SIAM J. CONTROL OPTIM.
Vol. 35, No. 2, pp. 562–584, March 1997
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BRACHISTOCHRONE WITH COULOMB FRICTION*

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Abstract. This paper formulates and solves in closed form the problem of finding the minimum-time path of a particle between two points in a uniform gravitational field when motion of the particle is resisted by a force proportional to the normal force exerted on the particle by the path. This resistance to motion is the common mathematical form for Coulomb friction. The problem solution involves the reformulation of the classical brachistochrone of Bernoulli in terms of a singular control problem in which the time derivative of the heading angle of the particle is the control parameter. As such, this solution provides a unique approach to the solution of minimum-time path problems.

Key words. brachistochrone, singular control, extremal control, minimum-time control, corner conditions, optimal control

AMS subject classifications. Primary, 49K15, 49K30; Secondary, 49J15, 49J30

PII. S0363012995287957

1. Introduction.

Minimum-time optimal control problems are a common research area in analytical and numerical control system synthesis. Current research in robotics and automation is fraught with minimum-time optimization problems [4, 8, 16, 20]. Some of the earliest work in optimal control theory was concerned with minimum-time optimization [3]. Perhaps the earliest problem proposed in minimum-time optimization was the brachistochrone of Bernoulli [1]. This problem may be stated as follows: A bead slides on a frictionless wire between points A and B in a constant-gravity field. The bead has an initial speed V_0 at point A. What is the shape of the wire that will produce a minimum-time path between the two points? Assuming the positive y axis points upward with $A = (x_0, y_0)$ and $B = (0, 0)$, the optimal path is a portion of a cycloid [2], the path generated by a point on the circumference of a circle as it rolls in the direction of the x axis. The minimum-time solution in this case requires that the time rate of change of heading angle of the bead is constant [2].

Euler proposed an extension to this problem. As quoted from Goldstine [7], Euler "...takes up the elegant problem of finding the shape of the brachystochrone [sic] curve in case the medium through which the heavy particle falls resists the motion depending only on the velocity v. Here he assumes that the resistance function R is proportional to v^{2n}." Euler's solution to this problem is an implicit solution [5]. Due to the fact that the friction is proportional to a function of the velocity only, this problem may be solved using the calculus of variations.

Current research into the brachistochrone has been primarily focused on different approaches to solving the classical brachistochrone problem. For example, Roomany [19] uses a graph theoretical approach to solve the classical brachistochrone problem. This technique is efficient and convergent but is only useful if the field through which the particle moves is derivable from a potential. In a similar manner, Razzaghi and Elnagar solve the classical brachistochrone problem using interpolating polynomials of appropriate degree [18]. Szarkowicz approaches the solution of the brachistochrone

*Received by the editors June 19, 1995; accepted for publication (in revised form) February 4, 1996. A version of this paper was presented at the 35th IEEE Conference on Decision and Control, Kobe, Japan, December 11–13, 1996.

http://www.siam.org/journals/sicon/35-2/28795.html

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problem with use of multistage Monte Carlo methods [22]. Some extensions of the classical brachistochrone problem have been discussed in the current literature as well. Hoskins extended and solved the classical brachistochrone problem, where the change in direction is allowed only along circles of a given radius R [9]. Perlick solved the classical brachistochrone problem extended to a stationary Lorentzian space-time [14].

The present paper examines the brachistochrone problem in which the friction force on the particle is a resistance proportional to the force exerted by the particle by its path (the normal force). This formulation is the equivalent of determining minimum-time paths in a uniform gravitational field with Coulomb friction resisting the motion. With this choice of motion resistance, the penalty for curvature in the path is high. In fact, the dynamics to be derived will demand that there be no abrupt changes in curvature. Therefore, if the optimal control problem is reformulated with an intermediate point as the initial point and the corresponding speed at that point as the initial speed, a *different* optimal path will result. This again differs from the classical brachistochrone result in that the position and speed at any point on an optimal cycloidal path determine the cycloid. It is the position and *velocity* which uniquely determine the time-optimal path when Coulomb friction is present in the brachistochrone problem.

A discretization of this problem was formulated and solved using a Davidon–Fletcher–Powell algorithm in 1990 [17]. A number of important points in this result were left unsolved, including the question of whether a solution exists and how close a solution is to the actual minimum-time solution. Some interesting features of the minimum-time path were not illustrated in the discretized solution. This paper will present the closed-form solution for the minimum-time path. Therefore, it will demonstrate the validity of the discretized approach and provide further insight into the limitations of that generic approach in its application to this problem.

2. Optimal control formulation. The classical brachistochrone problem may be formulated as an optimal control problem with the heading angle θ as the control variable. In the formulation of any optimal control problem, the first step is the formation of the scalar Hamiltonian [15]. The Hamiltonian is defined by

$$\mathcal{H} = \langle \boldsymbol{\lambda}, \boldsymbol{f}(\boldsymbol{x}, u) \rangle + L(\boldsymbol{x}, u), \tag{2.1}$$

where $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, u)$ are the state dynamics, $\dot{\boldsymbol{\lambda}} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{x}}$ are the "costate" dynamics, $L(\boldsymbol{x}, u)$ is the scalar function whose time integral is to be minimized, u is the control variable, and $\langle \cdot, \cdot \rangle$ denotes the scalar inner (dot) product. For minimum-time control problems, the scalar cost function $L(\boldsymbol{x}, u) \equiv 1$, since the minimization of the time integral of one corresponds to the minimization of the final time.

Pontryagin's "minimum" principle [15] states that the optimal control is the control u which minimizes the Hamiltonian almost everywhere. When the Hamiltonian is convex in the control variable u, the optimal control necessarily satisfies Euler's condition (the first variation) $\frac{\partial \mathcal{H}}{\partial u} = 0$ and sufficiency is guaranteed with the "strict" Legendre–Clebsch condition (the second variation) $\frac{\partial^2 \mathcal{H}}{\partial u^2} > 0$. The strict Legendre–Clebsch condition is guaranteed in the case of the classical brachistochrone.

The state dynamics for the standard brachistochrone may be derived in x and y coordinates using Newton's laws. Assume that positive x is to the right and positive y is up. Additionally, assume that a first quadrant heading angle θ corresponds to negative x-speed and y-speed. The particle speed is denoted by V. A diagram of the geometry is shown in Figure 2.1.

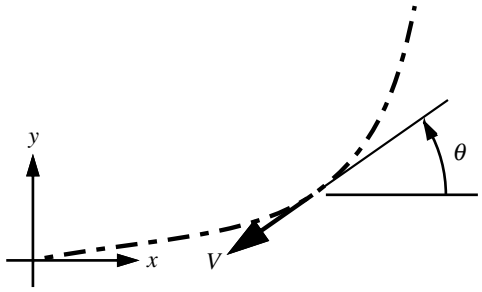

FIG. 2.1. *State-variable geometry.*

Using dimensionless variables

$$\begin{aligned} \dot{x} &= -V\cos\theta, \\ \dot{y} &= -V\sin\theta, \\ \dot{V} &= \frac{1}{2}\sin\theta\,. \end{aligned} \tag{2.2}$$

The dimensionless variables chosen were

$$x = \frac{x^*}{x^*(0)}, \qquad y = \frac{y^*}{x^*(0)}, \qquad V = \frac{V^*}{\sqrt{2gx^*(0)}}, \qquad t = t^*\sqrt{\frac{2g}{x^*(0)}}\,, \tag{2.3}$$

where the superscript $*$ indicates dimensional variables. Treating θ as the control variable, the initial and final conditions on the states are

$$x(0) = 1, \quad y(0) = y_0, \quad V(0) = V_0, \quad x(t_f) = 0, \quad y(t_f) = 0\,. \tag{2.4}$$

Setting

$$\boldsymbol{\lambda} = \left\{ \begin{array}{c} \lambda_x \\ \lambda_y \\ \lambda_V \end{array} \right\}, \tag{2.5}$$

the Hamiltonian may be written as

$$\mathcal{H} = -\lambda_x V\cos\theta - \lambda_y V\sin\theta + \frac{1}{2}\lambda_V\sin\theta + 1\,. \tag{2.6}$$

The costate dynamics as obtained from $\dot{\boldsymbol{\lambda}} = -\frac{\partial\mathcal{H}}{\partial \boldsymbol{x}}$ are

$$\begin{aligned} \dot{\lambda}_x &= \dot{\lambda}_y = 0, \\ \dot{\lambda}_V &= \lambda_x\cos\theta + \lambda_y\sin\theta\,. \end{aligned} \tag{2.7}$$

With the state boundary conditions given, the costates $\boldsymbol{\lambda}$ have free (unknown) boundary conditions except at the boundary where $\boldsymbol{x}$ is unconstrained. Thus the only boundary condition satisfied by $\boldsymbol{\lambda}$ is

$$\lambda_V(t_f) = 0\,. \tag{2.8}$$

This means that there are six differential equations and seven unknowns ($V(t_f)$, $\lambda_x(0)$, $\lambda_y(0)$, $\lambda_V(0)$, $\lambda_x(t_f)$, $\lambda_y(t_f)$, and t_f).

