LINE-INTEGRAL ESTIMATES AND MOTION PLANNING USING THE CONTINUATION METHOD

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Abstract. For smooth driftless control-affine systems, the approach to the nonholonomic path planning problem based on the continuation method leads to an ordinary differential equation in the space of open-loop controls, known as the "path-lifting differential equation" (PLE). Global existence of the solutions of the PLE is known for systems that satisfy R. Strichartz' Strong Bracket Generating Condition (SBGC). We extend this existence result by proving an estimate for certain line integrals along trajectories, and showing that this estimate implies the global existence result for the case when the SBGC is satisfied, and also in some cases where it is not. In particular, we apply our results to a model of a front-wheel-driven car —for which the SBGC does not hold— and show that the continuation method solves the path-planning problem for this model.

Key words. Nonholonomic path planning. Continuation method. Abnormal extremals. Sub-riemannian manifolds. Front-wheel-driven car.

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1. Introduction. In [24], a continuation method was proposed for nonholonomic path-finding problems. Later, in [25], this method was successfully applied to driftless control-affine systems that satisfy the *Strong Bracket Generating Condition* (SBGC). The SBGC had been introduced by Strichartz in his papers [19, 20], in order to make sure that all subriemannian minimizers satisfy the "geodesic equation." The necessary condition of the Pontryagin Maximum Principle (cf. [13]) says that every minimizer is an extremal, and extremals can either be "normal," if the "abnormal multiplier" λ_0 does not vanish, or "abnormal," if $\lambda_0=0$. Normal extremals satisfy the geodesic equation. The SBGC guarantees that nontrivial abnormal extremals do not exist, and hence implies that all minimizers satisfy the geodesic equation. Abnormal extremals are currently the object of active research due to their importance in optimal control and sub-Riemannian geometry (cf. [8, 9], [10], [14], [15], [16], [22], [23], [28]).

It has turned out that abnormal extremals play a significant role in many problems other than that of the structure of minimizers. For example, it has been shown in [3, 24, 25] that they are one of the main obstructions to an efficient application of the continuation method to solve path-finding problems. It is the purpose of the present paper to show how this obstruction can sometimes be overcome. We will discuss in detail a

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case where nontrivial abnormal extremals occur, and show how the continuation method can still provide complete answers to the *Motion Planning Problem* (MPP) and the MPP with obstacles (MPPO).

We will consider completely controllable control-affine systems without drift, of the form $\Sigma : \dot{x} = \sum_{i=1}^{m} u_i f_i(x)$, $x \in M$, where M is a smooth manifold. Precisely, a system will be specified by giving a pair $(M, \mathbf{f})$, where M is a smooth, connected manifold, and $\mathbf{f} = (f_1, \ldots, f_m)$ is a finite sequence of smooth vector fields on M. The number m is the *number of controls*, $\mathbb{R}^m$ is the *control space*, M is the *state space*, and we will use n to denote the dimension of M. We let H be the space of all square-integrable open-loop controls that are defined on the interval $[0,1]$, so $H = L^2([0,1], \mathbb{R}^m)$n, the set of square-integrable $\mathbb{R}^m$-valued functions on $[0,1]$.

We will assume that Σ satisfies the *Lie Algebra Rank Condition*:

(LARC) $\boxed{\begin{array}{l}\textit{If } \Lambda \textit{ is the Lie algebra of vector fields generated by } f_1, \ldots, f_m, \\ \textit{and } \Lambda(x) = \{V(x) : V \in \Lambda\}, \textit{ then } \dim \Lambda(x) = n \textit{ for all } x \in M.\end{array}}$

and the *No Explosions Condition*:

(NEC) $\boxed{\begin{array}{l}\textit{For every point } \bar{x} \in M \textit{ and every control } \eta \in H \textit{ there exists} \\ \textit{a trajectory } \xi : [0,1] \to M \textit{ of } \Sigma \textit{ for } \eta \textit{ satisfying the initial} \\ \textit{condition } \xi(0) = \bar{x}.\end{array}}$

It is then clear that ξ is unique. From now on, we use $\xi_{\bar{x},\eta}$ to denote the unique trajectory ξ of Σ for η such that $\xi(0) = \bar{x}$.

It follows from our assumptions that the *Complete Controllability* property (CC) holds for Σ: given any two points p, q in M there exists a control $\eta \in H$ such that $\xi_{p,\eta}(1) = q$.

