New Differential Geometric Methods in Nonholonomic Path Finding

HÉCTOR J. SUSSMANN +

Abstract

We outline three approaches for nonholonomic path finding —nilpotent approximation, highly oscillatory inputs and path deformation— that are based on the use of the techniques of modern geometric optimal control theory, as well as a more classical one —optimal control— where differential geometric methods are also beginning to play a significant role.

§1. Introduction

In this note we will outline and compare three approaches that have recently been pursued in our work on path finding for nonholonomic systems, and will suggest adding a fourth one to this list. One of the approaches —optimal control— is old, but has recently attracted renewed interest due to the fact that, by making systematic use of "differential geometric methods," it now appears possible to get explicit solutions of problems that could not be handled by means of the techniques developed in the era of "classical" control theory. Two others —nilpotent approximation and highly oscillatory inputs— have been discussed in our recent joint work with G. Lafferriere and W. Liu. The new addition —which is naturally more speculative than the others— is the *path deformation method*, outlined in §6.

It should be emphasized that this paper is *not* intended to be a survey of the whole subject of nonholonomic path finding. This rich area has produced in recent years many ingenious ideas and methods, that cannot possibly be covered in an article such as this one. (Cf. the work of R. Brockett, e.g. [3], J.-P. Laumond and coworkers, e.g. [5], [9], [10], S. Sastry

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and R. Murray, [12], [13], [14], Fernandes, Gurvits and Li, [4].) We will instead limit our discussion to the four approaches in whose development we have been directly involved. A fuller account, describing these approaches in full generality, with precise statements of technical conditions and rigorous convergence theorems, is given in other technical reports, conference proceedings, and journal articles now in preparation.

§2. Nonholonomic path finding

We consider systems of the form

$$(2.1) \qquad \Sigma: \quad \dot{x} = f_0(x) + u_1 f_m(x) + \ldots + u_m f_m(x)\ , \quad x \in M\ , \quad u \in U\ ,$$

where

1. the *state space* M is a smooth (i.e. C^∞) connected manifold,
2. $f_0, \ldots, f_m$ are smooth vector fields on M,
3. the set U of *admissible control values* is a closed, convex subset of $\mathbb{R}^m$ with nonempty interior. (The most common cases are $U = \mathbb{R}^m$, i.e. a totally unconstrained control, $U = B(m)$, where $B(m) = \{u \in \mathbb{R}^m : u_1^2 + \ldots + u_m^2 \le 1\}$ is the unit ball in $\mathbb{R}^m$, and $U = C(m)$, where $C(m) = \{u \in \mathbb{R}^m : |u_1| \le 1, \ldots, |u_m| \le 1\}$ is the unit cube in $\mathbb{R}^m$.)

By a *control* we mean a Lebesgue integrable U-valued function defined on a compact interval $[a, b]$ of the real line. If $\eta : [a, b] \to \mathbb{R}^m$ is a control, and $t \in [a, b]$, we will use η^t to denote the restriction of η to the interval $[a, t]$. Usually, we are interested in controls that are "nice" in some sense, e.g. smooth, or piecewise smooth, or piecewise constant, so more general highly pathological controls are to be avoided whenever possible. However, there are cases where these pathological controls cannot be avoided, so we choose to allow them in our definition. As usual, a *trajectory* for a control $\eta = (\eta_1, \ldots, \eta_m) : [a, b] \to \mathbb{R}^m$ is an absolutely continuous curve $\gamma : [a, b] \to M$ that satisfies the differential equation

$$(2.2) \qquad \dot{\gamma}(t) = f_0(\gamma(t)) + \eta_1(t) f_1(\gamma(t)) + \ldots + \eta_m(t) f_m(\gamma(t))$$

almost everywhere. Given any initial condition $\gamma(a) = x$, and a control $\eta : [a, b] \to \mathbb{R}^m$, the trajectory γ that corresponds to η and satisfies the given initial condition may fail to exist, but if it exists then it is unique, and in that case we will denote it by γ_x^η. Moreover, in any case there is a $t \in [a, b]$ such that $t > a$ and $\gamma_x^{\eta^t}$ exists. (More precisely, given any x there exists an $\alpha(x) > 0$ such that γ_x^η exists for every a, b, η such that

$\eta \in L^1([a,b],\mathbb{R}^m)$ and $||\eta||_{L^1} < \alpha(x)$. In addition, if $K \subseteq M$ is compact, then $\alpha(x)$ can be chosen independently of x for all $x \in K$.) Under various growth conditions, e.g. if M is compact, or if $M = \mathbb{R}^n$ and the f_i are globally Lipschitz, the trajectory γ_x^η exists for all choices of x and η. If $\eta : [a,b] \to R^m$ is a control, γ_x^η exists, and $\gamma_x^\eta(b) = y$, then we say that η *steers* (or *drives*) x to y in time $b-a$. If there is a control that steers x to y then we say that y is *reachable*, or *accessible* from x.

Roughly, the *path finding problem* is the following: given two points p and q, find a way to steer p to q as nicely as possible. Somewhat less vaguely, we want an *algorithm* that, for every pair (p,q) of points of M (or, at least, for every pair (p,q) in some large subset S), will produce a control $\eta_{p,q}$ that steers p to q. Naturally, the largest set for which such an algorithm might possibly exist is the "reachability set" $\mathcal{R}(\Sigma)$, i.e. the set of all pairs (p,q) such that q is reachable from p. If $\mathcal{R}(\Sigma) = M \times M$, then Σ is said to be *completely controllable* (CC). This is obviously the only case where we can expect to be able to solve the path finding problem by finding algorithms that work for *all* (p,q). Another property, much weaker than controllability but very important in the theory, is *accessibility*: Σ has the *accessibility property* (AP) if for every x the set of points that can be reached from x has a nonempty interior. (This terminology is due to D. Elliott.) For real-analytic systems, accessibility is equivalent to the *Lie algebra rank condition* (LARC), i.e. the condition that $L(\Sigma)(x) = T_xM$ for every $x \in M$, where (i) T_xM denotes the tangent space of M at x, (ii) $L(\Sigma)$ —known as the *accessibility Lie algebra* of Σ— is the Lie algebra of vector fields generated by $f_0, \ldots, f_m$, (iii) $L(\Sigma)(x) = \{X(x) : X \in L(\Sigma)\}$.