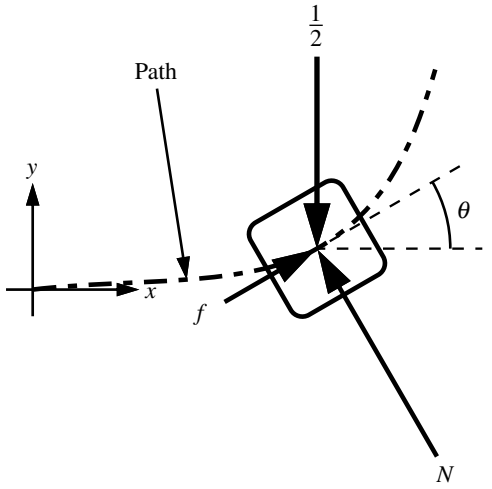

FIG. 2.2. *Free-body diagram of bead on path with Coulomb friction resisting motion.*

The final condition for determination of the optimal control is the transversality condition [15], which states that

$$\mathcal{H}(t_f) = \frac{\partial L(\boldsymbol{x}, u)}{\partial t} = 0\,. \tag{2.9}$$

With the Hamiltonian having no explicit time dependence, this means that $\mathcal{H}(t) \equiv 0$. Simple integration demonstrates that the first and second variation conditions hold for a cycloidal path. Therefore, the cycloidal path is the optimal minimum-time path for control.

When Coulomb friction is added as a resisting force to the motion, the dimensionless equations of motion may be determined from a free-body diagram of the particle (see Figure 2.2). In the figure, the dimensionless normal force N and dimensionless friction force f are given by

$$N = \frac{N^*}{2mg} \qquad \text{and} \qquad f = \frac{f^*}{2mg}\,. \tag{2.10}$$

Coulomb friction resistance has the constitutive law $f = \mu|N|$, where μ is the coefficient of friction. The normal force exerted by the path on the particle is equal to the component of the weight of the particle (1/2 in dimensionless terms) in the direction of the normal force minus the mass of the particle multiplied by the centripetal acceleration caused by the particle's speed and the curvature of the path. In dimensionless terms, this is $V\dot{\theta}$. Therefore, the dimensionless equations of motion are

$$\begin{aligned} \dot{x} &= -V\cos\theta, \\ \dot{y} &= -V\sin\theta, \\ \dot{V} &= \frac{1}{2}\sin\theta - \mu\left|\frac{1}{2}\cos\theta - V\dot{\theta}\right|\,. \end{aligned} \tag{2.11}$$

Unlike the classical or Euler's brachistochrone problem, $\dot{\theta}$ has entered the dynamics. In order to formulate the problem as an optimal control problem, θ will be adjoined as an extra state with $\dot{\theta} = \Omega$ as the control variable.

This formulation is troublesome due to the fact that the control objective and the state dynamics are not convex in the control variable. The second derivative of the Hamiltonian with respect to the control being zero is not sufficient for optimality. In fact, Euler's necessary condition is not satisfied initially for nonzero initial velocity. Recourse to Pontryagin's minimum principle yields extremal control initially but yields no information about the control when the system equations simplify with Euler's condition being satisfied. This problem requires the theory of singular control [10, 11].

When the partial derivative of the Hamiltonian with respect to the control variable equals zero but yields a relation in which there is no explicit control variable, then a singular control solution may exist for the system. Singular control methods yield the following constraint equations for determining the control:

$$\begin{aligned} &\frac{d^i}{dt^i}\left[\frac{\partial \mathcal{H}}{\partial u}\right] = 0, \qquad i = 0, 1, \ldots, m-1, \\ &\frac{d^m}{dt^m}\left[\frac{\partial \mathcal{H}}{\partial u}\right] = g(\boldsymbol{x}, \boldsymbol{\lambda}, u) = 0. \end{aligned} \tag{2.12}$$

The first $m-1$ derivatives are independent of the control u. The mth derivative determines the control u as a function of the states/costates. A result obtainable from this theory is [11]: (1) m is always even; (2) if the number of states is n, the singular control restricts state/costate space to dimension $2n-m$ (the singular hypersurface); and (3) it is necessary that

$$(-1)^{\frac{m}{2}} \frac{\partial}{\partial u}\left(\frac{d^m}{dt^m}\left[\frac{\partial \mathcal{H}}{\partial u}\right]\right) \geq 0\,.$$

The final condition is a generalized Legendre–Clebsch condition, a *necessary* condition for optimality. In general there are no sufficiency conditions for optimality of a singular control arc. Even the strict inequality does not guarantee optimality. Nevertheless, this method will prove invaluable in generating control trajectories for the brachistochrone with Coulomb friction. Furthermore, since the classical brachistochrone can be derived as a singular control problem and the Coulomb friction brachistochrone simplifies to the classical brachistochrone when the coefficient of friction $\mu \to 0$, the necessarily optimal path derived by singular control methods is sufficient in the limit.

3. Reformulation of the classical brachistochrone problem. The classical brachistochrone problem can also be formulated as a singular control problem where the control variable is Ω, the time rate of change of the slope of the time-optimal path. In this case, the state equations are

$$\begin{aligned} \dot{x} &= -V\cos\theta, \\ \dot{y} &= -V\sin\theta, \\ \dot{V} &= \frac{1}{2}\sin\theta, \\ \dot{\theta} &= \Omega\,. \end{aligned} \tag{3.1}$$

The dimensionless Hamiltonian is thus

$$\mathcal{H} = -\lambda_x V\cos\theta - \lambda_y V\sin\theta + \frac{1}{2}\lambda_V \sin\theta + \lambda_\theta \Omega + 1\,. \tag{3.2}$$

The costate equations are

$$\begin{aligned}
\dot{\lambda}_x &= -\frac{\partial \mathcal{H}}{\partial x} = 0,\\
\dot{\lambda}_y &= -\frac{\partial \mathcal{H}}{\partial y} = 0,\\
\dot{\lambda}_V &= -\frac{\partial \mathcal{H}}{\partial V} = \lambda_x \cos\theta + \lambda_y \sin\theta,\\
\dot{\lambda}_\theta &= -\frac{\partial \mathcal{H}}{\partial \theta} = V(\lambda_y \cos\theta - \lambda_x \sin\theta) - \frac{1}{2}\,\lambda_V \cos\theta
\end{aligned} \tag{3.3}$$

with the boundary conditions

$$\lambda_V(t_f) = 0, \qquad \lambda_\theta(0) = \lambda_\theta(t_f) = 0, \qquad \mathcal{H}(t_f) = 0\,. \tag{3.4}$$

With the Hamiltonian being linear in the control variable Ω, the optimal control will be singular control. The singular control is determined with the following conditions:

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial \Omega} &= \lambda_\theta = 0,\\
\frac{d}{dt}\,\frac{\partial \mathcal{H}}{\partial \Omega} &= V(\lambda_y \cos\theta - \lambda_x \sin\theta) - \frac{1}{2}\,\lambda_V \cos\theta = 0,\\
\frac{d^2}{dt^2}\,\frac{\partial \mathcal{H}}{\partial \Omega} &= -\frac{1}{2}\,\lambda_x + \left(\frac{1}{2}\,\lambda_V \sin\theta - V(\lambda_x \cos\theta + \lambda_y \sin\theta)\right)\Omega = 0\,.
\end{aligned} \tag{3.5}$$

These conditions restrict state/costate space to dimension six. This restriction in the dimensionality of the state/costate dynamics is known as the singular surface; the trajectory defined on this surface is known as the singular arc. The generalized Legendre–Clebsch condition for this singular control problem is

$$-\frac{\partial}{\partial \Omega}\left[\frac{d^2}{dt^2}\left(\frac{\partial \mathcal{H}}{\partial \Omega}\right)\right] = V(\lambda_x \cos\theta + \lambda_y \sin\theta) - \frac{1}{2}\,\lambda_V \sin\theta \geq 0\,. \tag{3.6}$$

Solving the differential equations for the states/costates and choosing θ as the independent parameter yields (see section 4.2 for more details)

$$\begin{aligned}
x(\theta) &= \frac{V_f^2}{2\cos^2\theta_f}\,(2(\theta - \theta_f) + \sin 2\theta - \sin 2\theta_f), & \lambda_x(\theta) &= \frac{\cos\theta_f}{V_f},\\
y(\theta) &= \frac{V_f^2}{2\cos^2\theta_f}\,(\cos 2\theta_f - \cos 2\theta), & \lambda_y(\theta) &= \frac{\sin\theta_f}{V_f},\\
V(\theta) &= V_f\,\frac{\cos\theta}{\cos\theta_f}, & \lambda_V(\theta) &= -\frac{2\sin(\theta - \theta_f)}{\cos\theta_f},\\
t(\theta) &= t_f - \frac{2V_f}{\cos\theta_f}\,(\theta - \theta_f), & \lambda_\theta(\theta) &= 0\,.
\end{aligned} \tag{3.7}$$

The values of V_f, θ_f, and θ_0 are determined from the initial conditions on x, y, and V. The fact that $\lambda_\theta(\theta) \equiv 0$ indicates the singular solution for the control exists throughout the control trajectory. The optimal control is

$$\Omega = -\frac{\cos\theta_f}{2V_f}, \tag{3.8}$$

satisfying the necessary generalized Legendre–Clebsch condition

$$V(\lambda_x \cos\theta + \lambda_y \sin\theta) - \frac{1}{2}\lambda_V \sin\theta = 1 > 0\,.$$

These results will serve as a check for the brachistochrone problem with Coulomb friction. When $\mu = 0$, the Coulomb friction results should reduce to the above results.