The *Motion Planning Problem* (MPP) is the problem of finding a procedure that, for every pair (p,q) of points of M, will produce a control $\eta_{p,q} \in H$ that steers p to q. A variant of this is the *Motion Planning Problem with Obstacles* (MPPO), in which a closed subset C of M —the "obstacle"— is specified, and one seeks a procedure yielding, for every p, q in $M \setminus C$, a control $\eta_{p,q} \in H$ that steers p to q and is such that the trajectory $\xi_{p,\eta_{p,q}}$ is entirely contained in $M \setminus C$. These questions are among the most challenging issues in Robotics and its real-world applications —such as mobile robots with wheels or arms— and have led in recent years to the development of many ingenious ideas and methods (cf. [2], [4], [5], [6], [7], [12], [27]).

Our approach to the MPP, already outlined in [24] and numerically implemented with success by Seereeman and Wen in several examples (cf. [17]), is one of several that make a systematic use of "differential geometric" methods. Other examples of these methods are

(i) nilpotent approximation (cf. [5]);
(ii) highly oscillatory inputs (cf. [26]);
(iii) optimal control (cf. [27]);
(iv) differential flatness (cf. [11]);
(v) use of generic loops (cf. [18]).

The continuation method (sometimes also called *deformation* or *homotopy method*) is well known for solving nonlinear equations of the form $F(x) = y$. In general, it proceeds by starting from a value $\bar{x}$ of x and its corresponding image $\bar{y} = F(\bar{x})$, joining $\bar{y}$ to the given y by a continuous path π, and trying to lift π to a path Π such that $F \circ \Pi = \pi$. To construct Π one differentiates the condition $F(\Pi(s)) = \pi(s)$ and obtains $DF(\Pi(s)) \cdot \dot{\Pi}(s) = \dot{\pi}(s)$, which can be satisfied by solving $\dot{\Pi}(s) = P(\Pi(s)) \cdot \dot{\pi}(s)$, if $P(x)$ is, for each x, a right inverse of $DF(x)$.

In the specific case of the MPP, we will apply the continuation method by taking F to be the *endpoint map* $\mathcal{E}_p : H \to M$ corresponding to a point $p \in M$. The map $\mathcal{E}_p$ is defined by letting

$$\mathcal{E}_p(\eta) \stackrel{\text{def}}{=} \xi_{p,\eta}(1)\,. \tag{1.1}$$

Since Σ satisfies the NEC, $\mathcal{E}_p(\eta)$ is defined for every $\eta \in H$ and $\mathcal{E}_p$ is surjective. Therefore, the MPP can be reformulated as follows: *for every $p, q \in M$, exhibit a control $\eta_{p,q}$ in H such that $\mathcal{E}_p(\eta) = q$.* In other words, if we fix p, we are asking to find a map $\mathcal{I}_p : M \to H$ such that $\mathcal{E}_p(\mathcal{I}_p(q)) = q$ for all q, i.e. such that

$$\mathcal{E}_p \circ \mathcal{I}_p = \text{identity}\,. \tag{1.2}$$

That is, we want to find a *right inverse* of $\mathcal{E}_p$.

It is clear that the existence of a smooth right inverse is guaranteed, locally, in a neighborhood of any point $\eta \in H$ such that $D\mathcal{E}_p(\eta)$ —the differential of $\mathcal{E}_p$ at η— is surjective as a linear map from H to $T_{\mathcal{E}_p(\eta)}M$. (From now on, we use $T_x M$ to denote the tangent space of M at x.)

So it is natural to expect that the *singular points* of $\mathcal{E}_p$ —i.e. the points $\eta \in H$ such that $\operatorname{rank} D\mathcal{E}_p(\eta) < n$— are going to cause difficulties. We will use $S_{\Sigma,p}$ to denote the set of singular points of the map $\mathcal{E}_p$.

