NOTATIONS AND DEFINITIONS:

- $\mathbb{R}^n$ is the space of all real *column* vectors $x = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix}$.
- $\mathbb{R}_n$ is the space of all real *row* vectors $p = (p_1, p_2, \ldots, p_n)$.
- $\mathbb{R}^{m\times n}$ is the space of all real *matrices* with m columns and n rows.
- Given a metric space U, a *U-control* is a function $[a,b] \ni t \mapsto \eta(t) \in U$ whose domain is a compact interval.
- A U-control $[a,b] \ni t \mapsto \eta(t) \in U$ is *piecewise continuous* if there exists a finite subset F of the open interval $]a,b[$ such that η is continuous at all points t of $[a,b]\backslash F$ and the left and right limits of $\eta(t)$ as t approaches τ exist for all $\tau \in F$. Notice that

  $$\boxed{\text{\textit{The closure of the graph} Gr}(\eta) \text{ \textit{of a piecewise continuous U-control η is a compact subset of} } U \times \mathbb{R}}.$$

  We use $C^{0,pc}([a,b],U)$ to denote the set of all piecewise continuous U-controls.
- A *trajectory* for a piecewise continuous control $\eta : [a,b] \mapsto U$ is a continuous map $\xi : [a,b] \mapsto \Omega$ such that

$$\dot{\xi}(t) = f(\xi(t), \eta(t), t) \tag{0.1}$$

  for all but finitely many points $t \in [a,b]$. Equivalently, a continuous map $\xi : [a,b] \mapsto \Omega$ is a trajectory for η if and only if

  $$\xi(t) - \xi(s) = \int_s^t f(\xi(r), \eta(r), r)\, dr \qquad \text{for all } s,t \in [a,b]\,.$$
- A *trajectory-control pair* for $(\Omega, U, \mathcal{U}, f)$ is a pair (ξ, η) such that $\eta \in \mathcal{U}$ and ξ is a trajectory for η.
- $TCP(\Omega, U, \mathcal{U}, f)$ is the set of all trajectory-control pairs for the data $(\Omega, U, \mathcal{U}, f)$.
- A *cone* in a real linear space X is a nonempty subset C of X such that $(\forall c \in C)(\forall r \in \mathbb{R})(r \geq 0 \Rightarrow rc \in C)$.
- If X is a normed real linear space (or, more generally, if X is a real topological linear space), and C is a cone in X, the *polar* of C is the set

  $$C^\dagger \stackrel{\text{def}}{=} \{p \in X^\dagger : p(c) \leq 0 \text{ for all } c \in C\}\,,$$

  where $X^\dagger$ is the dual of X, i.e., the space of all continuous linear functionals $p : X \mapsto \mathbb{R}$.
- For each function $h : \Omega \times \mathbb{R}_n \mapsto \mathbb{R}$ of class C^1, $\vec{h}$ is the *Hamilton vector field* corresponding to h, defined by

  $$\vec{h} = \sum_{k=1}^{n} \left(\frac{\partial h}{\partial p_k} \frac{\partial}{\partial x^k} - \frac{\partial h}{\partial x^k} \frac{\partial}{\partial p_k} \right).$$

ASSUMPTIONS FOR THEOREM OC-FT:

(A1) Ω is an open subset of $\mathbb{R}^n$.
(A2) U is a metric space.
(A3) $a, b \in \mathbb{R}$ and $a < b$.
(A4) $\Omega \times U \times [a,b] \ni (x,u,t) \mapsto f(x,u,t) \in \mathbb{R}^n$ and $\Omega \times U \times [a,b] \ni (x,u,t) \mapsto L(x,u,t) \in \mathbb{R}$ are continuous maps such that
  (A4.a) the maps $\Omega \ni x \mapsto f(x,u,t) \in \mathbb{R}^n$ and $\Omega \ni x \mapsto L(x,u,t) \in \mathbb{R}$ are of class C^1 for each fixed (u,t),
  (A4.b) the maps $\Omega \times U \times [a,b] \ni (x,u,t) \mapsto \frac{\partial f}{\partial x}(x,u,t) \in \mathbb{R}^{n\times n}$ and $\Omega \times U \times [a,b] \ni (x,u,t) \mapsto \frac{\partial L}{\partial x}(x,u,t) \in\in \mathbb{R}_n$ are continuous.
(A5) $\mathcal{U} = C^{0,pc}([a,b],U)$.
(A6) S is a subset of Ω.
(A7) $(\xi_*, \eta_*) \in TCP(\Omega, U, \mathcal{U}, f)$ and $\xi_*(b) \in S$.
(A8) C is a tangent cone of S at $\xi_*(b)$.
(A9) $J(\xi_*, \eta_*) \le J(\xi, \eta)$ for all $(\xi, \eta) \in TCP(\Omega, U, \mathcal{U}, f)$ such that $\xi(b) \in S$, where