4. Brachistochrone with Coulomb friction.

4.1. The equations of motion. The dimensionless equations of motion for the brachistochrone with Coulomb friction are obtained from (2.11) with $\dot\theta = \Omega$ appended. The boundary conditions for the states are

$$(x(0), y(0), V(0), \theta(0)) = (1, y_0, V_0, \theta_0)\,, \tag{4.1}$$

$$(x(t_f), y(t_f), V(t_f), \theta(t_f)) = (0,\ 0\ , V_f, \theta_f)\,, \tag{4.2}$$

where θ_0, V_f, and θ_f are unknown parameters.

The Hamiltonian for the system is

$$\mathcal{H} = -\lambda_x V\cos\theta - \lambda_y V\sin\theta + \lambda_V\left(\frac{1}{2}\sin\theta - \mu\left|\frac{1}{2}\cos\theta - V\Omega\right|\right) + \lambda_\theta\Omega + 1\,. \tag{4.3}$$

Therefore, the equations for the costates are

$$\begin{aligned}
&\dot\lambda_x = 0,\\
&\dot\lambda_y = 0,\\
&\dot\lambda_V = \lambda_x\cos\theta + \lambda_y\sin\theta - \mu s\lambda_V\Omega,\\
&\dot\lambda_\theta = -\lambda_x V\sin\theta + \lambda_y V\cos\theta - \frac{1}{2}\lambda_V(\cos\theta + \mu s\sin\theta)
\end{aligned} \tag{4.4}$$

with the boundary conditions

$$(\lambda_x(0), \lambda_y(0), \lambda_V(0), \lambda_\theta(0)) = (\lambda_{x_0}, \lambda_{y_0}, \lambda_{V_0}, 0)\,, \tag{4.5}$$

$$(\lambda_x(t_f), \lambda_y(t_f), \lambda_V(t_f), \lambda_\theta(t_f)) = (\lambda_{x_f}, \lambda_{y_f},\ \ 0\ \ , 0)\,, \tag{4.6}$$

where λ_{x_0}, λ_{y_0}, λ_{V_0}, λ_{x_f}, and λ_{y_f} are unknown. In the above equations

$$s = \operatorname{sgn}\left(\frac{1}{2}\cos\theta - V\Omega\right) = \operatorname{sgn}(N)\,. \tag{4.7}$$

Note that for $N = 0$, the above differential equations for λ_V and λ_θ are not properly defined. This case will be considered in section 4.3.

4.2. The singular control arc. The first step in synthesizing the singular control arc for this problem is to derive the constraint equations. To simplify the following analysis, three parameters will be introduced to replace three parameters in the state/costate equations. These parameters are defined as follows:

$$\sin\theta_\mu = \frac{s\mu}{\sqrt{1+\mu^2}},\quad \cos\theta_\mu = \frac{1}{\sqrt{1+\mu^2}},\quad \lambda_x = L\cos\theta_\lambda,\quad \lambda_y = L\sin\theta_\lambda \tag{4.8}$$

with $L > 0$. Noting that the time derivatives of λ_x and λ_y are both zero, the reduced set of six state/costate equations is

$$
\begin{aligned}
\dot{x} &= -V\cos\theta,\\
\dot{y} &= -V\sin\theta,\\
\dot{V} &= \frac{1}{2\cos\theta_\mu}\,\sin(\theta-\theta_\mu) + \tan\theta_\mu V\Omega,\\
\dot{\theta} &= \Omega,\\
\dot{\lambda}_V &= L\cos(\theta-\theta_\lambda) - \tan\theta_\mu\lambda_V\Omega,\\
\dot{\lambda}_\theta &= -LV\sin(\theta-\theta_\lambda) - \frac{1}{2\cos\theta_\mu}\,\lambda_V\cos(\theta-\theta_\mu)\,.
\end{aligned}
\tag{4.9}
$$

The seven unknown parameters for these six equations are θ_0, θ_f, V_f, L, θ_λ, λ_{V_0}, and t_f. There is also θ_μ, which is unknown because of s being unknown. In forthcoming equations, s is assumed constant; its sign will be determined from the generalized Legendre–Clebsch condition. The seventh condition is the transversality condition which, for minimum-time optimization, states that the Hamiltonian is zero at time $t = t_f$. Using the final conditions of the states and costates, equations (4.2) and (4.6), respectively, in the Hamiltonian, (4.3), with use of the parameter equations (4.8), the transversality condition may be written as

$$
LV_f\cos(\theta_f-\theta_\lambda) = 1\,. \tag{4.10}
$$

For this problem, the constraints for singular control may be written as

$$
\begin{aligned}
\frac{\partial\mathcal{H}}{\partial\Omega} &= \lambda_\theta + \tan\theta_\mu V\lambda_V = 0,\\
\cos\theta_\mu\,\frac{d}{dt}\,\frac{\partial\mathcal{H}}{\partial\Omega} &= -LV\sin(\theta-\theta_\mu-\theta_\lambda) - \frac{1}{2\cos\theta_\mu}\,\lambda_V\cos\theta = 0,\\
\cos^2\theta_\mu\,\frac{d^2}{dt^2}\,\frac{\partial\mathcal{H}}{\partial\Omega} &= \frac{L}{2}(\sin\theta_\mu\sin(2\theta-\theta_\mu-\theta_\lambda) - \cos\theta_\lambda)\\
&\quad - LV\Omega\cos(\theta-2\theta_\mu-\theta_\lambda) + \frac{1}{2\cos\theta_\mu}\,\lambda_V\Omega\sin(\theta+\theta_\mu) = 0\,.
\end{aligned}
\tag{4.11}
$$

With the condition that $\lambda_V(t_f) = \lambda_\theta(t_f) = 0$, it is evident that singular control will lead the system to the final state.

The generalized Legendre–Clebsch condition for the existence of singular arcs is

$$
-\frac{\partial}{\partial\Omega}\left[\frac{d^2}{dt^2}\left(\frac{\partial\mathcal{H}}{\partial\Omega}\right)\right] > 0\,.
$$

For this problem, this is equivalent to the condition

$$
-\frac{1}{2\cos\theta_\mu}\,\lambda_V\sin(\theta+\theta_\mu) + LV\cos(\theta-2\theta_\mu-\theta_\lambda) > 0\,. \tag{4.12}
$$

The assumptions upon which the singular control derivation is based are that (1) the speed, including the parameter V_f, is positive; (2) the parameter L is positive; (3) the heading angle θ lies in the interval $(-\pi/2, \pi/2)$; (4) the variable s as manifested in the parameter θ_μ is constant; and (5) the generalized Legendre–Clebsch condition holds.

The derivation will proceed assuming these conditions hold. When the singular-control state trajectories are finally derived, these assumptions will be verified.

The control as determined from the final condition is

$$\Omega = \frac{\frac{L}{2}(\sin\theta_\mu \sin(2\theta - \theta_\mu - \theta_\lambda) - \cos\theta_\lambda)}{LV\cos(\theta - 2\theta_\mu - \theta_\lambda) - \frac{1}{2\cos\theta_\mu}\lambda_V \sin(\theta + \theta_\mu)}\,. \tag{4.13}$$

If the second equation is solved for λ_V, the result is

$$\lambda_V = -\frac{2V}{\cos\theta}\, L\cos\theta_\mu \sin(\theta - \theta_\mu - \theta_\lambda)\,. \tag{4.14}$$

Substituting this result into the control equation,

$$\Omega = \frac{\cos\theta}{2V}\,\frac{\sin\theta_\mu \sin(2\theta - \theta_\mu - \theta_\lambda) - \cos\theta_\lambda}{\sin\theta_\mu \sin(2\theta - \theta_\mu - \theta_\lambda) + \cos(2\theta_\mu + \theta_\lambda)}\,. \tag{4.15}$$

Notice the equations are identical to the singular control equations for the classical brachistochrone when $\theta_\mu = 0$.

With λ_V obtained from (4.14), the generalized Legendre–Clebsch condition becomes

$$\frac{LV}{\cos\theta}\,(\sin\theta_\mu \sin(2\theta - \theta_\mu - \theta_\lambda) + \cos(2\theta_\mu + \theta_\lambda)) > 0\,. \tag{4.16}$$

With $-\pi/2 < \theta < \pi/2$, the constraint that $\lambda_V(t_f) = 0$ implies that if (4.14) is evaluated at t_f then

$$\theta_f = \theta_\mu + \theta_\lambda + k\pi,$$

where k is an integer. Substituting $\theta = \theta_f$ in the generalized Legendre–Clebsch condition yields

$$\frac{LV}{\cos\theta}\,(\sin\theta_\mu \sin(2\theta - \theta_\mu - \theta_\lambda) + \cos(2\theta_\mu + \theta_\lambda))\Big|_{t=t_f} = (-1)^k LV_f \cos\theta_\mu > 0\,.$$

Therefore, k must be an even integer. Without loss of generality, $k = 0$ and

$$\theta_\lambda = \theta_f - \theta_\mu\,. \tag{4.17}$$

Hereafter, the variable θ_λ will be replaced by the right-hand side of (4.17). Rewriting the generalized Legendre–Clebsch condition in terms of θ_f,

$$\frac{LV}{\cos\theta}\,\{\sin\theta_\mu \sin(2\theta - \theta_f) + \cos(\theta_\mu + \theta_f)\} > 0\,. \tag{4.18}$$

With L and V positive, and $-\pi/2 < \theta < \pi/2$, the generalized Legendre–Clebsch condition holds if the term in the braces is positive.