To apply the continuation method, the first step will be to identify the set $S_{\Sigma,p}$. The second step will then consist of "lifting" paths $\pi : [0,1] \to M$ that avoid the set $\mathcal{E}_p(S_{\Sigma,p})$ of singular values of $\mathcal{E}_p$, to paths $\Pi : [0,1] \to H$ such that

$$\forall s \in [0,1],\ \mathcal{E}_p\Big(\Pi(s)\Big) = \pi(s)\,. \tag{1.3}$$

Differentiating (1.3) yields the equation

$$D\mathcal{E}_p(\Pi(s)) \cdot \frac{d\Pi}{ds}(s) = \frac{d\pi}{ds}(s)\,. \tag{1.4}$$

If $D\mathcal{E}_p(\Pi(s))$ has full rank, then (1.4) can be solved for $\Pi(s)$ by choosing Π such that

$$\frac{d\Pi}{ds}(s) = P(\Pi(s)) \cdot \frac{d\pi}{ds}(s)\,, \tag{1.5}$$

where, for $\eta \in H \setminus S_{\Sigma,p}$, $P(\eta)$ is a right inverse of $D\mathcal{E}_p(\eta)$. (For example, we can choose $P(\eta)$ to be the Moore-Penrose pseudoinverse of $D\mathcal{E}_p(\eta)$.)

We are then led to study (1.5), the *Path Lifting Equation* (PLE), regarded as an ordinary differential equation in H.

It should be clear from our preceding discussion that the main issues that will have to be confronted to provide a theoretical justification of the continuation method are:

- *Nondegeneracy*: we have to choose the path π so that $\pi(s) \notin \mathcal{E}_p(S_{\Sigma,p})$ whenever $s \in [0,1]$, thus guaranteeing that $D\mathcal{E}_p(\Pi(s))$ will always be of full rank.
- *Explosions*: we have to be sure that no explosions occur, i.e. that the Path Lifting Equation has a global solution on the whole interval $[0,1]$. (Local existence and uniqueness of the solution of the PLE are usually easy to establish, e.g. if $\mathcal{E}_p$ is of class C^2. But in principle the solution may fail to be defined globally on $[0,1]$.)

It is not hard to show that the singular points of $\mathcal{E}_p$ are exactly the controls $\eta \in H$ such that $(\xi_{p,\eta}, \eta)$ is an abnormal extremal. Rather remarkably, it turns out that, when the SBGC (cf. Section 3) holds, not only do nontrivial abnormal extremals fail to exist, but in fact one can prove an estimate implying that *the PLE has no explosions*, so the continuation method works globally, cf. [3], [25].

This result can be generalized and extended to cases where the SBGC does not hold. The key step —carried out in Section 4 below— is to prove some estimates on line integrals along trajectories.

2. Obstacles. To handle the MPPO, we follow a strategy suggested by E.D. Sontag [18]. Let $M, \mathbf{f}$ be as before, and let C be a closed subset of M. We use $\widetilde{M}$ to denote $M \setminus C$. We replace our system Σ by another system $\widetilde{\Sigma}$,

$$\dot{x} = \sum_{i=1}^{m} u_i \hat{f}_i(x)\,,$$

where $\hat{f}_i(x) = \psi(x) f_i(x)$, $i = 1, \ldots, m$, and $\psi : M \to \mathbb{R}$ is a C^∞ function such that $\psi = 0$ on C and $0 < \psi \leq 1$ on $\widetilde{M}$.

It is then easy to verify that the new system, restricted to $\widetilde{M}$, satisfies the LARC and the NEC. (To prove that the NEC holds, let $p \in \widetilde{M}$, let

$\eta \in H$, and let $\xi : I \to \widetilde{M}$ be a maximally defined solution of

$$\dot{x}(t) = \sum_{i=1}^{m} \eta_i(t)\hat{f}_i(x(t)), \quad x(0) = p, \tag{2.1}$$

so I is a subinterval of $[0,1]$, $0 \in I$, and ξ cannot be extended to a solution on a strictly larger subinterval of $[0,1]$. Suppose that $I \neq [0,1]$. Then $I = [0, a[$ for some $a \in]0,1]$. Define $\zeta_i(t) = \eta_i(t)\psi(\xi(t))$ for $t \in I$, $\zeta_i(t) = 0$ for $t \in [0,1]\backslash I$. Then $\zeta \in H$ —because ψ is bounded— and ξ is a solution of the initial value problem $\dot{x}(t) = \sum_{i=1}^{m} \zeta_i(t) f_i(x(t))$, $x(0) = p$. By the NEC, this problem has a solution ξ^* defined on $[0,1]$, which of course must agree with ξ on I. But then $\lim_{t\uparrow a} \xi(t)$ exists, showing that ξ is extendable to a solution of (2.1) on $[0,a]$, and contradicting the maximality of ξ. To prove that the new system satisfies the LARC, we let $\widetilde{\Lambda}_1$ be the linear span of $\hat{f}_1, \ldots, \hat{f}_m$, and define $\widetilde{\Lambda}_k$ recursively for $k > 1$ to be the linear span of all the brackets $[g_1, g_2]$ with $g_i \in \widetilde{\Lambda}_{k_i}$ for $i = 1,2$, $k_1 + k_2 \leq k$. It then suffices to observe that every bracket $[\hat{f}_{i_1}, [\hat{f}_{i_2}, \ldots, [\hat{f}_{i_{k-1}}, \hat{f}_{i_k}] \ldots]]$ is equal to $\psi^k [f_{i_1}, [f_{i_2}, \ldots, [f_{i_{k-1}}, f_{i_k}] \ldots]]$ modulo $\widetilde{\Lambda}_{k-1}$.) So on each connected component of $\widetilde{M}$ our new system satisfies all our hypotheses.