The vector field f_0 is the *drift*. When Σ has no drift, i.e. $f_0 = 0$, then the AP and complete controllability are equivalent. If, in addition, Σ is real analytic, then it is CC if and only if it has the LARC. The equivalence is also true in certain cases with drift, e.g. if M is a compact Lie group and Σ is a translation-invariant system, or if the flow of f_0 is periodic. But, in general, the CC property is hard to characterize by means of "algebraic" conditions such as the LARC.

§3. Exact solutions for nilpotent systems

The *nilpotent approximation method* works for systems *without drift.* It is based on a procedure that produces an exact solution for nilpotent systems. In this section we discuss this exact solution, and in the next one we explain how to use it for more general systems.

The procedure for finding exact solutions for nilpotent systems is based on two observations. First, for a *complete nilpotent system*, the trajectory can be computed explicitly in terms of a finite number of iterated integrals of the control. (Recall that Σ is said to be *nilpotent* if the Lie algebra $L(\Sigma)$ is nilpotent. A Lie algebra L is *nilpotent* if there exists a positive integer k such that every bracket $[X_1, [X_2, \ldots, [X_{k-1}, X_k]]]$ of k elements of L vanishes. A system is *complete* if for every control η and every initial condition x the trajectory γ_x^η exists. For driftless nilpotent systems completeness is equivalent to the condition that the vector fields $f_1, \ldots, f_m$ are themselves complete, i.e. all their trajectories are defined for all times $t \in \mathbb{R}$.)

The second observation is that, once the formula for the solution has been expressed in terms of iterated integrals, then it is possible in various ways to invert it and obtain, for any given initial and terminal points p, q, a control for which the iterated integrals have the values needed to steer p to q.

We now explain these ideas in more detail, and begin by developing a formalism (similar to the "chronological calculus" of Agrachev and Gamkrelidze, cf. [1]) that makes the calculations simple. If f is a smooth vector field, let us use exponential notation for the flow of f, and let us write the flow on the right, so that $t \to xe^{tf}$ is the integral curve of f that goes through x at time $t = 0$. Consistently with this, let us also write vector fields on the right, so that xf is an alternative notation for $f(x)$. More generally, use the notation $x\Phi$ as an alternative to $\Phi(x)$ for any map Φ, whether it is a diffeomorphism such as e^{tf} or a vector field such as f. If $\Phi : M \to N$ is a smooth map, v is a tangent vector at x, and $y = \Phi(x)$, then $v\Phi$ will denote the tangent vector at y that is usually denoted by $\Phi_* v$, or $d\Phi(x)(v)$, i.e. the value at v of the differential $d\Phi(x) : T_x M \to T_y N$ of Φ at x. With these notations, it is not hard to prove the formula

$$\frac{d}{dt}(xe^{tg} f e^{-tg}) = xe^{tg}[g, f]e^{-tg} \ , \tag{3.1}$$

where $[g, f]$ is the Lie bracket of f and g. (Notice that, to begin with, the formula makes sense because the function $t \to xe^{tg} f e^{-tg}$ takes values in a fixed linear space, namely $T_x M$, so it is at least meaningful to differentiate with respect to t. Moreover, differentiating formally we get

$$\frac{d}{dt}(xe^{tg} f e^{-tg}) = xe^{tg} g f e^{-tg} - xe^{tg} f g e^{-tg} \ , \tag{3.2}$$

since $\frac{d}{dt} e^{tg} = e^{tg} g$. Since $[g, f] = gf - fg$, the desired identity follows, at least on the formal level. A complete justification requires some more work, in order to assign a precise meaning to terms such as $x\Phi fg\Psi$ when Φ and Ψ are diffeomorphisms and f, g are vector fields, and then justify all the steps in the calculation, but the formalism in which all this works is easy to construct and then all the steps become rigorous.) We remark in particular that, if $[f,g]=0$, then Equation (3.1) implies that $\frac{d}{dt}(xe^{tg}fe^{-tg})=0$, so that $xe^{tg}fe^{-tg}\equiv \text{constant}\equiv xf$, and therefore $xe^{tg}f=xfe^{tg}$.

Using this, let us show how to solve explicitly the equations for a nilpotent system. For simplicity, we will just consider the case of a system

$$\dot{x}=u_1f_1(x)+u_2f_2(x)\ ,\ \ x\in \mathbb{R}^n\ , \tag{3.3}$$

where f_1 and f_2 are smooth vector fields such that

$$[f_1,[f_1,f_2]]\equiv [f_2,[f_1,f_2]]\equiv 0\ . \tag{3.4}$$

(The case of a general nilpotent system is of course more complicated, but the idea is the same.) Assume we are given a control $t\rightarrow (u_1(t),u_2(t))$, $0\leq t\leq T$, and an initial condition $x(0)=\bar{x}$. We rewrite the equation as $\dot{x}=x(u_1f_1+u_2f_2)$, and try a solution of the form

$$x(t)=\bar{x}e^{a_1(t)f_1}e^{a_2(t)f_2}e^{a_3(t)f_3}\ , \tag{3.5}$$

where the $a_i(t)$ are functions to be chosen, and f_3 is defined to be $[f_1,f_2]$. Since we want $x(0)$ to be equal to $\bar{x}$, we stipulate that $a_1(0)=a_2(0)=a_3(0)$. Then we can differentiate with respect to t and find that

$$\begin{aligned}\dot{x}(t)=&\frac{d}{dt}\bar{x}e^{a_1f_1}e^{a_2f_2}e^{a_3f_3}\\ =&\quad \bar{x}e^{a_1f_1}\dot{a}_1f_1e^{a_2f_2}e^{a_3f_3}\\ &+\bar{x}e^{a_1f_1}e^{a_2f_2}\dot{a}_2f_2e^{a_3f_3}\\ &+\bar{x}e^{a_1f_1}e^{a_2f_2}e^{a_3f_3}\dot{a}_3f_3\ .\end{aligned} \tag{3.6}$$