$$J(\xi,\eta) \stackrel{\text{def}}{=} \int_a^b L(\xi(t), \eta(t), t)\, dt\,.$$

THEOREM OC-FT: ***(Optimal control theorem, fixed time.)*** *Assume that (A1,2,3,4,5,6,7,8,9) hold. Define the* ***Hamiltonian***

$$\Omega \times \mathbb{R}_n \times U \times \mathbb{R} \times [a,b] \ni (x,p,u,p_0,t) \mapsto H(x,p,u,p_0,t) \in \mathbb{R}$$

by letting

$$H(x,p,u,p_0,t) \stackrel{\text{def}}{=} p \cdot f(x,u,t) - p_0(x,u,t)\,.$$

Then there exists a pair (π, π_0) *such that*

(#) $\pi : [a,b] \mapsto \mathbb{R}_n$ *is a continuous map and* $\pi_0 \in \mathbb{R}$;
(AE) *(the "adjoint equation")*

$$\dot{\pi}(t) = -\pi(t) \cdot \frac{\partial f}{\partial x}(\xi_*(t), \eta_*(t), t) + \pi_0 \frac{\partial L}{\partial x}(\xi_*(t), \eta_*(t), t) \tag{0.2}$$

at all points of continuity t *of* η_*;
(HM) *(the "Hamiltonian maximization condition")*

$$H(\xi_*(t), \pi(t), \eta_*(t), \pi_0, t) = \max\Big\{H(\xi_*(t), \pi(t), u, \pi_0, t) : u \in U\}$$

at all points of continuity t *of* η_*;
(TC) *(the "transversality condition")*

$$\pi(b) \in -C^\dagger\,;$$

(NT) *(the "nontriviality condition")* $(\pi(t), \pi_0) \ne (0,0)$ *for some* t *(which implies that* $(\pi(t), \pi_0) \ne (0,0)$ *for all* t*);*
(NN) *(the "nonnegativtiy condition")* $\pi_0 \ge 0$.

REMARK: The adjoint equation (0.2) can also be written as

$$\dot{\pi}(t) = -\frac{\partial H}{\partial x}(\xi_*(t), \pi(t), \eta_*(t), \pi_0, t)\,. \tag{0.3}$$

The dynamical equation (0.1) is equivalent to

$$\dot{\xi}_*(t) = \frac{\partial H}{\partial p}(\xi_*(t), \pi(t), \eta_*(t), \pi_0, t)\,. \tag{0.4}$$

The pair of equations (0.4), (0.3) is equivalent to the single equation

$$\dot{\Theta}(t) = \vec{H}_{\eta_*(t),\pi_0,t}(\Theta(t))\,, \tag{0.5}$$

evolving in $\Omega \times \mathbb{R}_n$, i.e., (x,p) space (the cotangent bundle of Ω), where

$$\Theta(t) = (\xi_*(t), \pi(t))\,,$$

and

$$H_{u,p_0,t}(x,p) \stackrel{\text{def}}{=} H(x,p,u,p_0,t)\,.$$

THE INTRINSIC FORM OF THE ADJOINT EQUATION:
The adjoint equation (0.2) is equivalent to the following statement:

For every vector field X of class C^1 on Ω, the equality

$$\frac{d}{dt}\Big(\pi(t)\cdot X(\xi_*(t))\Big) = \pi(t)\cdot[f_{\eta_*(t),t}, X](\xi_*(t)) + \pi_0 X L_{\eta_*(t),t}(\xi_*(t))$$

holds, where

$$f_{u,t}(x) \stackrel{\text{def}}{=} f(x,u,t) \quad \text{and} \quad L_{u,t}(x) \stackrel{\text{def}}{=} L(x,u,t)\,.$$