The dimensionless normal force in the system is

$$N = \frac{1}{2}\,\cos\theta - V\Omega = \cos\theta\,\frac{\cos\theta_\mu \cos\theta_f}{\sin\theta_\mu \sin(2\theta - \theta_f) + \cos(\theta_\mu + \theta_f)}\,. \tag{4.19}$$

The generalized Legendre–Clebsch condition explicitly requires $s = \operatorname{sgn}(N) = +1$. In other words, there is no way of picking an optimal singular control for which $N < 0$ at

some point in the singular arc. Furthermore, the assumption of s constant is justified. Hereafter, the parameter θ_μ will be defined by the relations (4.8) with $s \equiv +1$.

It will be helpful to express the generalized Legendre–Clebsch condition, and other relationships in the equations, in a different form. First,

$$\begin{aligned}(4.20)\qquad &\frac{LV}{\cos\theta}\left(\sin\theta_\mu \sin(2\theta-\theta_f)+\cos(\theta_\mu+\theta_f)\right)\\ &= LV\cos\theta\cos\theta_f\cos\theta_\mu((\tan\theta+\tan\theta_\mu)^2-(\tan\theta_f+\tan\theta_\mu)^2+\sec^2\theta_f)>0\,.\end{aligned}$$

The dimensionless normal force may be written as

$$(4.21)\qquad N=\frac{\sec\theta}{(\tan\theta+\tan\theta_\mu)^2-(\tan\theta_f+\tan\theta_\mu)^2+\sec^2\theta_f}\,.$$

Similarly, (4.15) may be written as

$$(4.22)\qquad \Omega=-\frac{\cos\theta}{2V}\,\frac{(\tan\theta-\tan\theta_\mu)^2-(\tan\theta_f-\tan\theta_\mu)^2+\sec^2\theta_f}{(\tan\theta+\tan\theta_\mu)^2-(\tan\theta_f+\tan\theta_\mu)^2+\sec^2\theta_f}\,.$$

Using this last relationship and writing the differential equation for $V(\theta)$,

$$(4.23)\qquad \frac{dV}{d\theta}=-V\,\frac{\tan^3\theta+(1-2\tan\theta_f\tan\theta_\mu)\tan\theta-2\tan\theta_\mu}{(\tan\theta-\tan\theta_\mu)^2-(\tan\theta_f-\tan\theta_\mu)^2+\sec^2\theta_f}\,.$$

The solution of this differential equation is

$$(4.24)\qquad V_+(\theta)=\frac{V_f\sec\theta_f\sec\theta}{(\tan\theta-\tan\theta_\mu)^2-(\tan\theta_f-\tan\theta_\mu)^2+\sec^2\theta_f}\,.$$

The subscript + indicates that this and subsequent formulas are valid for θ near θ_f (singular control). The beauty of this result is its simplicity. With $\theta_\mu = 0$,

$$V(\theta)=V_f\,\frac{\sec\theta_f}{\sec\theta}=V_f\,\frac{\cos\theta}{\cos\theta_f},$$

which is the result for the classical brachistochrone.

With $V_+ > 0$, $\cos\theta > 0$, and the generalized Legendre–Clebsch condition, $\Omega(\theta) < 0$, which implies that θ decreases as a function of time. This means that $\theta \geq \theta_f$, where θ represents the heading angle at time $t \leq t_f$. The implicit assumption in the analysis to this point has been that $V_+ > 0$. In order for this to be true,

$$\begin{aligned}V_+(\theta)&=\frac{V_f\sec\theta_f\sec\theta}{(\tan\theta-\tan\theta_\mu)^2-(\tan\theta_f-\tan\theta_\mu)^2+\sec^2\theta_f}\\ &=V_f\,\frac{\cos\theta_f}{\cos\theta}\,\frac{1}{f^2(\theta)-f^2(\theta_f)+1}>0,\end{aligned}$$

where

$$(4.25)\qquad f(x)=\frac{\tan x-\tan\theta_\mu}{\sec\theta_f}=\cos\theta_f(\tan x-\mu)\,.$$

Given that θ and θ_f are each bounded between $-\pi/2$ and $\pi/2$, this implies that $f^2(\theta)-f^2(\theta_f)+1>0$. If $1-f^2(\theta_f)>0$, then the constraint holds regardless of the value of θ. If $1-f^2(\theta_f)<0$, then θ must be constrained. The fact that $V_+ > 0$ and

$\Omega(\theta) < 0$ translates to the following constraints on θ and θ_f:

$$\begin{aligned} &\theta_f > 2\theta_\mu - \frac{\pi}{2}\,, && \theta_f < \theta < \frac{\pi}{2}\,, \\ &\theta_f \le 2\theta_\mu - \frac{\pi}{2}\,, && \theta_f < \theta < \tan^{-1}\left(\tan\theta_\mu - \sqrt{(\tan\theta_f - \tan\theta_\mu)^2 - \sec^2\theta_f}\right). \end{aligned} \tag{4.26}$$

This constraint on the range of θ also implies the generalized Legendre–Clebsch condition is satisfied in general for the state trajectories to be computed.

The differential equations satisfied by x, y, and t are as follows:

$$\begin{aligned} \frac{dx}{d\theta} &= 2V_f^2 \cos\theta_f \frac{f^2(\theta) - 4f_0 f(\theta) - f^2(\theta_f) + 4f_0 f(\theta_f) + 1}{(f^2(\theta) - f^2(\theta_f) + 1)^3} f'(\theta)\,, \\ \frac{dy}{d\theta} &= 2V_f^2 \frac{(f(\theta) - f_0)(f^2(\theta) - 4f_0 f(\theta) - f^2(\theta_f) + 4f_0 f(\theta_f) + 1)}{(f^2(\theta) - f^2(\theta_f) + 1)^3} f'(\theta)\,, \end{aligned} \tag{4.27}$$

and

$$\frac{dt}{d\theta} = -2V_f \frac{f^2(\theta) - 4f_0 f(\theta) - f^2(\theta_f) + 4f_0 f(\theta_f) + 1}{(f^2(\theta) - f^2(\theta_f) + 1)^2} f'(\theta),$$

where $f_0 = f(0) = -\mu\cos\theta_f$.

The solution of each differential equation differs with θ_f greater than, equal to, or less than $2\theta_\mu - \pi/2$. The value for $\theta_f > 2\theta_\mu - \pi/2$ is presented first. The value for $\theta_f < 2\theta_\mu - \pi/2$ is obtained from this result by replacing each term $\sqrt{1 - f^2(\theta_f)}$ with $i\sqrt{f^2(\theta_f) - 1}$, and $\tan^{-1}(ix)$ with $i\coth^{-1}(x)$. For each solution the corresponding formula for $\theta_f = 2\theta_\mu - \pi/2$ is also presented. The solutions of the differential equations are

$$\begin{aligned} \frac{x_+(\theta)}{V_f^2} = \cos\theta_f \Bigg[&\frac{1 - f^2(\theta_f) + 3f(\theta_f)f_0}{(1 - f^2(\theta_f))^2} \left(\frac{f(\theta)}{f^2(\theta) - f^2(\theta_f) + 1} - f(\theta_f) \right. \\ &\left. + \frac{1}{\sqrt{1 - f^2(\theta_f)}} \left(\tan^{-1}\left(\frac{f(\theta)}{\sqrt{1 - f^2(\theta_f)}} \right) - \tan^{-1}\left(\frac{f(\theta_f)}{\sqrt{1 - f^2(\theta_f)}} \right) \right) \right) \\ &+ \frac{2f_0 f(\theta_f)}{1 - f^2(\theta_f)} \left(\frac{f(\theta)}{(f^2(\theta) - f^2(\theta_f) + 1)^2} - f(\theta_f) \right) \\ &- 2f_0 \left(1 - \frac{1}{(f^2(\theta) - f^2(\theta_f) + 1)^2} \right) \Bigg], \end{aligned} \tag{4.28}$$

$$\begin{aligned} \frac{y_+(\theta)}{V_f^2} = 1 &- \frac{1}{f^2(\theta) - f^2(\theta_f) + 1} + 2f_0(f(\theta_f) + f_0) \left(1 - \frac{1}{(f^2(\theta) - f^2(\theta_f) + 1)^2} \right) \\ &+ \frac{2f_0(1 - f^2(\theta_f) - f(\theta_f)f_0)}{1 - f^2(\theta_f)} \left(\frac{f(\theta)}{(f^2(\theta) - f^2(\theta_f) + 1)^2} - f(\theta_f) \right) \\ &- \frac{f_0(2 - 2f^2(\theta_f) + 3f(\theta_f)f_0)}{(1 - f^2(\theta_f))^2} \left(\frac{f(\theta)}{f^2(\theta) - f^2(\theta_f) + 1} - f(\theta_f) \right. \\ &\left. + \frac{1}{\sqrt{1 - f^2(\theta_f)}} \left(\tan^{-1}\left(\frac{f(\theta)}{\sqrt{1 - f^2(\theta_f)}} \right) - \tan^{-1}\left(\frac{f(\theta_f)}{\sqrt{1 - f^2(\theta_f)}} \right) \right) \right), \end{aligned} \tag{4.29}$$

and

$$\frac{t_+(\theta)}{2V_f} = \frac{t_f}{2V_f} - \frac{2f_0 f(\theta_f)}{1-f^2(\theta_f)}\left(\frac{f(\theta)}{f^2(\theta)-f^2(\theta_f)+1} - f(\theta_f)\right)$$
$$+\ 2f_0\left(1 - \frac{1}{f^2(\theta)-f^2(\theta_f)+1}) - (1 + \frac{2f_0 f(\theta_f)}{1-f^2(\theta_f)}\right)$$
$$\cdot \frac{1}{\sqrt{1-f^2(\theta_f)}}\left(\tan^{-1}\left(\frac{f(\theta)}{\sqrt{1-f^2(\theta_f)}}\right) - \tan^{-1}\left(\frac{f(\theta_f)}{\sqrt{1-f^2(\theta_f)}}\right)\right). \tag{4.30}$$