Since $[f_2,f_3]=0$, we have $yf_2e^{a_3f_3}=ye^{a_3f_3}f_2$ for every y, so in the second term of the above sum we can change the order of the factors $\dot{a}_2f_2$ and $e^{a_3f_3}$. Changing the order of $\dot{a}_1f_1$ and $e^{a_2f_2}$ is a bit more difficult. We have

$$yf_1e^{a_2f_2}=ye^{a_2f_2}e^{-a_2f_2}f_1e^{a_2f_2}=ze^{-a_2f_2}f_1e^{a_2f_2}\ , \tag{3.7}$$

if we let $z=ye^{a_2f_2}$. For any fixed z, we have

$$\frac{d}{dt}ze^{-a_2f_2}f_1e^{a_2f_2}=-ze^{-a_2f_2}\dot{a}_2[f_2,f_1]e^{a_2f_2}\ . \tag{3.8}$$

Since $[f_2,f_3]=0$, we can commute $\dot{a}_2[f_2,f_1]$ past $e^{a_2f_2}$, and end up with $\frac{d}{dt}ze^{-a_2f_2}f_1e^{a_2f_2}=-z\dot{a}_2[f_2,f_1]$. But this then gives

$$ze^{-a_2f_2}f_1e^{a_2f_2}=z(f_1-a_2[f_2,f_1])\ . \tag{3.9}$$

Letting $z = ye^{a_2 f_2}$ and $y = \bar{x}e^{a_1 f_1}$, we get

$$\bar{x}e^{a_1 f_1}\dot{a}_1 f_1 e^{a_2 f_2} e^{a_3 f_3} = \bar{x}e^{a_1 f_1} e^{a_2 f_2}\dot{a}_1(f_1 - a_2[f_2, f_1])e^{a_3 f_3} \ . \tag{3.10}$$

Since both f_1 and $[f_2, f_1]$ commute with f_3, we conclude that

$$\bar{x}e^{a_1 f_1}\dot{a}_1 f_1 e^{a_2 f_2} e^{a_3 f_3} = \bar{x}e^{a_1 f_1} e^{a_2 f_2} e^{a_3 f_3}\dot{a}_1(f_1 - a_2[f_2, f_1]) \ . \tag{3.11}$$

If we combine all these results we get

$$\dot{x}(t) = x(t)\Big(\dot{a}_1(t)f_1 - \dot{a}_1(t)a_2(t)[f_2, f_1] + \dot{a}_2(t)f_2 + \dot{a}_3(t)f_3\Big) \ . \tag{3.12}$$

To satisfy the equation $\dot{x}(t) = x(t)(u_1(t)f_1 + u_2(t)f_2)$, since $f_3 = [f_1, f_2]$, we must let

$$\dot{a}_1 = u_1 \ , \ \dot{a}_2 = u_2 \ , \ \dot{a}_3 = -u_1 a_2 \ . \tag{3.13}$$

So, if we let

$$\begin{aligned} U_1(t) &= \int_0^t u_1(s)ds \ , \\ U_2(t) &= \int_0^t u_2(s)ds \ , \\ U_3(t) &= -\int_0^t u_1(s)U_2(s)ds \ , \end{aligned} \tag{3.14}$$

we see that we need to choose $a_i(t) = U_i(t)$.

In other words, we have expressed our solution $x(t)$ as

$$x(t) = \bar{x}e^{U_1(t)f_1}e^{U_2(t)f_2}e^{U_3(t)f_3} \ , \tag{3.15}$$

where the U_i are defined by (3.14). We now show how this formula can be "inverted" so that, if we are given $p = \bar{x}$, a time T, and a desired terminal point q, we can find a control that steers p to q in time t. Assume that the system (3.3) satisfies the LARC. This means that at every point x the three vectors $f_i(x)$, $i = 1, 2, 3$, span $\mathbb{R}^n$. In particular, $n \leq 3$. Let $F(x) \in \mathbb{R}^{n\times 3}$ be the matrix whose columns are $f_1(x)$, $f_2(x)$ and $f_3(x)$, so $F(x)$ has rank n for every x. Let $M(x) = F(x)^\dagger(F(x)F(x)^\dagger)^{-1}$, where "$\dagger$" stands for transpose. (So $M(x)$ is a pseudoinverse of $F(x)$. In particular, $F(x)M(x)$ is the $n \times n$ identity matrix.)

Now consider the "extended system"

$$\dot{x} = v_1 f_1(x) + v_2 f_2(x) + v_3 f_3(x) \ . \tag{3.16}$$

For this system it is very easy to find a trajectory from p to q. We just find one path in $\mathbb{R}^n$ that goes from p to q in time T, for instance the segment given by $\sigma(t) = p + \frac{t}{T}(q-p)$. We then realize this path σ as a trajectory of (3.16) by writing $\dot{\sigma}(t)$ as a linear combination

$$\dot{\sigma}(t) = v_1(t) f_1(\sigma(t)) + v_2(t) f_2(\sigma(t)) + v_3(t) f_3(\sigma(t)) \ , \tag{3.17}$$

which is possible because the three vectors $f_i(\sigma(t))$ span $\mathbb{R}^n$. For instance, we can choose $v(t) = M(\sigma(t))\dot{\sigma}(t)$. For this particular choice of v, we now solve Equation (3.16) explicitly in terms of iterated integrals exactly as we did before for Equation (3.3). The only difference is that, in this case, we want $x(t)$ to satisfy $\dot{x}(t) = x(t)(v_1(t)f_1 + v_2(t)f_2 + v_3(t)f_3)$. So we must have

$$\dot{a}_1 = v_1 \ , \ \ \dot{a}_2 = v_2 \ , \ \ \dot{a}_3 = -v_1 a_2 + v_3 \ . \tag{3.18}$$

