Book Review

Singular Trajectories and their Role in Control Theory— Bernard Bonnard and Monique Chyba (New York, Springer-Verlag, 2003).

Reviewed by Héctor J. Sussmann

"Singular extremals" are, roughly, trajectories that satisfy the necessary conditions for optimality given by the Pontryagin Maximum Principle in a special ("singular," or "degenerate") way. In the optimal control literature, our vague phrase "in a special way" has been given several different precise meanings, as well as quite a few that are less than completely precise. It is one of the many virtues of this book that it presents the notion of "singular trajectory" in a clear, mathematically precise way, so that the reader can not only get the general picture of the very exciting mathematics that revolves around this idea but also follow the technical details, which to many of us are of interest in themselves.

The authors consider autonomous control systems of the form

$$\dot{x} = f(x,u)\,, \tag{1}$$

for which (a) the state variable x belongs to a smooth (i.e., C^∞) manifold M, (b) the control variable u takes values in a Euclidean space $\mathbb{R}^p$, (c) the map $f : M \times U \mapsto TM$ is of class C^∞ and such that $f(x,u) \in T_xM$ for each x, u (where we use TM to denote the tangent bundle of M and, for each x, T_xM is the tangent space to M at x) and (d) the class $\mathcal{U}$ of all admissible controls consists of all bounded measurable maps $u(\cdot) : [0, T(u)] \mapsto \mathbb{R}^p$ defined on a compact interval $[0, T(u)]$ such that the $u(\cdot)$-dependent terminal time $T(u)$ is positive. They then consider, for a fixed $x_0 \in M$ and a fixed T, the endpoint map $E^{x_0,T} : \mathcal{U}_T \mapsto M$ that sends a control $u(\cdot)$ (where we write $\mathcal{U}_T$ to denote the space $L^\infty([0,T],\mathbb{R}^p)$) to the time T point $x(T, x_0, u)$ of the trajectory $[0,T] \ni t \mapsto x(t, x_0, u)$ that corresponds to $u(\cdot)$ and the initial condition $x(0, x_0, u) = x_0$. A "singular trajectory" is then defined to be a singularity of this endpoint map, i.e., a point u of $\mathcal{U}_T$ where the Fréchet differential ${E'}^{x_0,T}_u : \mathcal{U}_T \mapsto T_{E(u)}M$ of $E^{x_0,T}$ at u fails to be surjective. (In this review, the word "singular" will often be used as applying to a *control-trajectory pair*—abbr. CTP–rather than to a trajectory or a control. This reviewer would have felt more comfortable if the authors had also talked about singularity of CTP's, because to talk about "singularity" one needs *both* the control and the trajectory— or, at a minimum, the control and the initial condition x_0.)

The singular CTP's are then exactly those CTP's $(u(\cdot), \xi)$ such that the linearized system

$$\dot{y}(t) = A(t)\cdot y(t) + B(t)\cdot v(t)\,,\ \ t \in [0, T(u)]$$

(where $A(t) = \frac{\partial f}{\partial x}(\xi(t), u(t))$ and $B(t) = \frac{\partial f}{\partial u}(\xi(t), u(t))$) is not controllable. They can also be characterized, using the Hamiltonian $H : (T^*M\backslash\{0\}) \times \mathbb{R}^p \mapsto \mathbb{R}$ given by

$$H(x,p,u) = \langle p, f(x,u)\rangle$$

(where $T^*M\backslash\{0\}$ is the bundle over M whose fiber at each $x \in M$ is $T^*_xM\backslash\{0\}$, and T^*_xM is the cotangent space of M at x), by the equivalant condition that there exists a solution $\Xi : [0,T] \mapsto T^*M\backslash\{0\}$ of the constrained Hamiltonian system

$$\dot{\Xi}(t) = \vec{H}_u(\Xi(t))\,,\ \ \frac{\partial H}{\partial u}(\Xi(t), u(t)) = 0$$

whose projection to M is ξ. (Here (a) for each fixed u, H_u is the function $(x,p) \mapsto H(x,p,u)$ and (b) if h is a smooth function on a symplectic manifold such as T^*M, we use $\vec{h}$ to denote the corresponding Hamilton vector field.) Naturally, if $\Xi(t) = (\xi(t), p(t))$, the last condition amounts to saying that there exists a field $[0,T] \ni t \mapsto p(t) \in T^*_{\xi(t)}M$ of nonzero covectors along ξ (usually called an "adjoint vector") that satisfies the "adjoint equation" $\dot{p}(t) = -\frac{\partial H}{\partial x}(\xi(t), p(t), u(t))$ and the "critical point condition" $\frac{\partial H}{\partial u}(\xi(t), p(t), u(t)) = 0$.

Singular trajectories are important, first of all, because, if a trajectory $\xi : [0,T] \mapsto M$ is such that $\xi(T)$ belongs to the boundary $\partial\mathcal{R}_{x_0,f,T}$ of the time T reachable set $\mathcal{R}_{x_0,f,T}$ from x_0 for the system (1), then ξ must be singular. Hence the study of the structure of reachable sets is intimately tied to the analysis of singular trajectories.

