1. The classical differential

If X, Y are normed spaces, S is a subset of X, $s_* \in S$, and L is a bounded linear map from X to Y, then we say that L is a ***Fréchet differential***—or, simply, a ***differential—at*** s_* of a ppd map $F : S \dashrightarrow Y$ if $F(s)$ is defined for all s in some relative neighborhood $S \cap \mathbb{B}_X(s_*, \delta) = \{s \in S : \|s - s_*\| < \delta\}$ of s_* in S, and

$$\lim_{s \to s_*, s \in S, s \neq s_*} \frac{\|F(s) - F(s_*) - L \cdot (s - s_*)\|}{\|s - s_*\|} = 0\,. \tag{1.1}$$

We say that F is ***differentiable at*** s_* if there exists a bounded linear map $L : X \mapsto Y$ such that L is a differential of F at s_*. Clearly, if F is differentiable at s_*, then the differential L of F at s_* is uniquely determined on the closed linear subspace

$$\Sigma(S, s_*) = \mathrm{Clos}_X\left(\mathrm{lin\,span} T^B_{s_*} S\right),$$

where $T^B_{s_*} S$ is the ***Bouligand tangent cone*** of S at s_*, i.e., the set of all limits $v = \lim_{j \to \infty} \frac{s_j - s_*}{h_j}$, for all sequences $\{s_j\}_{j=1}^\infty$, $\{h_j\}_{j=1}^\infty$, such that (i) $s_j \in S$ and $h_j > 0$ for all j, and (ii) $h_j \to 0$ as $j \to \infty$. (Indeed, if L_1, L_2 are two differentials of F at s_*, and $v \in T^B_{s_*} S$, then we can find sequences $\{s_j\}_{j=1}^\infty$ in S, $\{h_j\}_{j=1}^\infty$ in $]0, +\infty[$, such that $v = \lim_{j \to \infty} \frac{s_j - s_*}{h_j}$. Then, if $i = 1, 2$, and j is large enough, we have $L_i \cdot (s_j - s_*) = F(s_j) - F(s_*) + o(\|s_j - s_*\|)$, and $s_j - s_* = h_j v + o(h_j)$, so $L_i \cdot (s_j - s_*) = F(s_j) - F(s_*) + o(h_j)$, and then

$$L_i \cdot \left(\frac{s_j - s_*}{h_j}\right) = \frac{F(s_j) - F(s_*)}{h_j} + o(1) \;\text{ as }\; j \to \infty\,,$$

from which it follows that

$$L_i \cdot v = \lim_{j \to \infty} \left(\frac{F(s_j) - F(s_*)}{h_j}\right).$$

Hence $L_1 \cdot v = L_2 \cdot v$. So L_1 and L_2 agree on $T^B_{S_*} S$ and then they agree on $\Sigma(S, s_*)$, since they are linear and continuous.)

We use $\mathcal{D}F(s_*)$ to denote the set of all differentials of F at s_*. Then $\mathcal{D}F(s_*)$ consists of a single linear map if and only if F is differentiable at s_* and $\Sigma(S, s_*) = X$. In that case, the unique member of $\mathcal{D}F(s_*)$ will be referred to as ***the differential of*** F ***at*** s_*, and will be represented with the notation $DF(s_*)$.

More generally, if $S \subseteq S' \subseteq X$, $s_* \in S$, and $F : S' \dashrightarrow Y$, a ***differential of*** F ***at*** s_* ***in the direction of*** S is a differential at s_* of the restriction $F \lceil S$ of F to S. We use $\mathcal{D}F_S(s_*)$ to denote the set of all differentials of F at s_* in the direction of S. Naturally, $\mathcal{D}F_S(s_*)$ consists of a single linear map if and only if F is differentiable at s_* in the direction of S and $\Sigma(S, s_*) = X$. In that case, the unique member of $\mathcal{D}F_S(s_*)$ will be referred to as ***the differential of*** F ***at*** s_* ***in the direction of*** S, and will be represented with the notation $DF_S(s_*)$.

1.1. The chain rule.

PROPOSITION 1.1.1. *Let X_1, X_2, X_3 be normed real linear spaces, let S_1, S_2 be subsets of X_1, X_2, let F_1, F_2 be ppd maps from S_1 to X_2 and from S_2 to X_3, respectively, such that $F_1(S_1) \subseteq S_2$. Let $s_* \in S_1$ be such that F_1 is differentiable at s_*, so in particular $\sigma_* \stackrel{\mathrm{def}}{=} F_1(s_*)$ exists and belongs to S_2. Assume that F_2 is* *differentiable at* σ_*. *Let* $L_1 \in \mathcal{D}F_1(s_*)$, $L_2 \in \mathcal{D}F_2(\sigma_*)$. *Let* $F = F_2 \circ F_1$, $L = L_2 \circ L_1$. *Then* $L \in \mathcal{D}F(s_*)$.