The solutions of the differential equations for $f(\theta_f) = -1$ ($\theta_f = 2\theta_\mu - \pi/2$) are

$$\frac{x_+(\theta)}{V_f^2} = \sin 2\theta_\mu\left[-\frac{2}{3}\left(\frac{1}{f^3(\theta)}+1\right) + 2f_0\left(\frac{1}{f^4(\theta)}-1\right) + \frac{8f_0}{5}\left(\frac{1}{f^5(\theta)}+1\right)\right], \tag{4.31}$$

$$\frac{y_+(\theta)}{V_f^2} = 1 - \frac{1}{f^2(\theta)} + 2f_0(1-f_0)\left(\frac{1}{f^4(\theta)}-1\right)$$
$$+\ \frac{10f_0}{3}\left(\frac{1}{f^3(\theta)}+1\right) - \frac{8f_0^2}{5}\left(\frac{1}{f^5(\theta)}+1\right), \tag{4.32}$$

and

$$\frac{t_+(\theta)}{2V_f} = \frac{t_f}{2V_f} + \frac{1}{f(\theta)} + 1 + 2f_0\left(1 - \frac{1}{f^2(\theta)}\right) - \frac{4f_0}{3}\left(\frac{1}{f^3(\theta)}+1\right). \tag{4.33}$$

When $\theta_\mu = 0$, the equations simplify to

$$x(\theta) = \frac{V_f^2}{2\cos^2\theta_f}\left(2(\theta-\theta_f) + \sin 2\theta - \sin 2\theta_f\right),$$
$$y(\theta) = \frac{V_f^2}{2\cos^2\theta_f}\left(\cos 2\theta_f - \cos 2\theta\right),$$

and

$$t(\theta) = t_f - \frac{2V_f}{\cos\theta_f}\left(\theta - \theta_f\right).$$

Admittedly, the results were presented without exposition, but the fact that the results yielded are identical to classical brachistochrone results when $\mu = 0$ is at least reassuring.

4.3. The extremal control arc. With the derivation of the singular control arc, the optimal control has yet to be completely described. For one, each of the expressions derived contains two unknowns, V_f and θ_f. There must be relations connecting the states in the singular arc with states starting at time $t = 0$.

Initially the control will be extremal control, that is, control which lies at the boundary of the admissible domain of control. As demonstrated earlier, the optimal control for initial conditions in the first or fourth quadrant (x_0 positive) is control in which the normal force is nonnegative. Therefore

$$\frac{1}{2}\cos\theta - V\Omega \geq 0. \tag{4.34}$$

This constraint will yield the initial extremal control.

The partial derivative of the Hamiltonian with respect to the control variable Ω is given by

$$\frac{\partial \mathcal{H}}{\partial \Omega} = \lambda_\theta + s\mu V \lambda_V \,. \tag{4.35}$$

Examining this term at time $t = 0$, if $V_0 = V(0) \neq 0$ then this term is nonzero. Furthermore, the time derivative of the Hamiltonian is

$$\frac{d\mathcal{H}}{dt} = (\lambda_\theta + s\mu V \lambda_V)\dot{\Omega} - \mu \lambda_V \left(\frac{1}{2} \cos\theta - V\Omega \right) \dot{s} = 0 \,. \tag{4.36}$$

There are thus two options for the initial control Ω if $V_0 \neq 0$: (1) choose Ω constant with s constant (either + or − 1), or (2) choose $\Omega = \frac{\cos\theta}{2V}$ such that the normal force is zero and define $s = -\frac{\lambda_\theta}{\mu V \lambda_V}$ so that the coefficient of $\dot{\Omega}$ in the time derivative of the Hamiltonian is also zero.

Choosing case (1), the *constant* extremal control which "minimizes" the Hamiltonian must be $\Omega = \pm\infty$; that is, the initial control will be a jump change in the direction of the particle. By Pontryagin's principle [15], the optimal control is the control which extremizes the Hamiltonian. In this case, extremization of the Hamiltonian occurs with $\Omega = \pm\infty$ if and only if

$$\left| \frac{\lambda_\theta}{\mu V \lambda_V} \right| \geq 1 \,. \tag{4.37}$$

This constraint follows from the fact that λ_V, from (4.14), is less than zero and that λ_θ, from the first equation of (4.11), is greater than zero at the corner between singular control and extremal control. Thus, in order for constant control to be extremal control, the control reduces to a jump change in slope. After the jump in the control trajectory, the velocity is now zero and singular control yields the constraints. Intuitively, this possibility is trivial as setting the initial normal force infinite makes this problem identical to a zero-initial-velocity problem. All that is gained from having nonzero initial energy is lost immediately.

Therefore, the extremal control will be to let

$$\Omega = \frac{\cos\theta}{2V} \,. \tag{4.38}$$

This control minimizes the Hamiltonian if and only if

$$s = -\frac{\lambda_\theta}{\mu V \lambda_V} \qquad \text{with} \qquad |s| \leq 1 \,. \tag{4.39}$$

This initial extremal control is different from the singular control derived earlier. Due to the order of the singularity, the control *must* be discontinuous [21] at the corner between extremal control and singular control. Setting the normal force equal to zero means that the curve described by the motion of the particle is that of a particle in "free fall."

Starting with the free-fall equation for velocity,

$$\frac{dV}{d\theta} = V \tan\theta \,.$$

Integrating this result yields

$$V_-(\theta) = V_0 \frac{\cos\theta_0}{\cos\theta}\,. \tag{4.40}$$

The subscript $-$ indicates that this and subsequent formulas are valid for θ near θ_0 (extremal control). Using this result in the differential equations for x, y, and t yields

$$\begin{aligned} \frac{dx}{d\theta} &= -2V_0^2 \cos^2\theta_0 \sec^2\theta\,, \\ \frac{dy}{d\theta} &= -2V_0^2 \cos^2\theta_0 \sec^2\theta \tan\theta\,, \end{aligned} \tag{4.41}$$

and

$$\frac{dt}{d\theta} = 2V_0 \cos\theta_0 \sec^2\theta\,.$$

The solution of this set of differential equations is

$$\begin{aligned} x_-(\theta) &= 1 - 2V_0^2 \cos^2\theta_0(\tan\theta - \tan\theta_0) \\ &= 1 - 2V_0^2 \frac{\cos^2\theta_0}{\cos^2\theta_f} \cos\theta_f(f(\theta) - f(\theta_0))\,, \end{aligned} \tag{4.42}$$

$$\begin{aligned} y_-(\theta) &= y_0 - V_0^2 \cos^2\theta_0(\tan^2\theta - \tan^2\theta_0) \\ &= y_0 - V_0^2 \frac{\cos^2\theta_0}{\cos^2\theta_f}(f^2(\theta) - 2f(\theta)f_0 - f^2(\theta_0) + 2f(\theta_0)f_0), \end{aligned} \tag{4.43}$$

and

$$t_-(\theta) = 2V_0 \cos\theta_0(\tan\theta - \tan\theta_0) = 2V_0 \frac{\cos\theta_0}{\cos\theta_f}(f(\theta) - f(\theta_0))\,. \tag{4.44}$$

If θ is the heading angle at the time at which the control switches, then letting $x_+(\theta)$ from (4.28) equal $x_-(\theta)$ from (4.42), $y_+(\theta)$ from (4.29) equal $y_-(\theta)$ from (4.43), and $V_+(\theta)$ from (4.24) equal $V_-(\theta)$ from (4.40) yields three equations in unknowns θ_0, θ_f, V_f, and θ. Therefore it is necessary to derive a fourth equation in these unknowns. The fourth equation will come from meeting the first two conditions for the existence of a singular arc at heading angle θ.