Therefore any general method that solves the MPP immediately yields a general solution of the MPPO. On the other hand, a method that solves the MPP for a restricted class $\mathcal{C}$ of problems will solve the MPPO for those systems Σ and obstacles C such that the new system $\widetilde{\Sigma}$ belongs to $\mathcal{C}$. It turns out that certain algebraic properties such as the Strong Bracket Generating Condition (defined in Section 3 below) are necessarily true for $\widetilde{\Sigma}$ if they hold for Σ, so for systems satisfying those conditions a general solution of the MPP yields a general solution of the MPPO.

3. Definitions and notations. In this section we collect the technical tools necessary to present the application of the continuation method to nonholonomic control systems. We refer to [3] for all the details.

Throughout this paper, if n and m are positive integers, then $\mathbb{R}^{n\times m}$ denotes the space of real matrices with n rows and m columns, and $\mathbb{R}^n$ stands for $\mathbb{R}^{n\times 1}$, so the members of $\mathbb{R}^n$ are *column* vectors. We use $M^\dagger$ to denote the transpose of an $M \in \mathbb{R}^{n\times m}$. In particular, if $v \in \mathbb{R}^n$, then $v^\dagger$ is an n-dimensional *row* vector.

The word "smooth" will mean "of class C^∞." If M is a smooth manifold, and $x \in M$, we use T_xM, T_x^*M, TM and T^*M to denote, respectively, the tangent and cotangent spaces of M at x, and the tangent and cotangent bundles of M. We write $T^*M \setminus \{0\}$ to denote the set $\{(x,z) \in T^*M : z \neq 0\}$, and $C^\infty(M)$, $V(M)$ to denote, respectively, the spaces of smooth functions and smooth vector fields on M.

For $y \in T_xM$ and $z \in T_x^*M$, we write the duality product $z(y)$ as $\langle z, y\rangle$ or, simply, $z.y$ or zy. Relative to a coordinate chart $\kappa = (x^1, \ldots, x^n)^\dagger$ of M, tangent vectors and cotangent vectors will be viewed as column and row vectors, respectively. If $x \in \text{Domain}(\kappa)$, $z \in T_x^*M$ and $y \in T_xM$, then the components of z relative to κ are the numbers $z_i = z.\frac{\partial}{\partial x^i}$, and those of y are the numbers $y^i = dx^i(y)$, so $z.y$ is the ordinary matrix product $\sum_i z_i y^i$.

We fix an integer m such that $m \geq 1$, and a driftless control-affine system $\Sigma = (M, \mathbf{f})$ with dynamical equation

$$\dot{x} = \sum_{i=1}^{m} u_i f_i(x)\,,$$

whose state space M is a smooth connected n-dimensional manifold. We assume that $\mathbf{f} = (f_1, \ldots, f_m)$ is an m-tuple of vector fields of class C^∞, and Σ satisfies the LARC and the NEC.

An "open-loop control" —or, simply, a "control"— for Σ is a Lebesgue-integrable map $\eta : [a, b] \to \mathbb{R}^m$ which is defined on some compact subinterval $[a, b]$ of the real line. The specification of η includes in particular that of its domain, and we will write $a(\eta)$, $b(\eta)$ to denote the endpoints of the domain of η.

As before, we let $H = L^2([0, 1], \mathbb{R}^m)$, so H is a special class of controls, namely, that of those controls η that are square-integrable and such that $a(\eta) = 0$ and $b(\eta) = 1$.