PROOF. We assume, as we clearly may without loss of generality, that $s_* = 0$, $\sigma_* = 0$. Fix positive numbers $r_1^\#$, $\bar{r}_2^\#$, such that F_i is everywhere defined on $U_i^\# \cap S_i$ for $i = 1, 2$, where $U_i^\# = \mathbb{B}_{X_i}(0, r_i^\#)$. Since $\|s\|^{-1}\Big(F_1(s) - L_1(s)\Big)$ goes to zero as $s \to 0$ via values in $U_1^\# \cap S_1$, we may pick a positive $\tilde{r}_1$ such that $\tilde{r}_1 \le r_1^\#$ and $\|F_1(s) - L_1(s)\| \le \|s\|$ whenever $s \in S_1$ and $\|s\| \le \tilde{r}_1$. Let $\kappa = 1 + \|L_1\|$. Then $\|F_1(s)\| \le \kappa\|s\|$ whenever $s \in S_1$ and $\|s\| \le \tilde{r}_1$. Pick a positive r_1 such that $r_1 \le \tilde{r}_1$ and $\kappa m r_1 \le r_2^\#$. Then $F_1(s) \in U_2^\# \cap S_2$ whenever $s \in S_1$ and $\|s\| \le r_1$. Define $r_2 = r_2^\#$, and let $U_i = \mathbb{B}_{X_i}(0, r_i)$ for $i = 1, 2$. Then, for $i = 1, 2$, F_i is everywhere defined on $U_i \cap S_i$, and in addition $F_1(U_1 \cap S_1) \subseteq U_2 \cap S_2$.

For $i = 1, 2$, $0 < r \le r_i$, let

$$\theta_i(r) = \sup\left\{\frac{\|F_i(s) - L_i \cdot s\|}{\|s\|} : s \in S_i, 0 < \|s\| \le r\right\},$$

so that $\theta_i(r) \to 0$ as $r \downarrow 0$. Then, if $s \in U_1 \cap S_1$, and we write $\sigma = F_1(s)$, we have

$$\begin{aligned}
\|F(s) - L(s)\| &= \|F_2(F_1(s)) - L_2(L_1(s))\| \\
&\le \|F_2(F_1(s)) - L_2(F_1(s))\| + \|L_2(F_1(s)) - L_2(L_1(s))\| \\
&\le \theta_2(\|\sigma\|) + \|L_2\|\,\|F_1(s) - L_1(s)\| \\
&\le \theta_2(\|\sigma\|)\|\sigma\| + \|L_2\|\theta_1(\|s\|)\|s\|\,.
\end{aligned}$$

On the other hand,

$$\|\sigma\| = \|F_1(s)\| \le \|F_1(s) - L_1(s)\| + \|L_1(s)\| \le \theta_1(\|s\|)\,\|s\| + \|L_1\|\,\|s\|\,.$$

Therefore $\|F(s) - L(s)\| \le \Theta(\|s\|)\,\|s\|$, where

$$\Theta(r) \stackrel{\text{def}}{=} \theta_2\Big(r(\theta_1(r) + \|L_1\|)\Big)\Big(\theta_1(r) + \|L_1\|\Big) + \|L_2\|\theta_1(r)\,.$$

Clearly, $\lim_{r\downarrow 0}\Theta(r) = 0$, so $\|s\|^{-1}\|F(s) - L(s)\|$ goes to 0 as $s \to 0$ via values in S_1, and our proof is complete. ♢

1.2. Open mapping theorems for classical differentials.

DEFINITION 1.2.1. Let X, Y be topological spaces, let $F : X \mapsto Y$ be a map, and let $x \in X$. We say that F is ***open at*** x if F maps neighborhoods of x to neighborhoods of $F(x)$ (that is, whenever U is a subset of X such that $x \in \operatorname{Int} U$ it follows that $F(x) \in \operatorname{Int} F(U)$). ♢

Recall that a ***cone*** in a real linear space X is a nonempty subset C of X such that $r \cdot v \in C$ whenever $v \in C$, $r \in \mathbb{R}$, and $r \ge 0$. It follows from this definition that if C is a cone in X then automatically $0 \in C$.

DEFINITION 1.2.2. Let X, Y be normed real linear spaces, let $\bar{x}$, v be points of X, Y, respectively, let S be a subset of X such that $\bar{x} \in S$, and let $F : S \mapsto Y$ be a map. We say that F is ***directionally open at*** $\bar{x}$ ***along*** S ***in the direction of*** v if there exists a convex cone D in Y such that

(1) $v \in \operatorname{Int}(D)$,
(2) for every positive number α there exists a positive number $\beta = \beta(\alpha)$ such that

(2.i) the image under F of the set $\{\, x \in S : \|x - \bar{x}\| \le \alpha \,\}$ contains the set $\{\, F(\bar{x}) + y : y \in D, \|y\| \le \beta \,\}$.