In order to derive the two necessary conditions for a singular arc, the differential equations for λ_V and λ_θ need to be solved. Using the costate time differential equations (4.4) with $s = -\frac{\lambda_\theta}{\mu V \lambda_V}$ yields

$$\begin{aligned} \dot\lambda_V &= L\cos(\theta - \theta_f + \theta_\mu) + \frac{1}{2}\frac{\lambda_\theta \cos\theta}{V_-^2(\theta)}, \\ \dot\lambda_\theta &= -LV_-(\theta)\sin(\theta - \theta_f + \theta_\mu) - \frac{1}{2}\lambda_V \cos\theta + \frac{1}{2}\frac{\lambda_\theta \sin\theta}{V_-(\theta)}\,. \end{aligned} \tag{4.45}$$

Rewriting these time differential equations, with a change of dependent variable, in terms of θ

$$\begin{aligned} \frac{d\lambda_V}{d\theta} &= \frac{2LV_0 \cos\theta_0 \cos(\theta - \theta_f + \theta_\mu)}{\cos^2\theta} + \frac{\lambda_\theta}{V_-}, \\ \frac{d}{d\theta}\frac{\lambda_\theta}{V_-} &= -\frac{2LV_0 \cos\theta_0 \sin(\theta - \theta_f + \theta_\mu)}{\cos^2\theta} - \lambda_V\,. \end{aligned} \tag{4.46}$$

Furthermore, the first two necessary conditions for a singular solution may be written as

$$\lambda_\theta + \tan\theta_\mu V_-(\theta)\lambda_V = \frac{V_-(\theta)}{\cos\theta_\mu}\left(\frac{\lambda_\theta}{V_-(\theta)}\cos\theta_\mu + V_-(\theta)\lambda_V \sin\theta_\mu\right) = 0\,, \tag{4.47}$$

$$-LV_-(\theta)\sin(\theta-\theta_f) - \frac{1}{2\cos\theta_\mu}\lambda_V\cos\theta = 0\,. \tag{4.48}$$

Solving the differential equations for λ_V and λ_θ/V_- yields

$$\lambda_V(\theta) = \lambda_{V_0}\cos(\theta-\theta_0) + 2L(V_0\sin(\theta-\theta_0-\theta_f+\theta_\mu) + V_-(\theta)\sin(\theta_f-\theta_\mu))$$

and

$$\frac{\lambda_\theta(\theta)}{V_-(\theta)} = -\lambda_{V_0}\sin(\theta-\theta_0) + 2L(V_0\cos(\theta-\theta_0-\theta_f+\theta_\mu) - V_-(\theta)\cos(\theta_f-\theta_\mu)),$$

where $\lambda_{V_0} = \lambda_V(0)$.

With these two quantities determined, the value of the switching function given by (4.47) and the constraint given by (4.48) is

$$-\frac{V_-(\theta)}{\cos\theta_\mu}\left(\lambda_{V_0}\sin(\theta-\theta_0-\theta_\mu) + 2LV_0\,\frac{\sin(\theta-\theta_0)\sin(\theta-\theta_f)}{\cos\theta}\right) = 0\,, \tag{4.49}$$

$$-\frac{1}{2\cos\theta_\mu}\left[\lambda_{V_0}\cos(\theta-\theta_0)\cos\theta + L\left(\frac{\cos\theta_f\cos\theta_\mu + \cos\theta\cos(\theta-\theta_f+\theta_\mu)}{\cos\theta}\sin(\theta-\theta_0)\right.\right.$$

$$\left.\left. + \frac{\cos\theta_0\cos\theta_\mu + \cos\theta\cos(\theta-\theta_0+\theta_\mu)}{\cos\theta}\sin(\theta-\theta_f)\right)V_0\right] = 0\,. \tag{4.50}$$

These two linear equations have a nontrivial solution in λ_{V_0} and L if and only if the determinant of the coefficient matrix is zero. The coefficients of λ_{V_0} and L in the second equation are *not* (scaled) time derivatives of the corresponding coefficients in the first equation, despite the derivation given for the conditions in the singular solution. This is due to the fact that the costates on the (constrained) extremal trajectory have different governing equations than the singular trajectory paths (s is *not* constant here). Expanding and simplifying the determinant of the coefficient matrix yield

$$\frac{V_0^2\cos\theta_0}{\cos^2\theta}[\sin(\theta-\theta_0)(\sin(\theta-\theta_0)-\mu\cos(\theta-\theta_0))\cos\theta_f - \mu\sin(\theta-\theta_f)\cos\theta] = 0\,. \tag{4.51}$$

Provided that the (dimensionless) initial velocity V_0 is not zero, the term in brackets being zero is the fourth equation that must hold. The constraints on these four equations come from the fact that $\theta > \theta_0$, $\theta > \theta_f$, and $V_f > 0$ and the restrictions on θ given with the states, inequality (4.26).

4.4. Abnormal trajectories. The derivation up to this point has implicitly assumed "normality" of the minimum-time trajectories. The complete formulation of an optimal control problem in the calculus of variations requires the formation of the Hamiltonian as

$$\mathcal{H} = \langle \boldsymbol{\lambda}, \boldsymbol{f}(\boldsymbol{x}, u)\rangle + \lambda_0 L(\boldsymbol{x}, u), \tag{4.52}$$

where λ_0 is a nonnegative constant. To this point the assumption has been that λ_0 was strictly positive. Thus equation (2.1) is obtained by scaling the remaining costates by the positive constant λ_0.

If, however, λ_0 is zero, then a degenerate condition exists which may yield optimal trajectories. For example, as noted in a novel case in [13], a completely controllable optimal control system with k linear controls and n states, $k < n$, and a cost function L which is quadratic positive-definite in the controls yields an optimal solution which is abnormal. The full definition of the optimal control problem is as follows [12, 23]:

(1) For any t, $\boldsymbol{\lambda}(t)$ and λ_0 are not both zero.

(2) At $(t, \boldsymbol{x}(t), \boldsymbol{\lambda}(t), \lambda_0)$, the function of u,

$$\mathcal{H} = \langle \boldsymbol{\lambda}, \boldsymbol{f}(\boldsymbol{x}, u) \rangle + \lambda_0 L(\boldsymbol{x}, u),$$

assumes for $u = u(t)$ its minimum. With the cost and the state dynamics having no explicit time dependence, the minimum value of the function $\mathcal{H}$ is constant.

(3) For the arguments $t, \boldsymbol{x}(t), \boldsymbol{\lambda}(t), \lambda_0$ the canonical Euler equations

$$\dot{\boldsymbol{x}} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}}, \qquad \dot{\boldsymbol{\lambda}} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{x}}$$

hold.

(4) If T_0 and T_f are the tangent spaces to the state manifold at $t = t_0$ and $t = t_f$, respectively, then

$$\boldsymbol{\lambda}(t_0) \perp T_0 \qquad \text{and} \qquad \boldsymbol{\lambda}(t_f) \perp T_f \,.$$

This is the *transversality* condition.

The abnormal case corresponds to choosing $\lambda_0 = 0$ and finding nontrivial $\boldsymbol{\lambda}(t)$ which satisfy the Euler equations everywhere except possibly at points where the state dynamics $\boldsymbol{f}$ are discontinuous [6]. An equivalent definition of abnormality [23] is that for an n-dimensional system $\boldsymbol{x}(t)$, there do not exist $2n$ linearly independent permissible variations $\delta P, \delta Q$, where

$$(P, Q) = ((t_0, \boldsymbol{x}(t_0)), (t_f, \boldsymbol{x}(t_f))) \,.$$

This second definition indicates the nature of abnormal trajectories; they are trajectories where the states/control cannot be admissibly varied about a given trajectory.

The singular control trajectory or the extremal control trajectory limits the admissible variations of the states and control about the given trajectory. For example, the extremal control trajectory requires that the normal force, given by

$$\frac{1}{2} \cos\theta - V\Omega,$$

be zero along the entire path. This specifies the control Ω as a function of the states V and θ. If V or θ is varied, the control varies. The control itself does *not* restrict the variation of V and θ. If it did, this would indicate an abnormal trajectory.

Similarly, the singular control trajectory requires the states/costates/control to satisfy the three constraints (4.11), thus restricting the state/costate manifold to dimension six, rather than the unconstrained eight. Thus in the singular trajectory, there are only six admissible variations in the states/costates, rather than the eight

of the unconstrained system. Nevertheless, the derivation of the singular control relies upon the cost function L in that the generalized Legendre–Clebsch condition (4.18) implicitly uses the transversality condition for admissible variations of t_f to ensure positive definiteness. With $\lambda_0 = 0$, there are no admissible variations of t_f and the generalized Legendre–Clebsch condition is trivial (zero). The singular paths therefore require "normal" trajectories, that is, those in which all seven unconstrained parameters are allowed to vary in order to realize a minimum.

The question then remains: is there a control strategy such that the states/costates may not be admissibly varied to the full dimension of the space. This question may be answered by examining the "smoothness" of the state manifold. The state manifold consists of two smooth submanifolds joined at the corner where

$$\frac{1}{2}\cos\theta - V\Omega = 0\,.$$

The corner is defined by the control Ω. Furthermore, there are no stationary points in the state-space for $V > 0$. Thus with $V > 0$, the states and the cost may be varied on the corner and on both sides of the corner due to the definition of the corner in terms of the control Ω. The minimization of the final time t_f trivially requires $V > 0$ at all interior points in a state trajectory. Thus at every point in the interior of a state trajectory, there exist admissible variations in the states/costates allowing the normal minimization.

5. The brachistochrone when the initial velocity is zero. If $V_0 = 0$, then the solution of the optimization problem is much simpler than the general case. In this case $\theta_0 = \theta = \pi/2$. The optimal control becomes the solution of the two equations

$$x_+(\pi/2) = 1 \qquad \text{and} \qquad y_+(\pi/2) = y_0\,.$$

The velocity equation and (4.51) both hold. Furthermore,

$$f(\theta) = f(\pi/2) \to \infty \qquad \text{and} \qquad \theta_f > 2\theta_\mu - \pi/2\,.$$

Taking the ratio of y_+ to x_+ will eliminate V_f from the equations, yielding one equation in one unknown, θ_f. When this is solved, the corresponding value of V_f can be solved for in either of the two equations as a positive scaling factor.