Clearly, a control η can be thought of as an m-tuple $\eta = (\eta_1, ..., \eta_m)^\dagger$ with entries η_i in $L^1([a(\eta), b(\eta)], \mathbb{R}))$. For any subinterval $[a, b]$ of $\mathbb{R}$, we let

$$||\eta||^2_{L^2([a,b])} \stackrel{\text{def}}{=} \sum_{i=1}^{m} \int_a^b \eta_i(t)^2\, dt,$$

so $||\eta||^2_{L^2([a,b])} < \infty$ iff η is square-integrable. If $\eta \in H$, we write

$$||\eta||^2_H \stackrel{\text{def}}{=} ||\eta||^2_{L^2([0,1])} = \sum_{i=1}^{m} \int_0^1 \eta_i(t)^2\, dt\,.$$

If $\eta = (\eta_1, ..., \eta_m)^\dagger$ is a control, a *trajectory of* Σ *corresponding to* η is an absolutely continuous curve $\xi : [a(\eta), b(\eta)] \to M$ such that

$$\dot{\xi}(t) = \sum_{i=1}^{m} \eta_i(t) f_i(\xi(t)) \quad \text{for a.e.} \quad t \in [a(\eta), b(\eta)]. \tag{3.1}$$

A *trajectory-control pair* of Σ is a pair $\gamma = (\xi, \eta)$ such that η is an open-loop control for Σ and ξ is a trajectory of Σ corresponding to η. We use $TCP(\Sigma)$ to denote the set of all trajectory-control pairs of Σ. If $\gamma = (\xi, \eta) \in TCP(\Sigma)$, we write $a(\gamma) \stackrel{\text{def}}{=} a(\eta)$ and $b(\gamma) \stackrel{\text{def}}{=} b(\eta)$.

For $X \in V(M)$, we use H_X to denote the *Hamiltonian function*, or *momentum function* corresponding to X, i.e. the function $H_X : T^*M \to \mathbb{R}$ given by

$$H_X(x,z) = \langle z, X(x)\rangle, \quad \text{for} \;\; x \in M, \;\; z \in T^*_x M. \tag{3.2}$$

For $X \in V(M)$, we let TX, the *variational vector field of* X (cf. [21]) be the element of $V(TM)$ which is the infinitesimal generator of the 1-parameter pseudogroup of maps $\{\Phi_t^{(1)}\}_{t\in\mathbb{R}}$, where $\{\Phi_t\}_{t\in\mathbb{R}}$ is the 1-parameter pseudogroup of maps $\Phi_t : M \to M$ generated by the vector field X, and $\Phi_t^{(1)}(x,y) = (\Phi_t(x), D\Phi_t(x).y)$ for $(x,y) \in TM$. Similarly, T^*X, the *variational covector field*, or *Hamiltonian lift*, of X, is the element of $V(T^*M)$ which is the infinitesimal generator of the 1-parameter pseudogroup of maps $\{\Phi_t^{(1,*)}\}_{t\in\mathbb{R}}$, where $\Phi_t^{(1,*)}(x,z) = (\Phi_t(x), ((D\Phi_t(x))^\dagger)^{-1}.y)$ whenever (x,z) belongs to T^*M.

If $\kappa = (x^1,\ldots,x^n)^\dagger$ is a coordinate chart of M with domain Ω, then we can use κ to identify Ω with its image $\kappa(\Omega) \subseteq \mathbb{R}^n$. Then the set $\{(x,z) \in T^*M : x \in \Omega\}$ can be identified with the product $\kappa(\Omega)\times\mathbb{R}^n$, which is a subset of $\mathbb{R}^{2n}$. If $X(x) = \sum_i X^i(\kappa(x))\frac{\partial}{\partial x^i}$ for $x \in \Omega$, then the $2n$-dimensional vector $(T^*X(x,z))_\kappa$ of components of $T^*X(x,z)$ is given by

$$((T^*X)(x,z))_\kappa = \begin{bmatrix} X_\kappa(x) \\ -(zDX_\kappa(x))^\dagger \end{bmatrix}, \tag{3.3}$$

where $X_\kappa(x) \stackrel{\text{def}}{=} (X^1(x),\ldots,X^n(x))^\dagger$, and $DX_\kappa(x)$ is the Jacobian matrix of X_κ, regarded as a map from Ω to $\mathbb{R}^n$, at $x\in\Omega$.