If the function $\alpha \mapsto \beta(\alpha)$ can be chosen so that there exist positive numbers α^*, μ such that $\beta(\alpha) = \mu\alpha$ for $0 < \alpha \le \alpha^*$, then F is said to be ***directionally open with linear rate at*** $\bar{x}$ ***along*** S ***in the direction of*** v. $\diamondsuit$

THEOREM 1.2.3. (The directional open mapping theorem for the classical differential). *Let X, Y be normed real linear spaces such that Y is finite-dimensional, let $\bar{x} \in X$, $v \in Y$, let C be a convex cone in X, and let*

$$S = \bar{x} + C = \{\, x \in X : x - \bar{x} \in C \,\}.$$

Let U be a neighborhood of $\bar{x}$ in X, let $F : U \cap S \mapsto Y$ be a continuous map, and let $L : X \mapsto Y$ be a bounded linear map such that $L \in \mathcal{D}F(\bar{s})$. Assume that $v \in \mathrm{Int}(L \cdot C)$. Then F is directionally open with linear rate at $\bar{x}$ along S in the direction of v.

PROOF. We assume, without loss of generality, that $\bar{x} = 0$ and $F(\bar{x}) = 0$. We let $m = \dim Y$. If $w \in Y$ and $\rho > 0$, we use $\bar{\mathbb{B}}_Y(w,\rho)$ to denote the closed ball $\{\, y \in Y : \|y - w\| \le \rho \,\}$. We let $(\tilde{e}_1, \ldots, \tilde{e}_m)$ be a basis of Y, write $\tilde{e}_0 = -(\tilde{e}_1 + \ldots + \tilde{e}_m)$, and pick an r such that $r > 0$ and $v + r\tilde{e}_j \in L \cdot C$ for every $j \in \{0, \ldots, m\}$. We then let $e_j = v + r\tilde{e}_j$, and observe that the vectors $e_0, \ldots, e_m$ are affinely independent, i.e., that whenever a linear combination $a_0e_0 + a_1e_1 + \ldots + a_me_m$ with $a_0 + a_1 + \ldots + a_m = 0$ is equal to zero, it follows that $a_0 = a_1 = \ldots = a_m = 0$. (Indeed, if $a_0e_0 + a_1e_1 + \ldots + a_me_m = 0$ and $a_0 + a_1 + \ldots + a_m = 0$, we get $(a_0+a_1+\ldots+a_m)v + ra_0\tilde{e}_0 + ra_1\tilde{e}_1 + \ldots + ra_m\tilde{e}_m = 0$. Then $a_0\tilde{e}_0 + a_1\tilde{e}_1 + \ldots + a_m\tilde{e}_m = 0$, so $(a_1 - a_0)\tilde{e}_1 + \ldots + (a_m - a_0)\tilde{e}_m = 0$, which implies $a_1 - a_0 = a_2 - a_0 = \ldots = a_m - a_0 = 0$, so $a_0 = a_1 = a_2 = \ldots = a_m$, and then $0 = a_0 + a_1 + \ldots + a_m = (m+1)a_0$, so $a_0 = a_1 = a_2 = \ldots = a_m = 0$.) We then remark that every member y of Y can be expressed in a unique way as an affine combination (that is, a linear combination whose coefficients add up to 1) $a_0(y)e_0 + a_1(y)e_1 + \ldots + a_m(y)e_m$ of the e_j. (Indeed, we can find $b_1, \ldots, b_m$ such that $r^{-1}(y - v) = b_1\tilde{e}_1 + \ldots + b_m\tilde{e}_m$, since the $\tilde{e}_j$ form a basis of Y. Then we can let $a_0 = (m+1)^{-1}(1 - b_1 - \ldots - b_m)$ and $a_j = a_0 + b_j$ for $j = 1, \ldots, m$. It is then easy to verify that $y = a_0e_0 + a_1e_1 + \ldots + a_me_m$ and $a_0 + a_1 + \ldots + a_m = 1$. Uniqueness follows from the fact that $e_0, \ldots, e_m$ are affinely independent.) It is clear that the functions $Y \ni y \mapsto a_j(y) \in \mathbb{R}$ are continuous.

Since $v = (m+1)^{-1}(e_0 + \ldots + e_m)$, we have $a_j(v) = (m+1)^{-1}$ for $j = 0, \ldots, m$. Therefore we can fix a ρ such that $\rho > 0$ and $a_j(y) > 0$ for all $j \in \{0, \ldots, m\}$ and all $y \in \bar{\mathbb{B}}_Y(v,\rho)$. Let $W = \bar{\mathbb{B}}_Y(v,\rho)$. Then $W \subseteq L \cdot C$, because if $y \in \bar{\mathbb{B}}_Y(v,\rho)$ then $y = \sum_{j=0}^m a_j(y)e_j$, the e_j are in $L \cdot C$, $\sum_{j=0}^m a_j(y) = 1$, and $a_j(y) > 0$, so y is a convex combination of the e_j.