The one equation in θ_f can be used to obtain θ_f as a function of y_0 and μ. Expressing this relationship in one equation,

$$\begin{aligned} &1 + \frac{f_0(2f(\theta_f) - f_0)}{1 - f^2(\theta_f)} + \frac{3f_0^2}{(1 - f^2(\theta_f))^2} - f_0\left(\frac{2}{1 - f^2(\theta_f)}\right. \\ &\quad \left. + \frac{3f(\theta_f)f_0}{(1 - f^2(\theta_f))^2}\right)\frac{1}{\sqrt{1 - f^2(\theta_f)}}\cos^{-1}(f(\theta_f)) = y_0\cos\theta_f\left[\frac{f_0 - f(\theta_f)}{1 - f^2(\theta_f)}\right. \\ &\quad \left. - \frac{3f_0}{(1 - f^2(\theta_f))^2} + \left(\frac{1}{1 - f^2(\theta_f)} + \frac{3f(\theta_f)f_0}{(1 - f^2(\theta_f))^2}\right)\frac{1}{\sqrt{1 - f^2(\theta_f)}}\cos^{-1}(f(\theta_f))\right]. \end{aligned} \tag{5.1}$$

Expanding this equation in a power series in θ_f about $\theta_f = 2\theta_\mu - \pi/2$ will yield y_0 as a function of θ_f. Then, reverting this power series will yield θ_f as a function of y_0.

The result of this analysis is

$$\theta_f = 2\theta_\mu - \frac{\pi}{2} + 3\,\frac{y_0-\mu}{1+\mu^2} + \frac{3}{\mu}\,(1-2\mu^2)\left(\frac{y_0-\mu}{1+\mu^2}\right)^2 - \frac{3\sqrt{6}}{\pi\sqrt{\mu}}\,(3-\mu^2)\left(\frac{y_0-\mu}{1+\mu^2}\right)^{\frac{5}{2}} + O((y_0-\mu)^3). \tag{5.2}$$

Immediately apparent from this series expansion is that $y_0 = \mu$ corresponds to a final angle $\theta_f = 2\theta_\mu - \pi/2$. If this angle is substituted into equation (4.30), it is evident that the time-to-go $t_f - t(\theta)$ is infinite. Furthermore, the gradient $\frac{\partial \theta_f}{\partial y_0}$ is positive. Thus initial conditions for y_0 which are less than or equal to μ will not yield the origin in any time. This may also be argued on the grounds of the energy content of the system.

A further point to be noted is that the formal power series derived for θ_f does not converge if $\mu = 0$. This is due to the fact that the power series was expanded about $\theta_f = 2\theta_\mu - \pi/2$. In order to obtain a convergent power series for $\mu = 0$, (5.1) would need to be solved about $\mu = 0$. In this case a power series expansion for θ_f in terms of μ with y_0 as a parameter would be obtained. The initial value for θ_f would be the solution for θ_f in the classical frictionless brachistochrone. Due to the difficulty of the solution of these implicit transcendental equations, this power series solution will not be attempted.

An examination of the brachistochrones computed for $V_0 = 0$, $y_0 = 1$, and varying values of μ (Figure 5.1) demonstrates that the curves cross. The crossing occurs because near the starting point $(x, y) = (1, 1)$, the curvature of the brachistochrones is an increasing function of μ, with the curvature being infinite for $\theta = \pi/2$ and $\mu = 1$. With the increasing curvature, the brachistochrones lie closer to the line $y = y_0 x$, which, in the limit, is the "curve" for $\mu = y_0 = 1$. However, at the final point $(x, y) = (0, 0)$ of each brachistochrone, the slope $\tan\theta_f$ is a *decreasing* function of μ. This observation was not predicted correctly by Ramamani, Lu, and Tabarrok [17] in their discretization of this problem.

The final heading θ_f, the final velocity V_f, and the time-to-go t_f as a function of the coefficient of friction μ are plotted in Figure 5.2. The figure uses an initial condition of $y_0 = 1$. Although the trends displayed in the figures are consistent with different values of y_0, there is no linear relationship between figures with different values of y_0. Demonstrated in the figure is that θ_f is a decreasing function of μ. Furthermore, this function also has negative concavity. This negative concavity, which becomes infinite as $\mu \to y_0^-$, means that $t_f \to \infty$ "slowly" as $\mu \to y_0^-$. The slowness of this approach to infinity can be determined by substituting the power series for θ_f into the relationship for t_f, and by noting the implicit relationship for V_f in this expression. The result is

$$t_f = 4\sqrt{\frac{\pi}{3}}\left(\frac{6(y_0-\mu)}{\mu(1+\mu^2)}\right)^{-\frac{1}{4}} + O((y_0-\mu)^{\frac{3}{4}}). \tag{5.3}$$

This slow divergence is demonstrated in Figure 5.2.

A point not immediately apparent in Figure 5.2 is that although V_f decreases with increasing μ, its concavity has an inflection point around $\mu = 0.65$. Referring to the differential equation for $V(\theta)$, (4.23), for a given value of y_0, there is a value

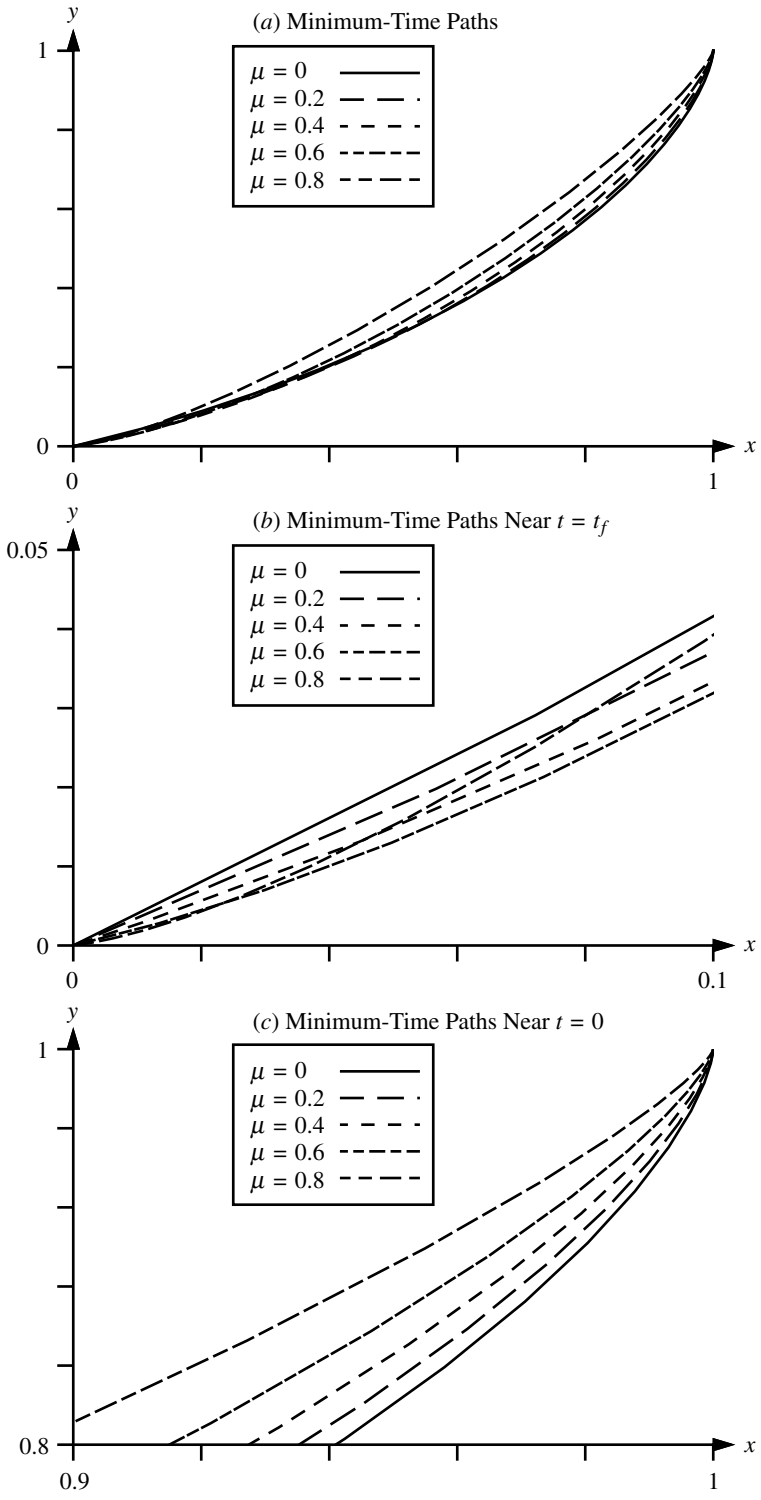

FIG. 5.1. *Coulomb friction brachistochrones with* $V_0 = 0$, $y_0 = 1$, *and differing values for* μ.