ASSUMPTIONS FOR THEOREM OC-VT:

- ($\tilde{\text{A}}$1) Ω is an open subset of $\mathbb{R}^n$.
- ($\tilde{\text{A}}$2) U is a metric space.
- ($\tilde{\text{A}}$3) $T_* \in \mathbb{R}$ and $T_* > 0$.
- ($\tilde{\text{A}}$4) $\Omega \times U \ni (x,u) \mapsto f(x,u) \in \mathbb{R}^n$ and $\Omega \times U \ni (x,u) \mapsto L(x,u) \in \mathbb{R}$ are continuous maps such that
  - ($\tilde{\text{A}}$4.a) the maps $\Omega \ni x \mapsto f(x,u) \in \mathbb{R}^n$ and $\Omega \ni x \mapsto L(x,u) \in \mathbb{R}$ are of class C^1 for each fixed u,
  - ($\tilde{\text{A}}$4.b) the maps $\Omega \times U \ni (x,u) \mapsto \frac{\partial f}{\partial x}(x,u) \in \mathbb{R}^{n\times n}$ and $\Omega \times U \ni (x,u,t) \mapsto \frac{\partial L}{\partial x}(x,u) \in\in \mathbb{R}_n$ are continuous.
- ($\tilde{\text{A}}$5) $\alpha > 0$ and $\mathcal{U} = \bigcup_{T_*-\alpha\le T\le T_*+\alpha} C^{0,pc}([0,T],U)$.
- ($\tilde{\text{A}}$6) S is a subset of Ω.
- ($\tilde{\text{A}}$7) $(\xi_*,\eta_*) \in TCP(\Omega,U,\mathcal{U},f)$, $\eta_* \in C^{0,pc}([0,T_*],U)$, and $\xi_*(T_*) \in S$.
- ($\tilde{\text{A}}$8) C is a tangent cone to S at $\xi_*(T_*)$.
- ($\tilde{\text{A}}$9) $J(\xi_*,\eta_*) \le J(\xi,\eta)$ for all $(\xi,\eta) \in TCP(\Omega,U,\mathcal{U},f)$ such that $\xi(b) \in S$, where

$$J(\xi,\eta) \stackrel{\text{def}}{=} \int_0^T L(\xi(t),\eta(t),t)\,dt \text{ if } \eta \in C^{0,pc}([0,T],U)\,..$$

THEOREM OC-VT: ***(Optimal control theorem, variable time.)*** *Assume that ($\tilde{\text{A}}$1,2,3,4,5,6,7,8,9) hold. Define the* ***Hamiltonian*** $\Omega\times\mathbb{R}_n\times U\times\mathbb{R} \ni (x,p,u,p_0,t) \mapsto H(x,p,u,p_0) \in \mathbb{R}$ *by letting*

$$H(x,p,u,p_0) \stackrel{\text{def}}{=} p\cdot f(x,u) - p_0(x,u)\,.$$

Then there exists a pair (π,π_0) such that

- (#) $\pi : [a,b] \mapsto \mathbb{R}_n$ *is a continuous map and* $\pi_0 \in \mathbb{R}$*;*

(AE) *(the "adjoint equation")*

$$\dot{\pi}(t) = -\pi(t) \cdot \frac{\partial f}{\partial x}(\xi_*(t), \eta_*(t)) + \pi_0 \frac{\partial L}{\partial x}(\xi_*(t), \eta_*(t)) \tag{0.6}$$

at all points of continuity t of η_*;

(HM) *(the "Hamiltonian maximization condition")*

$$H(\xi_*(t), \pi(t), \eta_*(t), \pi_0) = \max\Big\{H(\xi_*(t), \pi(t), u, \pi_0) : u \in U\}$$

at all points of continuity t of η_*;

(ZH) *(the "zero Hamiltonian condition")*

$$H(\xi_*(t), \pi(t), \eta_*(t), \pi_0) = 0$$

at all points of continuity t of η_*;

(TC) *(the "transversality condition")*

$$\pi(b) \in -C^\dagger\,;$$

(NT) *(the "nontriviality condition")* $(\pi(t), \pi_0) \neq (0, 0)$ *for some t (which implies that $(\pi(t), \pi_0) \neq (0, 0)$ for all t);*

(NN) *(the "nonnegativtiy condition")* $\pi_0 \geq 0$.