We then let

$$\begin{aligned} V_1(t) &= \int_0^t v_1(s)ds \ , \\ V_2(t) &= \int_0^t v_2(s)ds \ , \\ V_3(t) &= \int_0^t (-v_1(s)V_2(s) + v_3(s))ds \ , \end{aligned} \tag{3.19}$$

and choose $a_i(t) = V_i(t)$. In particular, we have shown that

$$q = pe^{V_1(T)f_1} e^{V_2(T)f_2} e^{V_3(T)f_3} \ . \tag{3.20}$$

The above computations establish that, given any p, q, one can explicitly find numbers α_i, $i = 1,2,3$, such that $q = pe^{\alpha_1 f_1} e^{\alpha_2 f_2} e^{\alpha_3 f_3}$. The last step is then to show that one can explicitly find, for each triple $(\alpha_1, \alpha_2, \alpha_3)$, a control $u = (u_1, u_2)$, defined on some interval $[0, T]$, for which the three iterated integrals U_i defined above satisfy $U_i(T) = \alpha_i$. This can be done in a number of ways. For instance, we may choose

$$\begin{aligned} u_1(t) =& \frac{\alpha_1}{T} + \beta \sin \frac{2\pi t}{T} \ , \\ u_2(t) =& \frac{\alpha_2}{T} + \gamma \cos \frac{2\pi t}{T} \ , \end{aligned} \tag{3.21}$$

where

$$\beta = \sqrt{\left|\alpha_3 - \frac{\alpha_1\alpha_2}{2}\right|} \quad \text{and} \quad \gamma = \beta \operatorname{sign}\left(\alpha_3 - \frac{\alpha_1\alpha_2}{2}\right). \tag{3.22}$$

Although the algebra becomes much more cumbersome, the idea of the above example generalizes to arbitrary complete nilpotent systems. A more detailed explanation of this is given in [6], [7], [8]. The appropriate mathematical formalism turns out to be the use of *noncommutative power series.* For a general nilpotent system, one can always write the solution explicitly in the form

$$x(t) = x(0)e^{U_1(t)B_1(f)}e^{U_2(t)B_2(f)}\ldots e^{U_r(t)B_r(f)}\ , \tag{3.23}$$

where

a. $B_1, B_2, \ldots, B_r$ are a basis of $L_k(X_1, X_2, \ldots, X_m)$, the free Lie algebra generated by m noncommuting indeterminates $X_1, X_2, \ldots, X_m$,

b. k is chosen so that all the brackets of the f's of degree $> k$ vanish,

c. for a formal bracket B, $B(f)$ is the result of plugging in the f_i for the X_i (so that, for instance, if $B = [X_1, X_2]$, then $B(f) = [f_1, f_2]$),

d. the $U_i(t)$ are certain functionals of the control u.

It turns out that, if the basis $\mathcal{B} = (B_1, B_2, \ldots, B_r)$ is chosen to be a *P. Hall basis* of $L_k(X_1, X_2, \ldots, X_m)$, then the U_i are given by very simple formulae as *iterated integrals* (cf. Sussmann [18]).

One then proceeds in general as we did in the simple case considered here. We find a suitably rich Lie bracket extension of the given system, then pick a (not necessarily admissible) path $\gamma : [0, T] \to M$ that joins p to q, then find a control v that gives rise to γ for the extended system, then express $x(T)$ as in [3.23]. (This turns out to be possible in a simple way even for extended controls.) The last step is then to choose the u_i so that the numbers $U_i(T)$ will be equal to the $V_i(T)$ found by the above procedure. This can be done in several different ways. One such way, involving bang-bang controls, has been proposed in [7] and [8]. Other methods, e.g. involving sinusoids, are proposed by Murray and Sastry in [14] for special classes of nilpotent systems, and can be extended to general nilpotent systems using the formalism of our work with W. Liu. cf. [19], [20], [21].

§4. Nilpotent approximation of general systems

So far, we have only discussed nilpotent systems. It turns out, however, that the method can be used to obtain an algorithm that works for more general systems. Let us explain this by referring once again to the system (3.3), where f_1, f_2 are no longer assumed to satisfy (3.4).

Assume that the three brackets $f_1(x)$, $f_2(x)$, $f_3(x)$ span $\mathbb{R}^n$ for every x. Then we can still join any two points p, q by a path σ, and then compute v from σ as we did above. We can then compute u from v exactly as above. (That is, we define the V_i by (3.19), then we let $\alpha_i = V_i(T)$, and finally choose the u_i according to (3.21).) The only difference, of course, is that now Equations (3.15) and (3.20) do not follow, so we are no longer entitled to conclude that u steers p to q. Let $\tilde{q}$ be the point to which u steers p. One can then prove an estimate $||\tilde{q} - q|| \leq C||p - q||^{3/2}$. (Precisely: given any compact set K there exists a $C > 0$ such that the estimate holds whenever p and q are in K.) In particular, this implies that for every compact K there is a constant Δ such that the bound $||\tilde{q} - q|| \leq \frac{1}{2}||p - q||$ holds if $p \in K$, $q \in K$, and $||p-q|| \leq \Delta$. (The number Δ was called the "'critical distance" in [7] and [8].) If we want to join p to q and $||p - q||$ is already $\leq \Delta$, then we may apply the "nilpotent" algorithm to produce a control that steers p to a point q_1, then apply it again starting at q_1 rather than p (but always with target q) to get a point q_2, and so on. Each time the distance will go down by a factor of at least $1/2$, so that convergence will be very fast.

In the general case, if p and q are far, it suffices to choose a $\hat{\Delta}$ such that $\hat{\Delta} \leq \Delta$, and apply the nilpotent algorithm starting at p but with target q_1 rather than q, where q_1 lies in the straight line segment from p to q, but at a distance $\hat{\Delta}$ from p. The resulting control will steer p to a point p_1 such that $||p_1 - q_1|| \leq \frac{1}{2}\hat{\Delta}$. On the other hand, since

$$||q_1 - q|| = ||p - q|| - \hat{\Delta} \ , \tag{4.1}$$

the triangle inequality implies that

$$||p_1 - q|| \leq ||p - q|| - \frac{1}{2}\hat{\Delta} \ . \tag{4.2}$$

We can then repeat the whole procedure starting at p_1, to produce a point p_2, and then continue to iterate. Each step will decrease the distance to q by at least $\frac{1}{2}\hat{\Delta}$, until a point p_k is attained whose distance to q is $\leq \hat{\Delta}$. From then on we just continue to apply the algorithm with target q, and each step will bring down the distance by a factor $\frac{1}{2}$, as before.