Since $e_j \in L \cdot C$, we can pick $c_j \in C$ such that $L \cdot c_j = e_j$. Let $M : Y \mapsto X$ be the map that sends $y \in Y$ to the affine combination $M \cdot y \stackrel{\text{def}}{=} \sum_{j=0}^m a_j(y)c_j$. Then M is a continuous (and in fact affine, but not necessarily linear) map, and $L \circ M = \text{identity}_Y$, because $L(M \cdot y) = \sum_{j=0}^m a_j(y)L \cdot c_j = \sum_{j=0}^m a_j(y)e_j = y$.

Let $\kappa = \sup\{\|M \cdot y\| : y \in W\}$. For $s > 0$, let

$$\theta(s) = \sup\left\{ \frac{\|F(x) - L \cdot x\|}{\|x\|} : x \in C, \|x\| \le s \right\},$$

so $\theta(s) \to 0$ as $s \downarrow 0$.

Now fix a ρ^* such that (i) $0 < \rho^* < \rho$, and (ii) if $v \neq 0$ then $\rho^* < \|v\|$. Then define $W^* = \bar{\mathbb{B}}_Y(v, \rho^*)$. Also, let δ^* be such that $\delta^* > 0$ and $\bar{\mathbb{B}}_X(0, \delta^*) \subseteq U$. Then pick a τ^* such that $0 < \tau^*$, $\kappa\tau^* \leq \delta^*$, and $\kappa\theta(\tau^*\kappa) \leq \rho - \rho^*$. Let $\alpha^* = \kappa\tau^*$, so $0 < \alpha^* \leq \delta^*$. For each positive number α, define $V_\alpha = \bar{\mathbb{B}}_X(0, \tilde{\alpha})$ and $\tau_\alpha = \frac{\tilde{\alpha}}{\kappa}$, where $\tilde{\alpha} = \min(\alpha, \alpha^*)$. Then $V_\alpha \subseteq U$, and

$$tM \cdot y \in V_\alpha \cap C \quad \text{whenever} \quad \alpha > 0\,, y \in W\,,\, 0 \leq t \leq \tau^*\,.$$

We then define

$$G_t(y) = \frac{1}{t}\Big(F(tM \cdot y)\Big) \quad \text{for} \quad y \in W\,,\ 0 < t \leq \tau^*\,.$$

Clearly, G_t is a continuous map from W to Y, if $t \in]\,0, \tau^*]$, and

$$\begin{aligned}
\|G_t(y) - y\| &= \frac{1}{t}\Big(\|F(tM \cdot y) - ty\|\Big) \\
&\leq \frac{1}{t}\Big(\|F(tM \cdot y) - \|t(L \circ M) \cdot y)\|\Big) \\
&= \frac{1}{t}\Big(\|F(tM \cdot y) - \|L(tM \cdot y)\|\Big) \\
&\leq \frac{1}{t}\Big(\theta(t\kappa)t\kappa\Big) \\
&= \kappa\theta(t\kappa)\,.
\end{aligned}$$

If $y \in W^*$ and $0 < t \leq \tau^*$, we can consider the map $\mu_{t,y} : W \mapsto Y$ given by

$$\mu_{t,y}(z) = z - G_t(z) + y\,.$$

Then

$$\|\mu_{t,y}(z) - v\| = \|z - G_t(z) + y - v\| \leq \|z - G_t(z)\| + \|y - v\| \leq \kappa\theta(t\kappa) + \rho^* \leq \rho\,,$$

since $\kappa\theta(\tau^*\kappa) \leq \rho - \rho^*$. Therefore $\mu_{t,y}$ is a continuous map from W to W. By the Brouwer fixed point theorem, there exists, for each $y \in W^*$ and each $t \in]0, t^*]$, a $z = z(t, y) \in W$ such that $\mu_{t,y}(z) = z$. Then $z - G_t(z) + y = z$, so $G_t(z) = y$. But then $F(tM \cdot (z(t, y))) = ty$. Since this is true for every $y \in W^*$ and every $t \in]0, \tau^*]$, we have shown that

$$tW^* \subseteq F(tM(W)) \ \text{ whenever } \ 0 < t \leq \tau^*\,.$$

Furthermore, the inclusion of the previous line also holds for $t = 0$, since $F(0) = 0$. It follows that

$$\bigcup_{0 \leq t \leq \tau_\alpha} tW^* \subseteq F(V_\alpha \cap C) \ \text{ whenever } \ \alpha > 0\,.$$