ASSUMPTIONS FOR THEOREM SEP-FT:

(S1) Ω is an open subset of $\mathbb{R}^n$.

(S2) U is a metric space.

(S3) $a, b \in \mathbb{R}$ and $a < b$.

(S4) $\Omega \times U \times [a, b] \ni (x, u, t) \mapsto f(x, u, t) \in \mathbb{R}^n$ is a continuous map such that

  (S4.a) the map $\Omega \ni x \mapsto f(x, u, t) \in \mathbb{R}^n$ is of class C^1 for each fixed (u, t),

  (S4.b) the map $\Omega \times U \times [a, b] \ni (x, u, t) \mapsto \frac{\partial f}{\partial x}(x, u, t) \in \mathbb{R}^{n\times n}$ is continuous.

(S5) $\mathcal{U} = C^{0,pc}([a, b], U)$.

(S6) S is a subset of Ω.

(S7) $(\xi_*, \eta_*) \in TCP(\Omega, U, \mathcal{U}, f)$ and $\xi_*(b) \in S$.

(S8) C is a tangent cone of S at $\xi_*(b)$.

(S9) $\mathcal{R}_{\Omega,U,\mathcal{U},f}(\xi_*(a)) \cap S = \{\xi_*(b)$, where

$$\mathcal{R}_{\Omega,U,\mathcal{U},f}(\xi_*(a)) \stackrel{\text{def}}{=} \Big\{\xi(b) : (\xi, \eta) \in TCP(\Omega, U, \mathcal{U}, f)\Big\}\,.$$

THEOREM SEP-FT: ***(Separation theorem, fixed time.)*** *Assume that conditions (S1,2,3,4,5,6,7,8,9) hold. Define the* ***Hamiltonian***

$$\Omega \times \mathbb{R}_n \times U \times \mathbb{R} \times [a, b] \ni (x, p, u, p_0, t) \mapsto H(x, p, u, t) \in \mathbb{R}$$

by letting

$$H(x, p, u, t) \stackrel{\text{def}}{=} p \cdot f(x, u, t)\,.$$

Then there exists a π such that

(#) $\pi : [a, b] \mapsto \mathbb{R}_n$ *is a continuous map;*

(AE) *(the "adjoint equation")*

$$\dot{\pi}(t) = -\pi(t) \cdot \frac{\partial f}{\partial x}(\xi_*(t), \eta_*(t), t) \tag{0.7}$$

at all points of continuity t of η_*;

(HM) *(the "Hamiltonian maximization condition")*

$$H(\xi_*(t),\pi(t),\eta_*(t),t)=\max\Big\{H(\xi_*(t),\pi(t),u,t):u\in U\Big\}$$

at all points of continuity t of η_*;

(TC) *(the "transversality condition")*

$$\pi(b)\in -C^{\dagger}\,;$$

(NT) *(the "nontriviality condition") $\pi(t)\neq 0$ for some t (which implies that $\pi(t)\neq 0$ for all t);*

REMARK: The adjoint equation (0.7) can also be written as

$$\dot\pi(t)=-\frac{\partial H}{\partial x}(\xi_*(t),\pi(t),\eta_*(t),t)\,. \tag{0.8}$$

The dynamical equation (0.1) is equivalent to

$$\dot\xi_*(t)=\frac{\partial H}{\partial p}(\xi_*(t),\pi(t),\eta_*(t),t)\,. \tag{0.9}$$

The pair of equations (0.9), (0.8) is equivalent to the single equation

$$\dot\Theta(t)=\vec H_{\eta_*(t),t}(\Theta(t))\,, \tag{0.10}$$

evolving in $\Omega\times\mathbb{R}_n$, i.e., (x,p) space (the cotangent bundle of Ω, where $\Theta(t)=(\xi_*(t),\pi(t))$, $H_{u,t}(x,p)\stackrel{\mathrm{def}}{=}H(x,p,u,t)$,

THE INTRINSIC FORM OF THE ADJOINT EQUATION:

The adjoint equation (0.7) is equivalent to the following statement:

For every vector field X of class C^1 on Ω, the equality

$$\frac{d}{dt}\Big(\pi(t)\cdot X(\xi_*(t))\Big)=\pi(t)\cdot[f_{\eta_*(t),t},X](\xi_*(t))$$

holds, where

$$f_{u,t}(x)\stackrel{\mathrm{def}}{=}f(x,u,t)\,.$$

ASSUMPTIONS FOR THEOREM SEP-VT:

($\tilde{\rm S}$1) Ω is an open subset of $\mathbb{R}^n$.
($\tilde{\rm S}$2) U is a metric space.
($\tilde{\rm S}$3) $T_*\in\mathbb{R}$ and $T_*>0$.
($\tilde{\rm S}$4) $\Omega\times U\ni(x,u)\mapsto f(x,u)\in\mathbb{R}^n$ is a continuous map such that
  ($\tilde{\rm S}$4.a) the map $\Omega\ni x\mapsto f(x,u)\in\mathbb{R}^n$ is of class C^1 for each fixed u,
  ($\tilde{\rm S}$4.b) the map $\Omega\times U\ni(x,u)\mapsto\frac{\partial f}{\partial x}(x,u)\in\mathbb{R}^{n\times n}$ is continuous.
($\tilde{\rm S}$5) $\alpha>0$ and $\mathcal{U}=\bigcup_{T_*-\alpha\le T\le T_*+\alpha}C^{0,pc}([0,T],U)$.
($\tilde{\rm S}$6) S is a subset of Ω.
($\tilde{\rm S}$7) $\eta_*\in C^{0,pc}([0,T_*],U)$, $(\xi_*,\eta_*)\in TCP(\Omega,U,\mathcal{U},f)$, and $\xi_*(T_*)\in S$.
(S8) C is a tangent cone of S at $\xi_*(b)$.

(S9) $\mathcal{R}_{\Omega,U,\mathcal{U},f}(\xi_*(0)) \cap S = \{\xi_*(T)$, where

$$\mathcal{R}_{\Omega,U,\mathcal{U},f}(\xi_*(0)) \stackrel{\text{def}}{=} \Big\{\xi(T) : \eta \in C^{0,pc}([0,T],U), (\xi,\eta) \in TCP(\Omega,U,\mathcal{U},f)\Big\}\,.$$

THEOREM SEP-VT: ***(Separation theorem, variable time.)*** *Assume that ($\tilde{\text{S}}$1,2,3,4,5,6,7,8,9) hold. Define the* ***Hamiltonian***

$$\Omega \times \mathbb{R}_n \times U \times \mathbb{R} \times [a,b] \ni (x,p,u,p_0,t) \mapsto H(x,p,u) \in \mathbb{R}$$

by letting

$$H(x,p,u) \stackrel{\text{def}}{=} p \cdot f(x,u)\,.$$

Then there exists a π such that

(#) $\pi : [a,b] \mapsto \mathbb{R}_n$ *is a continuous map;*

(AE) *(the "adjoint equation")*

$$\dot{\pi}(t) = -\pi(t) \cdot \frac{\partial f}{\partial x}(\xi_*(t), \eta_*(t)) \tag{0.11}$$

at all points of continuity t of η_*;

(HM) *(the "Hamiltonian maximization condition")*

$$H(\xi_*(t), \pi(t), \eta_*(t)) = \max\Big\{H(\xi_*(t), \pi(t), u) : u \in U\}$$

at all points of continuity t of η_*;

(ZH) *(the "zero Hamiltonian condition")*

$$H(\xi_*(t), \pi(t), \eta_*(t)) = 0$$

at all points of continuity t of η_*;

(TC) *(the "transversality condition")*

$$\pi(b) \in -C^\dagger\,;$$

(NT) *(the "nontriviality condition") $\pi(t) \neq 0$ for some t (which implies that $\pi(t) \neq 0$ for all t).*

$$\boxed{\begin{array}{ccc} \text{Theorem SEP}-\text{FT} & \Longrightarrow & \text{Theorem SEP}-\text{VT} \\ \Downarrow & & \Downarrow \\ \text{Theorem OC}-\text{FT} & \Longrightarrow & \text{Theorem OC}-\text{VT} \end{array}}$$