We use $T^*\Sigma$ to denote the *Hamiltonian lift of* Σ, i.e. the control system $T^*\Sigma = (T^*M, T^*\mathbf{f})$, where $T^*\mathbf{f} = (T^*f_1,\ldots,T^*f_m)$. Then a trajectory of $T^*\Sigma$ corresponding to a control $\eta = (\eta_1,\ldots,\eta_m)$ is a pair $\Xi = (\xi,\lambda)$ where ξ is a trajectory of Σ corresponding to η and $\lambda : [a(\eta), b(\eta)] \to T^*M$ is a field of covectors along ξ (i.e. a map that satisfies $\lambda(t) \in T^*_{\xi(t)}M$ for each value of t) which is absolutely continuous and such that, in coordinates,

$$\dot\lambda(t) = -\lambda(t).\left(\sum_{i=1}^m \eta_i(t) Df_i(\xi(t))\right) \qquad \text{for a.e. } t \text{ in } [a(\eta), b(\eta)]. \tag{3.4}$$

If $\gamma = (\xi,\eta) \in TCP(\Sigma)$, then a field of covectors λ along ξ such that $\Xi = (\xi,\lambda)$ is a trajectory of $T^*\Sigma$ corresponding to η is called an *adjoint vector along* γ, and (3.4) is the *adjoint equation along* γ. Since this equation is linear in λ, for any terminal condition $\lambda(b(\gamma))$ in $T^*_{\xi(1)}M$ the solution of (3.4) is well-defined on $[a(\gamma), b(\gamma)]$.

Recall that a smooth vector field g on M is —or "can be canonically identified with," depending on how one defines "vector field"— a first-order differential operator $\psi \to g\psi$ on functions, so $g\psi \in C^\infty(M)$ if $\psi \in C^\infty(M)$

and $g \in V(M)$. We will write $L_g\psi$, or $\nabla_g\psi$, to denote the function $g\psi$, and refer to this function as the *g-Lie derivative*, or the *g-directional derivative*, of ψ. More generally, if $r \geq 1$, then for any r-tuple $\mathbf{g} = (g_1, \ldots, g_r)$ of elements of $V(M)$ and any function $\psi \in C^\infty(M)$, the $\mathbb{R}^r$-valued map $\nabla_{\mathbf{g}}\psi : M \to \mathbb{R}^r$ given by $\nabla_{\mathbf{g}}\psi(x) = ((g_1\psi)(x), \cdots, (g_r\psi)(x))^\dagger$ will be called the **g**-*Lie derivative* of ψ.

With this notation, the time derivative of ψ along a trajectory ξ of Σ corresponding to a control $\eta = (\eta_1, ..., \eta_m)^\dagger$ is given by

$$\begin{aligned}\frac{d}{dt}\left[\psi(\xi(t))\right] &= d\psi(\xi(t)) \cdot \dot{\xi}(t) \\ &= \sum_{i=1}^{m} \eta_i(t)(f_i\psi)\Big(\xi(t)\Big) \\ &= \eta(t)^\dagger . \nabla_{\mathbf{f}}\psi(\xi(t)) .\end{aligned}$$

We write $(\eta^\dagger \cdot \nabla_{\mathbf{f}}\psi) \circ \xi$ to denote the function $t \to \eta(t)^\dagger . \nabla_{\mathbf{f}}\psi(\xi(t))$. Similarly, if $\mathbf{g} = (g_1, \cdots, g_m)$ is an m-tuple of elements of smooth vector fields on M, and $\eta = (\eta_1, \cdots, \eta_m)^\dagger$ belongs to H, then we will use the expressions

$$\begin{array}{lcl}(\eta^\dagger \cdot \mathbf{g}) \circ \xi\,, & & t \to \sum_{i=1}^{m} \eta_i(t) g_i\Big(\xi(t)\Big)\,, \\ (\eta^\dagger \cdot D\mathbf{g}) \circ \xi\,, & \text{to denote the functions} & t \to \sum_{i=1}^{m} \eta_i(t) Dg_i\Big(\xi(t)\Big)\,, \\ (\eta^\dagger \cdot T^*\mathbf{g}) \circ \Xi\,, & & t \to \sum_{i=1}^{m} u_i(t) T^* g_i\Big(\Xi(t)\Big)\,,\end{array}$$

respectively.