Let D be the convex cone generated by W^*. Then v is an interior point of D. Furthermore, D is closed. (Indeed, if $v = 0$ then $D = Y$, so D is closed. Assume that $v \neq 0$. If $\{y_j\}_{j=1}^\infty$ is a sequence of members of D that converges to a limit y, then we can write $y_j = t_j w_j$, $t_j \in [0, \infty\,[\,$, $w_j \in W^*$. After passing to a subsequence, if necessary, we may assume that $w = \lim_{j\to\infty} w_j$ exists and belongs to W^*. Since $v \neq 0$, the inequality $\rho^* < \|v\|$ implies that $0 \notin W^*$, so w and the w_j are all $\neq 0$. Then $t_j = \frac{\|y_j\|}{\|w_j\|}$, and $\lim_{j\to\infty} t_j = t = \frac{\|y\|}{\|w\|}$. It follows that $y = tw$, so $y \in D$.)

Now fix a constant ν such that $\mathbb{B}_Y(0, 1) \cap D \subseteq \bigcup_{0 \leq t \leq \nu} tW^*$. (The existence of ν is trivial. Indeed, if $v = 0$ we can pick $\nu = \frac{1}{\rho^*}$. If $v \neq 0$, then every $w \in W^*$ satisfies $\|w - v\| \geq \rho^*$, so $\|v\| \leq \rho^* + \|w\|$, and then $\|w\| \geq \|v\| - \rho^*$. Since $\rho^* < \|v\|$, we can take ν to be any nmuber such that $\nu \geq (\|v\| - \rho^*)^{-1}$, and conclude that if $y \in \mathbb{B}_Y(0,1) \cap D$, and we write $y = tw$, with $w \in W^*$, then $t = \frac{\|y\|}{\|w\|} \leq \frac{1}{\|v\|-\rho^*} \leq \nu$.) It follows that $\mathbb{B}_Y(0,s) \cap D \subseteq \bigcup_{0 \leq t \leq \nu s} tW^*$ for every positive s.

Now, if α is an arbitrary positive number, and we let $\beta(\alpha) = \nu^{-1}\tau_\alpha$, we have

$$\mathbb{B}_Y(0, \beta(\alpha)) \cap D \subseteq \bigcup_{0 \leq t \leq \nu\beta(\alpha)} tW^* = \bigcup_{0 \leq t \leq \tau_\alpha} tW^* \subseteq F(V_\alpha \cap C)\,.$$

The proof of Theorem 1.2.3 is thus complete. ♢

THEOREM 1.2.4. (The open mapping theorem for the classical differential). *Let X, Y be normed real linear spaces such that Y is finite-dimensional, let $\bar{x} \in X$, let U be a neighborhood of $\bar{x}$ in X, and let $F : U \mapsto Y$ be a continuous map. Assume that F is differentiable at $\bar{x}$, and let $L = DF(\bar{x})$. Assume that L is surjective. Then F is open at $\bar{x}$.*

PROOF. It suffices to apply Theorem 1.2.3 with $v = 0$, $C = X$. ♢

1.3. The transversal intersection theorem for the classical differential.

DEFINITION 1.3.1. Assume that Y is a normed real linear space, S is a subset of Y, $s_* \in S$, and C is a convex cone in Y. We say that C is an ***approximating cone to S at s_**** if there exist a normed space X, a neighborhood U of 0 in X, a convex cone D in X, a continuous map $F : U \cap D \mapsto Y$, and a bounded linear map $L \in \mathrm{Lin}(X,Y)$, such that $F(0) = s_*$, $F(U \cap D) \subseteq S$, $LD = C$, and $L \in \mathbb{B}DF(0)$.

We say that C is a ***limiting approximating cone to S at s_**** if there exists an increasing sequence $\{C_j\}_{j=1}^{\infty}$ of convex cones in Y such that $C = \mathrm{Clos} \bigcup_{j=1}^{\infty} C_j$ and each C_j is an approximating cone to S at s_*. ♢

DEFINITION 1.3.2. Let X be a real linear space. We say that two convex cones C^1, C^2 in X are ***transversal***, and write

$$C^1 \overline{\pitchfork} C^2\ ,$$

if $C^1 - C^2 = X$. We say that C^1 and C^2 in X are ***strongly transversal***, and write

$$C^1 \overline{\pitchfork\!\!\!|}\, C^2\ ,$$

if $C^1 \overline{\pitchfork} C^2$ and $C^1 \cap C^2 \neq \{0\}$. ♢

DEFINITION 1.3.3. Assume that Y is a normed real linear space, S^1 and S^2 are subsets of Y, and $s_* \in S^1 \cap S^2$. We say that S^1 and S^2 are ***locally separated at s_**** if every neighborhood U of s_* contains a point s of $S^1 \cap S^2$ such that $s \neq s_*$. ♢

