$$B(\alpha,\beta) = \begin{pmatrix} 2\cos(\beta-\alpha) \\ 2\sin(\beta-\alpha) \\ 0 \end{pmatrix}. \tag{53}$$

If we write $\hat{A} = A(a,b)$, $\hat{B} = B(a,b)$, then it follows easily that $\hat{A}$ and $\hat{B}$ are linearly independent. (Suppose that $\mu\hat{A} + \nu\hat{B} = 0$. Then by equating third components we find $\mu = 0$, so $\nu\hat{B} = 0$. Since $\hat{B} \neq 0$, we have $\nu = 0$.)

Therefore we may assume, by shrinking N even further if necessary, that, as (α,β) varies in N, the points $q(\alpha,\beta)$ describe a surface S. We now try to find a trajectory of DU that passes through the point $\hat{p}$ and is entirely contained in S. We do this by first finding two smooth functions $\mu(\alpha,\beta)$, $\nu(\alpha,\beta)$ such that

$$E(\alpha,\beta) = \mu(\alpha,\beta)A(\alpha,\beta) + \nu(\alpha,\beta)B(\alpha,\beta)$$

is a convex combination of $(f-g)(q(\alpha,\beta))$ and $(f+g)(q(\alpha,\beta))$. In order to do this, we first seek to find two smooth functions μ and ν such that $E(\alpha,\beta)$ is a linear combination of $f(q(\alpha,\beta))$ and $g(q(\alpha,\beta))$.

Since f, g, and h form a basis at every point of M, the vectors $A(\alpha,\beta)$, $B(\alpha,\beta)$ can be expressed as linear combinations of $f(q(\alpha,\beta))$, $g(q(\alpha,\beta))$, and $h(q(\alpha,\beta))$. We then seek a linear combination of A and B such that the coefficient of h vanishes. The inner products of A and B with h are given by

$$\langle A(\alpha,\beta), h(q(\alpha,\beta))\rangle = -2\sin\beta, \tag{54}$$

and

$$\langle B(\alpha,\beta), h(q(\alpha,\beta))\rangle = 2\sin\beta. \tag{55}$$

We want $\langle \mu A + \nu B, h\rangle$ to vanish. So $(\mu(\alpha,\beta) - \nu(\alpha,\beta))\sin\beta$ must vanish. Clearly, the choice $\mu = \nu = 1$ will do. This means that $\tilde{E}(\alpha,\beta) = A(\alpha,\beta) + B(\alpha,\beta)$ is a linear combination of $f(q(\alpha,\beta))$ and $g(q(\alpha,\beta))$. We now find the coefficient of f in $\tilde{E}(\alpha,\beta)$, i.e. the inner product

$\langle \tilde{E}, f\rangle$. Since $\langle A(\alpha,\beta), f(q(\alpha,\beta))\rangle = 3 - 2\cos\beta$ and $\langle B(\alpha,\beta), f(q(\alpha,\beta))\rangle = 2\cos\beta$, we get $\langle \tilde{E}(\alpha,\beta), f(q(\alpha,\beta))\rangle = 3$. So, if instead of $\tilde{E} = A + B$ we define $E = \frac{1}{3}(A+B)$, then the f-component of E is exactly equal to 1. The h-component is 0, and the g-component is $-\frac{1}{3}$. Therefore $E(q) = f(q) - \frac{1}{3}g(q)$ for $q \in S$. Hence every integral curve of E is in fact a trajectory of DU.

We are now ready to construct an envelope δ. Pick a small $\varepsilon > 0$, let I denote the interval $[-\varepsilon, 0]$, and let $\delta : I \to S$ be the integral curve of E such that $\delta(0) = \hat{p}$. Since the curve δ is entirely contained in S, we can write $\delta(s) = q(\alpha(s), \beta(s))$ for all $s \in I$. By shrinking ε if necessary, we may assume that $(\alpha(s), \beta(s)) \in N$ for all $s \in I$. For each $s \in I$, let $\hat{\gamma}_s$ be the curve obtained as follows: first follow the $R_{\alpha(s)}L_{\beta(s)}R_{\beta(s)}$ curve $\gamma_{\alpha(s),\beta(s)}$, which steers $\bar{p}$ to $q(\alpha(s),\beta(s))$, and then follow δ from time s to time 0. Then $\hat{\gamma}_s$ goes from $\bar{p}$ to $\hat{p}$ in time $\tau(s) = \alpha(s) + 2\beta(s) - s$.

If we use prime to denote differentiation with respect to s, we have

$$\delta'(s) = E(\delta(s)) = \frac{1}{3}\Big(A(\alpha(s),\beta(s)) + B(\alpha(s),\beta(s))\Big), \tag{56}$$

whereas, using $\delta(s) = q(\alpha(s),\beta(s))$, the chain rule, and the definition of A and B, we get

$$\delta'(s) = \alpha'(s)A(\alpha(s),\beta(s)) + \beta'(s)B(\alpha(s),\beta(s)). \tag{57}$$

Since A and B are linearly independent, this implies that $\alpha'(s) = \beta'(s) = \frac{1}{3}$. So $\tau'(s) = \frac{1}{3} + \frac{2}{3} - 1 = 0$. Therefore the function $s \to \tau(s)$ is in fact a constant.

In particular, the trajectory $\gamma_{-\varepsilon}$ steers $\bar{p}$ to $\hat{p}$ in exactly the same time as γ does. Thus $\gamma_{-\varepsilon}$ is optimal if γ is. Since δ is a part of $\gamma_{-\varepsilon}$, we conclude in particular that δ is optimal as well. On the other hand, $\delta'(s) = f(\delta(s)) - \frac{1}{3}g(\delta(s))$, i.e. δ is a trajectory of DU corresponding to the singular control $\frac{1}{3}$. But we know that along any optimal trajectory of DU v can only take the values 1, -1, and 0. Hence δ cannot be optimal, and we have reached a contradiction. ∎

Lemma 24 easily implies

Lemma 25 *Let $a > 0$, $b > 0$. Then a $B_bB_bB_a$-trajectory of DU is not time-optimal.*

Proof: For each γ, let γ^{-1} be γ run in reverse (i.e. with both the direction of motion of the car and the car orientation reversed). Then γ is $B_bB_bB_a$if and only if γ^{-1} is $B_aB_bB_b$. Lemma 24, applied to γ^{-1}, yields the desired conclusion. ∎

It then follows that

Lemma 26 *A strict BBBB-trajectory for DU cannot be time-optimal.*

Proof: Let γ be an optimal strict BBBB-trajectory. Let a, b, c, d be the times along the corresponding bang arcs. Then Lemma 22 implies that $b = c$. Hence γ contains a subarc $B_aB_bB_b$, which is not time-optimal because of Lemma 24. ∎

Theorem 13 clearly follows from Lemmas 22, 23, 24, 25 and 26. ∎

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