It then follows that, for any prescribed error margin ε, the algorithm described above will produce a point $\hat{q}$ such that $||\hat{q}-q|| \leq \varepsilon$ in at most N steps, where

$$N = 2\left[\frac{||q-p||}{\hat{\Delta}}\right] + \log_2\frac{\hat{\Delta}}{\varepsilon}\ , \tag{4.3}$$

where, as usual, $[x]$ is the largest integer ν such that $\nu \leq x$.

The procedure described above generalizes to arbitrary real analytic completely controllable driftless systems. Naturally, in this more general setting one has to consider a richer "Lie bracket extension" of the original system, and add enough brackets to the right-hand side to span the tangent space at each point. Such an extension always exists locally because of the LARC, and therefore given any compact set K one can find an extension that works for K. In many cases, e.g. for systems in $\mathbb{R}^n$ with polynomial vector fields, there is an extension that works globally. The main difference is that the estimate for the general case is $||\tilde{q}-q|| \leq C||p-q||^{1+\frac{1}{k}}$, where k is the smallest degree such that the Lie brackets up to order k span the whole space. (Cf. [7] and [8] for more details.) If k is large, the exponent is very close to 1, which means that convergence is very slow and Δ very small.

The main drawback of the above method is that it requires knowing a lower bound for Δ. In practice, one can simply try a value for $\hat{\Delta}$ and see if it produces an improvement. If it does not, one may try a smaller value and keep lowering until the algorithm starts producing improvements.

§5. Highly oscillatory inputs

Here again we deal with completely controllable real analytic systems without drift that satisfy the LARC. (The results also apply in some cases where a drift is present, e.g. if the associated driftless system obtained by setting $f_0 = 0$ still satisfies the LARC.)

The idea of this method is to pick a path $\gamma : [0,T] \to M$ such that $\gamma(0) = p$, $\gamma(T) = q$, that need not be admissible, and try to approximate the whole path by a sequence $\{\gamma_j\}$ of admissible paths, also defined on $[0,T]$, and also satisfying $\gamma_j(0) = p$. Exactly as in the previous section, one chooses a Lie bracket extension of the original system, and then expresses the path γ as a trajectory of the extended system corresponding to some "extended input" v. One then tries to find a sequence of "ordinary inputs" u^j whose trajectories converge to those of the "extended input" v. At first it may appear strange that such a convergence is possible, but we will illustrate how this can happen by referring again to the special case of System (3.3) and its extension (3.16). Let η_1, η_2, η_3, η_4 be smooth real-valued functions on the interval $[0,T]$, and consider the controls u^j defined by $u^j = (u_1^j, u_2^j)$, where $u_1^j(t) = \eta_1(t) + \sqrt{j}\,\eta_3(t)\sin jt$, $u_2^j(t) = \eta_2(t) + \sqrt{j}\,\eta_4(t)\sin jt$. Fix an initial point p, and let x^j be the resulting trajectory for the input u^j. (For simplicity, assume that the f_i and all their derivatives are bounded, so there are no explosions for (3.3) or any of its Lie bracket extensions.) We can then write

$$x^j(t) = \xi^j(t) + \zeta_1^j(t) + \zeta_2^j(t) \ , \tag{5.1}$$

where

$$\xi^j(t) = p + \int_0^t \eta_1(s) f_1(x^j(s))ds + \int_0^t \eta_2(s) f_2(x^j(s))ds \ , \tag{5.2}$$

and

$$\zeta_1^j(t) = \sqrt{j}\int_0^t \eta_3(s)\sin js f_1(x^j(s))ds \ , \tag{5.3}$$

$$\zeta_2^j(t) = \sqrt{j}\int_0^t \eta_4(s)\cos js f_2(x^j(s))ds \ , \tag{5.4}$$

Integration by parts yields

$$\begin{aligned}
&\zeta_1^j(t) = \theta_1^j(t) f_1(x^j(t)) \\
&\quad - \int_0^t \theta_1^j(s)\cdot Df_1(x^j(s))\cdot\Big(u_1^j(s)f_1(x^j(s)) + u_2^j(s)f_2(x^j(s))\Big)ds \ , \\
&\zeta_2^j(t) = \theta_2^j(t) f_2(x^j(t)) \\
&\quad - \int_0^t \theta_2^j(s)\cdot Df_2(x^j(s))\cdot\Big(u_1^j(s)f_1(x^j(s)) + u_2^j(s)f_2(x^j(s))\Big)ds \ ,
\end{aligned}$$

where

$$\theta_1^j(t) = \sqrt{j}\int_0^t \eta_3(s)\sin js\,ds \ , \tag{5.5}$$

and

$$\theta_2^j(t) = \sqrt{j}\int_0^t \eta_4(s)\cos js\,ds \ . \tag{5.6}$$

Another integration by parts shows that

$$\theta_1(t) = \frac{1}{\sqrt{j}}(\eta_3(0) - \eta_3(t)\cos jt) + \frac{1}{\sqrt{j}}\int_0^t \eta_3'(s)\cos js\,ds \ . \tag{5.7}$$

This implies in particular that $\theta_1^j(t) = O(\frac{1}{\sqrt{j}})$ uniformly in t, so in particular $\theta_1^j \to 0$ uniformly as $j \to \infty$. Moreover, the integral $\int_0^t \eta_3'(s)\cos js\,ds$ goes to 0 uniformly by the Riemann-Lebesgue Lemma, which implies that

$$\theta_1(t) = \frac{1}{\sqrt{j}}(\eta_3(0) - \eta_3(t)\cos jt) + \frac{1}{\sqrt{j}}\nu_1^j(t) \ , \tag{5.8}$$

where $\nu_1^j(t) \to 0$ uniformly. We can then write

$$\begin{aligned} \zeta_1^j(t) = & \int_0^t \Big(\eta_3(s)\cos js - \eta_3(0)\Big)\eta_3(s)\sin js(Df_1 \cdot f_1)(x^j(s))ds \\ & + \int_0^t \Big(\eta_3(s)\cos js - \eta_3(0)\Big)\eta_4(s)\cos js(Df_1 \cdot f_2)(x^j(s))ds + \mu_1^j(t) \end{aligned} \tag{5.9}$$