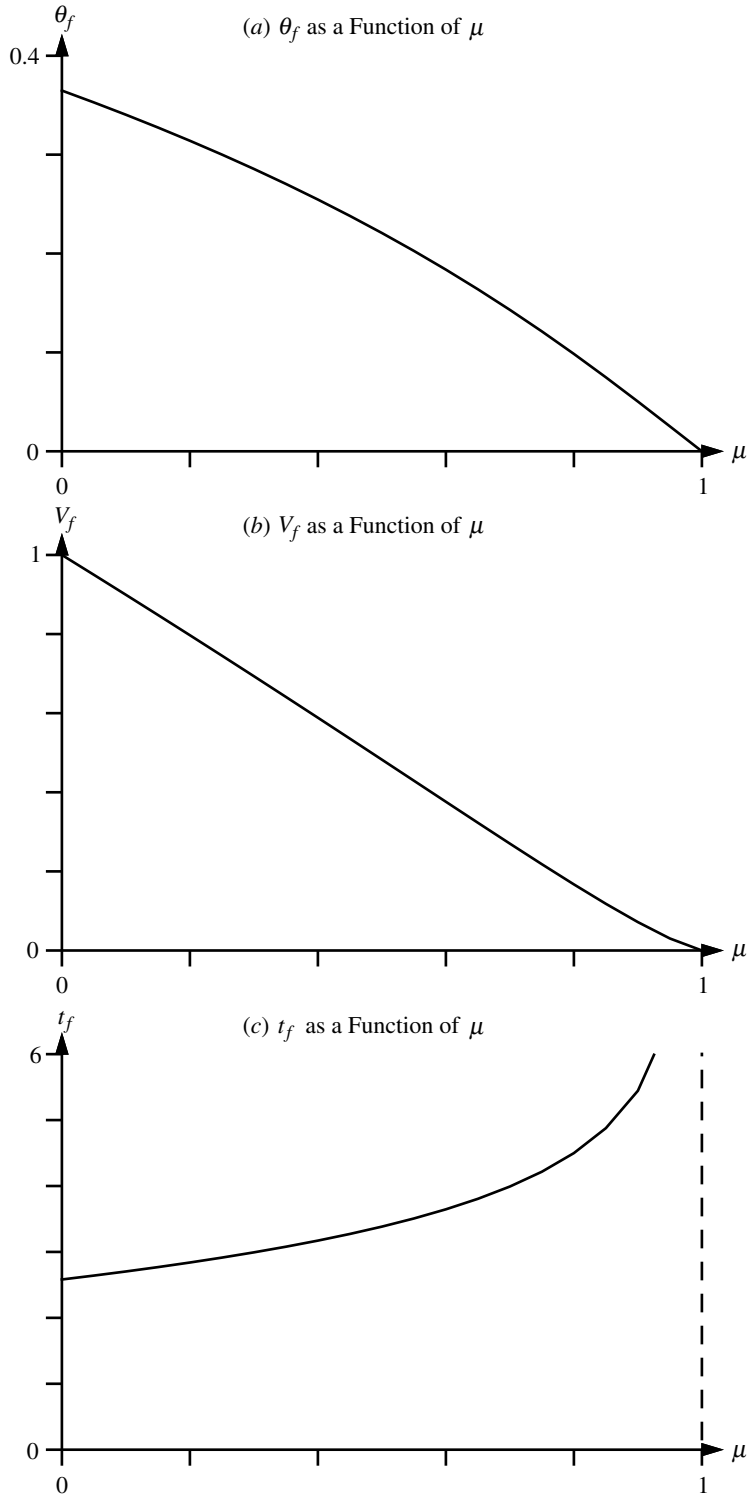

FIG. 5.2. (a) *Final heading angle* θ_f, (b) *final speed* V_f, *and* (c) *minimum time* t_f *for* $V_0 = 0$, $y_0 = 1$, *and variable* μ.

of μ at which the derivative $\frac{dV(\theta_f)}{d\theta} = 0$. This occurs when $\tan\theta_f = 2\mu$. For $y_0 = 1$, this corresponds to $\mu = 0.162\ldots$. Thus, for $y_0 = 1$ and $\mu > 0.162\ldots$, the value of V_f is no longer the largest value of V for the brachistochrone; that is, the speed of the particle initially increases but then decreases as the origin is approached. As the value of μ increases, the point of maximum velocity moves toward $\theta = \pi/2$. Due to the larger difference in angle between $V_{\max}$ and V_f for increasing μ, the plot of V_f as a function of μ will begin concave downward, inflect, and finish concave upward.

6. Some observations about minimum-time paths with nonzero initial velocity. As derived in section 4.3, the minimum-time path for a particle with nonzero initial velocity in a uniform gravitational field with Coulomb friction resisting motion consists of a "free fall" segment followed by a singular arc. The corner conditions joining the two segments are satisfied when (4.51) is satisfied. Setting $\theta = \theta_0$ in (4.51) and solving for θ_0 (and V_0) yields the minimum-time solution for which the entire path is a singular trajectory. The result of this substitution is the zero initial velocity paths presented in the previous section. These paths require $\theta = \theta_0 = \pi/2$.

Setting $\theta = \theta_f$ in (4.51) will yield the solution for a minimum-time path in which the optimal trajectory is entirely a "free fall" trajectory, that is, a trajectory in which the normal force exerted by the path on the particle is zero. Solving for θ_f yields

$$\theta_f = \theta_0 + \theta_\mu \,. \tag{6.1}$$

Substituting this result for θ in (4.42) and (4.43) and solving for θ_0 and V_0 yields

$$\theta_0 = \tan^{-1}\left(y_0 + \mu^{-1} - \sqrt{1 + y_0^2 + \mu^{-2}}\right) \tag{6.2}$$

and

$$V_0 = \sqrt{\frac{\sqrt{1 + y_0^2 + \mu^{-2}} - y_0}{2}} \,. \tag{6.3}$$

The question then is whether velocities greater than this value will yield optimal free-fall trajectories. The answer to this question comes in the solution of $s = -\frac{\lambda_\theta}{\mu V \lambda_V}$ when λ_θ, V, and λ_V are chosen to satisfy the boundary conditions at $t = 0$ and $t = t_f$. The solutions to the differential equations (4.46) with boundary conditions (4.6) are

$$\lambda_V(\theta) = -2\lambda_x V_0 \cos\theta_0 \cos\theta(\tan\theta_f - \tan\theta)(1 + \tan\theta\tan\theta_0) \,, \tag{6.4}$$

$$\frac{\lambda_\theta(\theta)}{V(\theta)} = 2\lambda_x V_0 \cos\theta_0 \cos\theta(\tan\theta_f - \tan\theta)(\tan\theta - \tan\theta_0) \,. \tag{6.5}$$

Thus

$$s = -\frac{\lambda_\theta}{\mu V \lambda_V} = \frac{\tan(\theta - \theta_0)}{\mu} \qquad \text{and} \qquad s_{\max} = \frac{\tan(\theta_f - \theta_0)}{\mu} \,. \tag{6.6}$$

Hence, if a free-fall path were time optimal, $s_{\max} \le 1$. Solving equation (4.42) equals zero and (4.43) equals zero for $\tan\theta_0$ and $\tan\theta_f$,

$$\tan\theta_0 = \sqrt{(2V_0^2 + y_0)^2 - 1 - y_0^2} - 2V_0^2, \quad \tan\theta_f = 2(V_0^2 + y_0) - \sqrt{(2V_0^2 + y_0)^2 - 1 - y_0^2} \,.$$

This means that

$$s_{\max} = \frac{\tan(\theta_f - \theta_0)}{\mu} = \frac{1}{\mu\sqrt{(2V_0^2 + y_0)^2 - 1 - y_0^2}} \le 1 \,. \tag{6.7}$$

Thus for V_0 greater than the critical V_0 given by equation (6.3), the minimum-time path is exclusively a free-fall path.

7. Conclusions. This paper has presented the time-optimal path of a particle between two points in a uniform gravitational field when motion of the particle is resisted by a force proportional to the normal force exerted on the particle by the path. This problem is equivalent to finding a brachistochrone in a uniform gravitational field with Coulomb friction resisting motion. The construction of the brachistochrone in the general case of nonzero initial velocity less than the critical initial velocity given by equation (6.3) becomes the solution of four nonlinear equations in four unknowns. For velocities greater than or equal to the critical initial velocity, the minimum-time path to the origin is a free-fall path. The special case of zero initial velocity is discussed in detail. This case simplifies the analysis to the solution of two transcendental equations in two unknowns. With a simplification, the solution becomes (5.1), which is a single transcendental equation in the unknown θ_f, the final heading angle.

A set of Coulomb friction brachistochrones with zero initial velocity are presented. The features displayed for the specific case plotted are general features of all zero initial velocity Coulomb friction brachistochrones, namely: (1) the (x, y) path is concave upward, (2) the initial heading is always directly downward, (3) the initial curvature is an increasing function of μ for a fixed initial position, (4) the final heading is a concave-downward decreasing function of the coefficient of friction μ for a fixed initial position, (5) the final velocity is a decreasing function of μ with a value of zero at $\mu = y_0$, and (6) the time-to-go is an increasing function of μ with $t_f \to +\infty$ for $\mu \to y_0^-$. Points (3) and (4) above indicate that if two brachistochrones between the same initial and final positions are drawn with different values of μ, the paths cross. This is not a common occurrence in time-optimal path problems. Typically, a change in a parameter in a time-optimal control problem does not produce a path which intersects the original path. The crossing of the paths with a change in the coefficient of friction μ is due to the lack of smoothness in the dynamics of the problem.

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