DEFINITION 1.3.4. Let Y be a normed real linear space, and let $Y^\dagger$ denote the dual space of Y. Let C be a convex cone in Y. The ***polar*** of C is the set

$$C^\dagger \stackrel{\mathrm{def}}{=} \{\lambda \in Y^\dagger : \lambda(c) \leq 0 \text{ for all } c \in C\}\,. \quad \diamondsuit \tag{1.2}$$

DEFINITION 1.3.5. Let Y be a normed real linear space, and let C^1, C^2 be two convex cones in Y. We say that the cones C^1 and C^2 are ***linearly separated*** if $(C^1)^\dagger \cap (-C_2)^\dagger \neq \{0\}$. ♢

In other words, C^1 ***and*** C^2 ***are linearly separated if and only if there exists a nontrivial continuous linear functional*** $\lambda : Y \mapsto \mathbb{R}$ ***such that*** $\langle \lambda, v\rangle \leq 0$ ***for all*** $v \in C^1$ ***and*** $\langle \lambda, v\rangle \geq 0$ ***for all*** $v \in C^2$***.***

PROPOSITION 1.3.6. *Let C^1, C^2 be convex cones in a finite-dimensional real linear space Y. Then the following three conditions are equivalent:*

(1) $C^1 \overline{\pitchfork} C^2$*;*
(2) C^1 *and* C^2 *are not linearly separated;*
(3) *either*
  (a) C^1 *are* C^2 *are linear subspaces of* Y *such that* $Y = C^1 \oplus C^2$*,*
 or
  (b) $C^1 \overline{\pitchfork\!\!\!\pitchfork} C^2$.

PROOF. The equivalence of (1) and (2) is very easy to prove: if (2) does not hold, then there exists a $\lambda \in Y^\dagger \backslash \{0\}$ such that $\langle \lambda, c\rangle \leq 0$ for all $c \in C^1$ and $\langle \lambda, c\rangle \geq 0$ for all $c \in C^2$, so that $\langle \lambda, c\rangle \leq 0$ for all $c \in C^1 - C^2$ and then $C^1 - C^2$ cannot be the whole space because λ is nontrivial, so (1) does not hold; conversely, if (1) does not hold, then the convex cone $C = C^1 - C^2 2$ is not the whole space, and then the closure $\bar{C}$ of C is not the whole space either, because Y is finite-dimensional; therefore the Hahn-Banach theorem implies the existence of a nontrivial linear functional λ which is nonpositive on $C^1 - C^2$; then λ is nonpositive on C^1 and nonnegative on C^2, so $\lambda \in (C^1)^\dagger \cap (-C^2)^\dagger$ and $\lambda \neq 0$, showing that (2) does not hold.

The implication (3)$\Rightarrow$(2) is trivial. To prove that (2)$\Rightarrow$(3), it suffices to show that if C^1 and C^2 are transversal but not strongly transversal then (a) holds. So assume that $C^1 \overline{\pitchfork} C^2$ but C^1 and C^2 are not strongly transversal. It then follows that $C^1 \cap C^2 = \{0\}$. We will show that C^1 and C^2 are linear subspaces. This will then imply—since $C^1 - C^2 = Y$ and $C^1 \cap C^2 = \{0\}$—that $C^1 + C^2 = Y$ and $C^1 \cap C^2 = \{0\}$, so $C^1 \oplus C^2 = Y$, establishing (a).

To prove that C^1 and C^2 are linear subspaces, we will just do it for C^1, the other case being similar. Since C^1 is a convex cone, it suffices to show that

$$c \in C^1 \Longrightarrow -c \in C^1\,.$$

Pick $c \in C^1$. Using the fact that $C^1 \overline{\pitchfork} C^2$, i.e., that $C^1 - C^2 = X$, express $-c$ as a difference $-c = c^1 - c^2$, $c^i \in C^i$. Then $c^1 + c = c^2$. But $c^1 + c \in C^1$ and $c^2 \in C^2$. Since $C^1 \cap C^2 = \{0\}$, we conclude that $c^1 + c = 0$. So $-c = c^1$, showing that $-c \in C^1$, and completing our proof. $\diamondsuit$

THEOREM 1.3.7. (The transversal intersection theorem for classical differentials). *Assume that Y is a finite-dimensional real linear space, S^1 and S^2 are subsets of Y, $s_* \in S^1 \cap S^2$, and C^1, C^2 are limiting approximating cones to S^1, S^2 at s_*. Assume that the cones C^1 and C^2 are strongly transversal. Then S^1 and S^2 are not locally separated at s_*.*