where the μ_1^j go to 0 uniformly. A similar calculation yields

$$\begin{aligned} \zeta_2^j(t) = & -\int_0^t \eta_4(s)\eta_3(s)\sin^2 js\,(Df_2 \cdot f_1)(x^j(s))ds \\ & - \int_0^t \eta_4(s)\sin js\ \cos js\,\eta_4(s)\,(Df_2 \cdot f_2)(x^j(s))ds + \mu_2^j(t) \end{aligned} \tag{5.10}$$

where the μ_2^j also go to 0 uniformly. In particular, the integrands that appear in the integrals of the formulas for ζ_1^j and ζ_2^j are uniformly bounded, so the integrals are equicontinuous. The μ_i^j are also equicontinuous, because they converge to 0, and the ξ^j are equicontinuous as well. Therefore the x^j are equicontinuous. By passing to a subsequence, we may assume that they converge uniformly to a limit x^∞. If we let

$$\rho_1^j(s) = \Big(\eta_3(s)\cos js - \eta_3(0)\Big)\eta_3(s)\sin js \ , \tag{5.11}$$

$$\rho_2^j(s) = \Big(\eta_3(s)\cos js - \eta_3(0)\Big)\eta_4(s)\cos js \ , \tag{5.12}$$

$$\rho_3^j(s) = \eta_4(s)\eta_3(s)\sin^2 js \ , \tag{5.13}$$

$$\rho_4^j(s) = \eta_4(s)\eta_4(s)\sin js\ \cos js \, , \tag{5.14}$$

then the four sequences of functions $\{\rho_1^j\}$, $\{\rho_2^j\}$, $\{\rho_3^j\}$, $\{\rho_4^j\}$ are uniformly bounded and converge weakly to 0, $\frac{1}{2}\eta_3(s)\eta_4(s)$, 0, and $\frac{1}{2}\eta_3(s)\eta_4(s)$, respectively. This means that x^∞ satisfies

$$\begin{aligned} x^\infty(t) =& p + \int_0^t \eta_1(s) f_1(x^\infty(s))ds + \int_0^t \eta_2(s) f_2(x^\infty(s))ds \\ &+ \frac{1}{2}\int_0^t \eta_3(s)\eta_4(s)(Df_1 \cdot f_2 - Df_2 \cdot f_1)(x^\infty(s))ds \ , \end{aligned} \tag{5.15}$$

i.e.

$$\begin{aligned} x^\infty(t) =& p + \int_0^t \eta_1(s) f_1(x^\infty(s))ds + \int_0^t \eta_2(s) f_2(x^\infty(s))ds \\ &+ \frac{1}{2}\int_0^t \eta_3(s)\eta_4(s)[f_2, f_1](x^\infty(s))ds \ . \end{aligned} \tag{5.16}$$

In other words, x^∞ is a solution of the initial value problem

$$\begin{aligned} \dot{x}(t) = \eta_1(t) f_1(x(t)) &+ \eta_2(t) f_2(x(t)) + \frac{1}{2}\eta_3(t)\eta_4(t)[f_2, f_1](x(t)) \ , \\ x(0) =& p \ . \end{aligned} \tag{5.17}$$

Although so far we have only asserted this for a subsequence, the argument actually proves that every subsequence of our original sequence has a subsequence that converges to a limit that is a solution of the above initial value problem. Since the initial value problem has a unique solution, it follows that the original sequence $\{x^j\}$ converges to this solution.

It is now easy to find a sequence $\{u^j\}$ of inputs whose trajectories converge to those of a given "extended input" v. It suffices to choose the u^j as above, with $\eta_1 \equiv v_1$, $\eta_2 \equiv v_2$, $\eta_3\eta_4 \equiv 2v_3$. Notice that the inputs u^j are *not* uniformly bounded, because of the factor $\sqrt{j}$. This is no accident: for a uniformly bounded (or even L^1-bounded) sequence $\{u^j\}$, if the trajectories converge to those of an "extended input" v as above for every choice of smooth f_1, f_2, then it follows that $v_3 \equiv 0$.

The above procedure generalizes to arbitrary systems that satisfy the LARC. One finds a suitable Lie bracket extension, as before, and then constructs a sequence of "highly oscillatory inputs" given as combinations of complex exponentials $e^{ij\omega t}$ for various frequencies ω with time-varying coefficients η_k to be chosen later, but one takes care to choose the frequencies and the coefficients so that the resulting controls are real. One then computes the limit, and expresses it in terms of the η_k. It then becomes apparent that the η_k can *always* be chosen so as to match any desired limit. However, to do all this it becomes necessary to introduce a considerable amount of algebraic machinery. Some of the details are outlined in [19] and [20], and a complete presentation of the results and proofs will be given in [21].

The main drawback of the method is that it requires the use of very large highly oscillatory controls. On the other hand, the method can be applied to obtain a rough approximation of a given path, to be then supplemented by the use of other methods to get much closer to the desired target point. This is likely to be particularly useful for problems with obstacles, in which case the primary purpose of the approximation is to produce an admissible path that avoids the obstacles.

§6. Optimal Control

Optimal control is a natural method for path finding problems. In many cases there is a function that one would like to minimize or nearly minimize. And, even if there is no obvious choice of such a function, artificially transforming the path finding problem into one of optimization is likely to make the problem much less underdetermined. Moreover, optimal control leads to *feedback* rather than just families of open-loop controls. On the other hand, the feedback typically cannot be smooth. For example, if a problem (2.1) has no drift and the number m of controls is less than the state space dimension, then the necessary condition for smooth local stablizability discovered by Brockett in [2] will necessarily be violated at every point x where the rank function $y \rightarrow \text{rank}\{f_1(y), \ldots, f_m(y)\}$ has a local maximum. (In particular, it will be violated on an open dense set.)

The main drawback of optimal control methods is that, except in some very special cases, the optimal controls are hard to compute. However, using recently developed differential geometric methods, some progress has been made for a number of systems that could not be dealt with by classical optimal control methods. In several cases, it turns out that, although one cannot actually compute the optimal controls exactly, one can determine a rather small *sufficient family of trajectories*, i.e. a family $\mathcal{F}$ of trajectories such that for every pair (p, q) of points there is an optimal trajectory in $\mathcal{F}$ that goes from p to q.