Second, as the authors point out, singular trajectories are natural candidates in the search for solutions of a minimum time problem in which the controls are further restricted to take values in a subset Ω of $\mathbb{R}^p$. (To be precise, if Ω is open then every time-minimizer is a singular trajectory. If Ω is not open, then it is still true that every time minimizer must be singular in all directions in which it is not an extreme point of Ω. That is, if the pair $(\xi(\cdot), u(\cdot))$ is a minimizer, then there must exist an adjoint vector $p(\cdot)$ such that the directional derivative $\nabla_u H(\xi(t), p(t), u(t)) \cdot v$ vanishes for every $v \in \mathbb{R}^p$ such that $\{u(t) + \varepsilon v : \varepsilon \in [-\bar{\varepsilon}, \bar{\varepsilon}]\} \subseteq \Omega$ for some $\bar{\varepsilon} > 0$. This follows from the Pontryagin Maximum Principle, which yields the existence of a nontrivial solution $p(\cdot)$ of the adjoint equation such that the function $\Omega \ni w \mapsto H(\xi(t), p(t), w)$ is minimized at $w = u(t)$ for a.e. t.)

Third, singular trajectories show up for more general optimal control problems, in which there are no restrictions on the control values, and the cost functional is a Lagrangian integral $\int_0^T L(\xi(t), u(t))\,dt$. For a problem of that kind, if the time T is fixed, the Pontryagin Maximum Principle requires that we consider the Hamiltonian H given by

$$H(x,p,p_0,u) = \langle p, f(x,u)\rangle + p_0 L(x,u)\,,$$

where p_0 is a new scalar variable known as the "abnormal multiplier." Furthermore, p_0 is required to be a nonnegative constant, but it could be zero, in which case we obtain a whole class of extremals characterized by conditions that do not involve L at all. These are known as "abnormal extremals," and

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turn out to be exactly the singular trajectories for the system (1). Abnormal extremals are known to play an important role in the study of the regularity properties of the value function for real-analytic optimal control problems.

Finally, singular trajectories are feedback invariants, and for large classes of control problems thay can actually be used to generate complete sets of feedback invariants.

This impressive book discusses the general theory of singular trajectories, and presents in detail a large number of important results, many of which are due to the authors. The book is, however, much broader and comprehensive than the title may suggest, and the authors have gone out of their way to make it accessible to an audience far wider than a narrow circle of specialists. In fact, it could be regarded as a general introduction to the field of differential-geometric optimal control theory, and would be useful for a two-semester graduate course taught on a high mathematical level, as long as it is supplemented with some additional readings on basic material for which Bonnard and Chyba do not provide all the necessary background.

The first two chapters introduce classical material on linear systems, controllability, time-optimal control and optimal synthesis, and on optimal control, the Pontryagin Maximum Principle, and the second variation.

Chapter 3 gives a brief introduction to symplectic geometry and Poisson brackets, and then moves on to discuss minimum-time control for scalar-control systems of the form $\dot{x} = X(x) + uY(x)$, $|u| \leq 1$, showing how to find singular optimal controls as feedback laws depending on the state and the adjoint vector, using the Poisson bracket formalism.

In Chapter 4 the authors show how to use singular trajectories to obtain feedback classifications of affine systems and of distributions.

Chapter 5 deals with controllability and high-order necessary conditions for optimality, and in particular presents the celebrated Legendre-Clebsch and Goh conditions as necessary conditions for rigidity of trajectories. Chapter 6 then gives an extremely clear discussion of conjugate points, with applications to time-optimal synthesis in two dimensions (which can be read as an excellent introduction to the more detailed work by this reviewer and by Bressan and Piccoli, cf. [6], [7], [8] and [2], [3]), optimal control in $\mathbb{R}^3$, and a very helpful study of an example of an optimal abnormal subriemannian extremal introduced by this reviewer and W. Liu in [4].

Chapter 7 shows the power of the methods introduced in the book, and the importance of singular trajectories, by applying them to the study of minimum time control for chemical batch reactors. This chapter is a must for any graduate course on optimal control whose instructor wants to show to the students examples of serious applications of hard mathematics.

In Chapter 8, the authors study generic properties of singular extremals. For systems of the form $\dot{x} = F_0(x) + uF_1(x)$, with a scalar control, they prove that, for generic pairs (F_0, F_1) of C^∞ vector fields, only minimal order singular extremals can occur.

Chapter 9 deals with subriemannian geometry, and contains a detailed proof of a spectacular result, due to the authors together with A. Agrachev and I. Kupka, cf. [1], on the structure of the cut locus for the Martinet case and in particular (in Theorem 29, page 267) of the non-subanalyticity of the corresponding subriemannian spheres.

Chapter 10 discusses Lagrangian submanifolds and the stratifications of the cotangent bundle associated to the Hamilton-Jacobi-Bellman equation. Finally, in Chapter 11 the authors present the results of numerical computations, showing elegant pictures of conjugate loci and cut loci.

Finally, in a brief concluding chapter (Ch. 12) some conjectures and open problems are presented.

This is a diffcult book, because it contains lots of technical, complicated proofs, that take time and effort to read and understand. It is not recommended reading for the mathematically faint-hearted. But those who appreciate really deep mathematics, written with passion, on a high technical level, by authors who have themselves made significant contributions, will find this book rewarding and will read it—and struggle and suffer with its many long and hard arguments— with enormous pleasure.

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