PROOF. Let $C^1 = \mathrm{Clos}\bigcup_{j=1}^{\infty} C_j^1$, $C^2 = \mathrm{Clos}\bigcup_{j=1}^{\infty} C_j^2$, where each C_j^i is an approximating cone to S^i at s_* and $C_j^i \subseteq C_{j+1}^i$ for all i, j. Fix a basis $(e_1, \ldots, e_m)$

of Y, and then let $e_0 = -(e_1 + \ldots + e_m)$. Using the fact that $C_1 \overline{\pitchfork} C_2$, find e^i_k, for $i = 1, 2$, $k = 0, \ldots, m$, such that $e^i_k \in C^i$ and $e^1_k - e^2_k = e_k$. Then find sequences $\{e^i_{k,j}\}_{j=1}^{\infty}$, for each i and each k, such that $e^i_{k,j} \in C^i_j$ for every i, j, k, and $\lim_{j\to\infty} e^i_{k,j} = e^i_k$. Let $e_{k,j} = e^1_{k,j} - e^2_{k,j}$. Then $\lim_{j\to\infty} e_{k,j} = e_k$ for $k = 0, \ldots, m$. Since 0 is an interior point of the convex hull of $e_0, \ldots, e_m$, it is clear that there exists a j^* such 0 is an interior point of the convex hull Σ_j of $e_{0,j}, \ldots, e_{m,j}$ if $j \geq j^*$. Since $\Sigma_j \subseteq C^1_j - C^2_j$, we see that $C^1_j \overline{\pitchfork} C^2_j$ if $j \geq j^*$.

We now claim that

(&) *There exists a $j^{\#}$ such that $j^{\#} \geq j^*$ and $C^1_j \overline{\pitchfork\!\!\!\pitchfork} C^2_j$ whenever $j \geq j^{\#}$.*

If (&) was not true, there would exist an infinite subset J of $\{j \in \mathbb{N} : j \geq j^*\}$ such that C^1_j is not strongly transversal to C^2_j for $j \in J$. In view of Proposition 1.3.6, since $C^1_j \overline{\pitchfork} C^2_j$ whenever $j \in J$, we can conclude that for every $j \in J$ the cones C^1_j and C^2_j are in fact linear subspaces and $C^1_j \oplus C^2_j = Y$. Since the sequences $\{C^1_j\}_{j=1}^{\infty}$, $\{C^2_j\}_{j=1}^{\infty}$, are increasing, we have $\bigcup_{j=1}^{\infty} C^1_j = \bigcup_{j\in J}^{\infty} C^1_j$ and $\bigcup_{j=1}^{\infty} C^2_j = \bigcup_{j\in J}^{\infty} C^2_j$. Furthermore, since the families $\{C^i_j\}_{j\in J}^{\infty}$ are increasing sequences of linear subspaces, and Y is finite-dimensional, there exists a $j^{\dagger}$ such that $\{C^i_j\} = C^i_{j^{\dagger}}$ whenever $j \in J$, $j \geq j^{\dagger}$, $i = 1, 2$. But then $\bigcup_{j=1}^{\infty} C^1_j = \bigcup_{j\in J} C^1_j = C^1_{j^{\dagger}}$ and $\bigcup_{j=1}^{\infty} C^2_j = \bigcup_{j\in J} C^2_j = C^2_{j^{\dagger}}$. It follows that $\bigcup_{j=1}^{\infty} C^1_j$ and $\bigcup_{j=1}^{\infty} C^2_j$ are linear subspaces of Y. In particular, these two sets are closed, and then $C^1 = \bigcup_{j\in J} C^1_j = C^1_{j^{\dagger}}$ and $C^2 = \bigcup_{j\in J} C^1_j = C^2_{j^{\dagger}}$. But then $C^1_{j^{\dagger}} \overline{\pitchfork\!\!\!\pitchfork} C^2_{j^{\dagger}}$, since $C^1 \overline{\pitchfork\!\!\!\pitchfork} C^2$. This contradicts the fact that $C^1_{j^{\dagger}} \oplus C^2_{j^{\dagger}} = Y$. This contradiction proves (&).

If we pick any j such that $j \geq j^{\#}$, and substitute C^1_j, C^2_j for C^1, C^2, then C^1, C^2 are approximating cones to S^1, S^2 at s_*, and in addition $C^1 \overline{\pitchfork\!\!\!\pitchfork} C^2$. In other words, we have shown that we may assume, without loss of generality, that

(&&) *C^1 and C^2 are approximating cones to S^1 and S^2 at s_*.*

From now on, we assume that (&&) holds, and also that $s_* = 0$.

Fix a neighborhood V of 0 in Y. For each $i \in \{1, 2\}$, choose a normed space X^i, a convex cone D^i in X^i, a neighborhood U^i of 0 in X^i, a continuous map $F^i : U^i \cap D^i \mapsto Y$, and a continuous linear map $L^i : X^i \mapsto Y$, such that $C^i = L \cdot D^i$, $F(0) = 0$, $L \in \mathcal{D}F^i(0)$, and $F^i(U^i \cap D^i) \subseteq S^i$. After shrinking the U^i, if necessary, we may assume that $F^i(U^i \cap D^i) \subseteq V$ for $i = 1, 2$.