As an example of this, we consider the problem $\dot{x}_1 = u\cos\theta$, $\dot{x}_2 = u\sin\theta$, $\dot{\theta} = v$, where the controls u, v are subject to the constraints $|u| \leq 1$, $|v| \leq 1$. This problem was studied by Reeds and Shepp in [15]. They proved that a sufficient family is obtained by considering the set of all concatenations of at most five pieces, each one of which is either a circle (i.e. $(u, v) \equiv (1, 1)$ or $(u, v) \equiv (1, -1)$, or $(u, v) \equiv (-1, 1)$, or $(u, v) \equiv (-1, -1)$) or a straight line, with no more than one straight line and no more than four circles. (They actually proved a number of extra restrictions as well, cf. [15]. They used a different formulation, in which the restriction on u is $|u| = 1$ rather than

$|u| \leq 1$. Since this defines a nonconvex set, existence of optimal trajectories is not guaranteed by the usual existence theorems of control theory, so they could not apply the usual necessary conditions for optimality.)

Recently, the Reeds-Shepp car and a number of other related problems have been investigated from an optimal control theory point of view. It turns out that the tools of classical optimal control, combined with the newer methods of differential geometric nonlinear control, make it possible to analyze the trajectories in detail, and to get a lot of information about the optimal feedback (cf. [22]). So there is new hope that optimal control methods, properly enriched by the addition of the more recently developed techniques, may after all turn out to be more useful than they are currently believed to be.

§7. Path deformation

The *path deformation approach* is based on extending to the path finding problem the well known continuation method used to solve nonlinear equations. Let us first recall how this method works.

Suppose M and N are smooth manifolds and $\Phi : M \to N$ is a smooth map. Assume N is connected. Suppose we want to solve an equation $\Phi(x) = y$, for which we are given $y \in N$ and want to find $x \in M$. One possible approach is to start with some values $\bar{x}$, $\bar{y}$ such that $\Phi(\bar{x}) = \bar{y}$, find a smooth path $\pi : [0,1] \to N$ such that $\pi(0) = \bar{y}$ and $\pi(1) = y$, and then try to "lift" this path to M, i.e. to find a path $\Pi : [0,1] \to M$ such that $\Phi \circ \Pi = \pi$ and $\Pi(0) = \bar{x}$. Clearly, if we find such a path, then $x = \Pi(1)$ will be a solution of our problem.

The reason why this is useful is that, if certain nondegeneracy and nonexplosion conditions are satisfied, then Π can be found by solving an ordinary differential equation, which lends itself easily to numerical implementation. To see this, let us assume from now on, for simplicity, that M and N actually are open subsets of Euclidean spaces $\mathbb{R}^m$, $\mathbb{R}^n$. In view of the initial condition $\Pi(0) = \bar{x}$, the identity $\Phi \circ \Pi = \pi$, i.e. $\Phi(\Pi(t)) = \pi(t)$, is equivalent to the equation

$$D\Phi(\Pi(t)) \cdot \dot{\Pi}(t) = \dot{\pi}(t) \ . \tag{7.1}$$

If $D\Phi(\Pi(t))$ has rank n, then (7.1) can be solved for $\dot{\Pi}(t)$. Precisely, then, the assumption that we need is that Φ *is a submersion*, i.e. that $D\Phi(x)$ has rank n for all x. If this condition holds, then we can define the pseudoinverse

$$P(x) = D\Phi(x)^{\dagger} \left(D\Phi(x) D\Phi(x)^{\dagger} \right)^{-1} , \tag{7.2}$$

which is a smooth $\mathbb{R}^{m\times n}$-valued function on M, and then consider the *path lifting equation*

$$\dot{x}(t) = P(x(t))\dot{\pi}(t) \ . \tag{7.3}$$

If we can find a solution of (7.3) with initial condition $x(0) = \bar{x}$, *which is defined on the whole interval* $[0,1]$, then $x(1)$ will solve our original equation.

The problem now is to make sure that the solutions of (7.3) exist globally. This, of course, requires some other condition, e.g. that the map Φ be proper. In applications, the map Φ usually is not a submersion, but the "singular set" $S = \{x \in M : \operatorname{rank} D\Phi(x) < n\}$ is small. What one does then is work on the complement $\tilde{M} = M - S$ of this set. Then by definition the restriction $\tilde{\Phi}$ of Φ to this set is a (not necessarily surjective) submersion. So there is no problem with the local existence and uniqueness of the solutions of (7.3) on $\tilde{M}$. But now the main issue becomes that of explosions: a solution of (7.3) can blow up either because it "goes to infinity" (i.e. does not remain in any compact set) in M, *or* because it approaches the singular set. If the latter possibility actually occurs, then extra work is needed to avoid or bypass the singularities. Naturally, this is not always possible, as can be shown by simple examples. (Consider, e.g., the map $\Phi : \mathbb{R} \to \mathbb{R}$ given by $\Phi(x) = x^3 - 3x$. The singular set S consists of 1 and -1. The map Φ is a surjective submersion on $\mathbb{R} - S$, but the path $\pi : [0,1] \to \mathbb{R}$ given by $\pi(t) = -3 + 6t$ cannot be lifted.)

To see how these ideas apply to the path finding problem, we consider once again the simple system (3.3) introduced above. As before we assume, for simplicity, that the f_i and all their derivatives are bounded.

In this case the object that plays the role of M should be the space of all controls, or some sufficiently rich subset. It will be convenient to normalize so that all the controls are defined on $[0,1]$, and to use for M the space $L^2([0,1],\mathbb{R}^2)$ of *square integrable* $\mathbb{R}^2$-valued controls. Then N will be $\mathbb{R}^n$. Suppose we fix p, q in N and want to find a control that steers p to q. Given any control $\eta \in M$, we can consider the corresponding trajectory γ_p^η, and we let $\Phi(\eta) = \gamma_p^\eta(1)$. With this formulation, our problem now becomes that of solving the equation $\Phi(\eta) = q$ for a given q. Clearly, we are exactly in the situation described before, except that now M is an infinite-dimensional Hilbert space.