Using the fact that $C^1 \overline{\pitchfork\!\!\!\pitchfork} C^2$, choose a linear functional $\mu : Y \mapsto \mathbb{R}$ and a vector $\bar{y} \in C^1 \cap C^2$ such that $\mu(\bar{y}) > 0$. Let $\alpha = \mu(\bar{y})$. Write $\bar{y} = L^1\bar{v}^1 = L^2\bar{v}^2$, with $\bar{v}^1 \in D^1$, $\bar{v}^2 \in D^2$. Let $U = U^1 \times U^2$, $D = D^1 \times D^2$. Define a map $F : U \cap D \mapsto Y \times \mathbb{R}$ by letting

$$F(x^1, x^2) = (F^1(x^1) - F^2(x^2), \mu(F^1(x^1))) \ \text{.if} \ \ x^1 \in U^1 \cap D^1\,, \ x^2 \in U^2 \cap D^2\,.$$

Then F is clearly continuous. A simple calculation shows that F is differentiable at $(0, 0)$, and a differential of F at $(0, 0)$ is the linear map $L : X^1 \times X^2 \mapsto Y \times \mathbb{R}$ given by

$$L \cdot (v^1, v^2) = (L^1 \cdot v^1 - L^2 \cdot v^2, \mu(L^1 \cdot v^1))$$

for $(v^1, v^2) \in X^1 \times X^2$. Given any $k \in \{0, \ldots, m\}$, the fact that $C^1 \overline{\pitchfork} C^2$ implies that we can find $(\tilde{v}^1_k, \tilde{v}^2_k) \in D$ such that $L^1 \cdot \tilde{v}^1_k - L^2 \cdot \tilde{v}^2_k = e_k$. Then $L \cdot (\tilde{v}^1_k, \tilde{v}^2_k) = (e_k, r_k)$ for some $r_k \in \mathbb{R}$. If $r \geq r$, then we can find a pair $(v^1_k(r), v^2_k(r)) \in D$ such that

$L\cdot(v_k^1(r), v_k^2(r)) = (e_k, r)$ by letting $v_k^1(r) = \tilde{v}_k^1 + \beta\bar{v}_1$, $v_k^2 = \tilde{v}_k^2 + \beta\bar{v}_2$, where $\beta = \frac{r - r_k}{\alpha}$. Let $\doteq \max(r_0, \ldots, r_m)$. Then we have shown that the vector (e_k, r) belongs to $L\cdot D$ if $k \in \{0, \ldots .m\}$ and $r \geq \bar{r}$. Let Σ be the convex hull of the set $\{e_0, \ldots, e_m\}$. Then $\Sigma \times [,\infty[\subseteq L\cdot D$. In particular, this implies that the vector $(0, \bar{r}+1)$ belongs to the interior $\mathrm{Int}(L\cdot D)$ of $L\cdot D$. Therefore $(0,1) \in \mathrm{Int}(L\cdot D)$ as well, and then Theorem 1.2.3 implies that there exists a positive number ρ such that $(0,r) \in F(U\cap D)$ whenever $0 \leq r \leq \rho$. Pick any r such that $0 < r \leq \rho$. Then $(0,r) \in F(U \cap D)$, so there exist $x^1 \in U^1 \cap D^1$, $x^2 \in U^2 \cap D^2$, such that $F^1(x^1) - F^2(x^2) = 0$ and $\mu(F^1(x^1)) = r$. If we let $y = F^1(x^1) = F^2(x^2)$, then $y \in S^1 \cap S^2$. Since $\mu(y) = r \neq 0$, y cannot be the origin of Y. Since $y \in F^1(U^1 \cap D^1) \subseteq V$, y satisfies $y \in V$. Since V is arbitrary, we have shown that $S^1 \cap S^2$ contains nonzero points arbitrarily close to the origin, which is the desired conclusion. ♢

1.4. The linear separation theorem for the classical differential.

THEOREM 1.4.1. (The linear separation theorem for the classical differential). *Let Y be a finite-dimensional real normed space. Let S^1, S^2 be subsets of Y, let $s_* \in S^1 \cap S^2$, and let C^1, C^2 be limiting approximating cones to S^1, S^2 at s_*. Assume that the sets S^1, S^2 are locally separated at s_*. Then either*

(a) *C^1 and C^2 are linear subspaces and $Y = C^1 \oplus C^2$*

or

(b) *C^1 and C^2 are linearly separated.*

PROOF. Assume that both (a) and (b) are false. Then Proposition 1.3.6 implies that either (a) holds or $C^1 \pitchfork C^2$. But we are assuming that (a) is false. So $C^1 \pitchfork C^2$. Then Theorem 1.4.1 tells us that S^1 and S^2 are not locally separated at s_*, and we have reached a contradiction. ♢