The singularities of the map Φ are well understood. It is a general fact that for problems of the kind considered here the singularities of Φ are precisely the *abnormal extremals*, i.e. the controls that satisfy the necessary conditions of the Pontryagin Maximum Principle with $\lambda_0 = 0$. (Here λ_0 is the multiplier that is associated to the Lagrangian in the Maximum Principle. Since we are interested here only in the case $\lambda_0 = 0$, the particular choice of a Lagrangian is immaterial.) The zero control is an abnormal extremal. And, unfortunately, for general nonholonomic problems abnormal extremals other than zero can occur, as shown by Montgomery in [11]. However, for our simple system (3.3) they do not. (This follows from a much more general fact, namely, that nontrivial abnormal extremals do not occur for "fat sub-Riemannian metrics," cf. e.g. Strichartz [16], [17]) but can be proved easily in our case, as will be indicated below.)

So for (3.3), if we take $\tilde{M} = M - \{0\}$, then the restriction $\tilde{\Phi}$ of Φ to $\tilde{M}$ is a surjective submersion. (Surjectivity is trivial, because (3.3) is CC, and p can be reached from p by a nontrivial loop.) To make sure that the solutions of (7.3) do not approach the singular point 0 of M, it suffices to use a starting control $\bar{\eta}$ which is not 0, which will then give rise to a starting point $\bar{q} = \gamma_p^{\bar{\eta}}$, and then take the precaution of choosing a path $\pi : [0,1] \to N$ that does not go through p and satisfies $\pi(0) = \bar{q}$, $\pi(1) = q$. (Such a path always exists if $n > 1$, and can always be chosen of a simple kind, e.g. a segment or a circular arc.) The only remaining problem is to show that the solutions do not explode. This can be done, but requires some more delicate estimates, that will be omitted here.

We conclude by writing out in detail the differential equation (7.3) for the evolution of a path for our problem (3.3). If $\eta : [0,1] \to \mathbb{R}^n$ is a control, and $x(t) = \gamma_p^{\eta}(t)$, then we can consider the *variational equation*

$$\begin{aligned} \dot{y}(t) = &\Big(\eta_1(t) Df_1(x(t)) + \eta_2(t) Df_2(x(t))\Big) y(t) \\ &+ \zeta_1(t) f_1(x(t)) + \zeta_2(t) f_2(x(t)) \ , \end{aligned} \tag{7.4}$$

corresponding to a control variation ζ. (Precisely, $y(t)$ is the derivative with respect to s at $s = 0$ of $\gamma_p^{\eta+s\zeta}$. The fact that y satisfies (7.4) is a standard result in the theory of ordinary differential equations. Moreover, it is clear that $y(0) = 0$.) Clearly, (7.4) is, for fixed p and η, a linear time-varying system. The map μ that assigns to each ζ the value $y(1)$ is precisely the differential of Φ at η. This map μ is going to be onto if and only if the controllability Gramian G of the linear system (7.4) is nonsingular. If $(t,s) \to M(t,s)$ denotes the fundamental solution of (7.4) (that is, $M(t,s)$ is, as a function of t, an $n \times n$ matrix-valued solution of

$$\dot{Y}(t) = \Big(\eta_1(t) Df_1(x(t)) + \eta_2(t) Df_2(x(t))\Big) Y(t) \ , \tag{7.5}$$

and $M(s,s) = \text{identity}$), and v is a vector in $\mathbb{R}^n$, then $v^\dagger G v$ is given by

$$v^\dagger G v = \int_0^1 ||v^\dagger M(1,t) \mathbf{f}(t)||^2 dt \ , \tag{7.6}$$

where $\mathbf{f}(t)$ is the $n\times 2$ matrix whose columns are $f_1(x(t))$ and $f_2(x(t))$. So G will be nonsingular if and only if the function $\varphi_v : t \to v^\dagger M(1,t)\mathbf{f}(t)$ does not vanish identically on $[0,1]$ for any nonzero vector v. Assume that $\varphi_v(t) \equiv 0$. Then the derivative $\dot{\varphi}_v$ will also vanish. To compute this derivative, we observe that the derivative of $M(1,t)$ with respect to t is easily seen to be equal to $-M(1,t)(\eta_1(t)Df_1(x(t)) + \eta_2(t)Df_2(x(t)))$. Therefore

$$\begin{aligned} &\frac{d}{dt}M(1,t)f_1(x(t)) = \\ &\qquad M(1,t)\cdot Df_1(x(t))\cdot\Big(\eta_1(t)f_1(x(t)) + \eta_2(t)f_2(x(t))\Big) \\ &\quad - M(1,t)\Big(\eta_1(t)Df_1(x(t)) + \eta_2(t)Df_2(x(t))\Big)f_1(x(t)), \end{aligned} \tag{7.7}$$

so that

$$\begin{aligned} &\frac{d}{dt}M(1,t)f_1(x(t)) = \\ &\eta_2(t)M(1,t)\cdot\Big(Df_1(x(t))f_2(x(t)) - Df_2(x(t))f_1(x(t))\Big) \ , \end{aligned} \tag{7.8}$$

i.e.

$$\frac{d}{dt}M(1,t)f_1(x(t)) = \eta_2(t)M(1,t)[f_2,f_1](x(t)) \ . \tag{7.9}$$

Similarly, we have

$$\frac{d}{dt}M(1,t)f_2(x(t)) = \eta_1(t)M(1,t)[f_1,f_2](x(t)) \ . \tag{7.10}$$

The above equalities are, of course, valid almost everywhere. If η does not vanish almost everywhere, then there exists a t such that $\eta(t) \neq 0$ and both (7.9) and (7.10) hold there. It then follows that

$$v^\dagger M(1,t)[f_1,f_2](x(t)) = 0 \ . \tag{7.11}$$

On the other hand, the hypothesis that φ_v vanishes identically implies that

$$v^\dagger M(1,t)f_1(x(t)) = v^\dagger M(1,t)f_2(x(t)) = 0 \ . \tag{7.12}$$

Since we are assuming that f_1, f_2 and $[f_1,f_2]$ are independent at every point, it follows that

$$v^\dagger M(1,t) = 0 \ . \tag{7.13}$$

Since $M(1,t)$ is nonsingular, this implies that $v = 0$ as desired.

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Department of Mathematics
Rutgers University
New Brunswick, NJ 08903

E-mail: sussmann@hamilton.rutgers.edu