Then (3.1) and (3.4) become

$$\begin{aligned}\dot{\xi} &= (\eta^\dagger \cdot \mathbf{f}) \circ \xi, \\ \dot{\lambda} &= -\lambda(\eta^\dagger \cdot D\mathbf{f}) \circ \xi, \\ \dot{\Xi} &= (\eta^\dagger \cdot T^*\mathbf{f}) \circ \Xi .\end{aligned}$$

We say that the system Σ satisfies *the Strong Bracket Generating Condition* (SBGC) (cf. [19]) if

(SBGC) *Whenever $\theta = (\theta_1, \ldots, \theta_m) \in \mathbb{R}^m \setminus \{0\}$ and $x \in M$, if $\theta\mathbf{f}$ denotes the vector field $\sum_{i=1}^{m} \theta_i f_i$, then the $2m$ vectors $f_1(x), \cdots, f_m(x), [\theta\mathbf{f}, f_1](x), \cdots, [\theta\mathbf{f}, f_m](x)$ span T_xM.*

Let p be a point of M. We already defined in the introduction the endpoint map $\mathcal{E}_p$, which associates to every control η the point $\xi_{p,\eta}(1)$, where $\xi_{p,\eta}$ is the trajectory of Σ for the control η that goes through p at time $t = 0$. It is not hard to show that *$\mathcal{E}_p$ is Fréchet differentiable at every $\eta \in H$*, and the differential $D\mathcal{E}_p(\eta)$ is the linear map from H to $T_{\mathcal{E}_p(\eta)}M$ that sends $v = (v_1, \ldots, v_m) \in H$ to the vector $D\mathcal{E}_p(\eta) \cdot v$ defined as follows:

For every $z \in T^*_{\mathcal{E}_p(\eta)}M$*, let* $\lambda_{p,z,\eta}$ *be the adjoint vector* λ *along* $(\xi_{p,\eta},\eta)$ *that satisfies the terminal condition* $\lambda(1) = z$*. For* $i = 1,\ldots,m$*, let* $\varphi_{i,p,z,\eta} : [0,1] \to \mathbb{R}$ *be the function defined by*

$$\varphi_{i,p,z,\eta}(t) = \lambda_{p,z,\eta}(t) \cdot f_i(\xi_{p,\eta}(t)). \tag{3.5}$$

Then $D\mathcal{E}_p(\eta)\cdot v$ *is the unique vector* $w \in T_{\mathcal{E}_p(\eta)}M$ *such that*

$$\langle z, w\rangle = \sum_{i=1}^{m}\int_0^1 v_i(t)\varphi_{i,p,\eta,z}(t)dt \quad \text{for all} \quad z \in T^*_{\mathcal{E}_p(\eta)}M\,. \tag{3.6}$$

The functions $\varphi_{i,p,\eta,z}$ are the *switching functions* along $(\xi_{p,\eta},\eta)$, corresponding to the adjoint vector $\lambda_{p,z,\eta}$. Clearly,

$$\varphi_{i,p,z,\eta}(t) = H_{f_i}(\xi_{p,\eta}(t),\lambda_{p,z,\eta}(t))\,.$$

(The phrase "switching function" comes from optimal control. These functions play a fundamental role in sub-Riemannian geometry and in the analysis of the structure of optimal trajectories for control-affine systems. We refer the reader to [8, 10, 14, 15, 16, 22, 23, 27, 28] for recent developments in these areas.)

We define $G_p(\eta) = D\mathcal{E}_p(\eta)\circ D\mathcal{E}_p(\eta)^\dagger$. Then $G_p(\eta)$ is a linear map from $T^*_{\mathcal{E}_p(\eta)}M$ to $T_{\mathcal{E}_p(\eta)}M$, called the *controllability Grammian* of the trajectory $\xi_{p,\eta}$. It is clear from our previous characterization of $D\mathcal{E}_p(\eta)$ that, if $z \in T^*_{\mathcal{E}_p(\eta)}M$, then $D\mathcal{E}_p(\eta)^\dagger z$ is the linear functional on H that sends $v = (v_1,\ldots,v_m)^\dagger \in H$ to $\sum_{i=1}^m\int_0^1 v_i(t)\varphi_{i,p,\eta,z}(t)dt$. Therefore, using the natural identification of the real Hilbert space H with its dual, we have

$$D\mathcal{E}_p(\eta)^\dagger z = (\varphi_{1,p,z,\eta},\ldots,\varphi_{m,p,z,\eta})^\dagger\,. \tag{3.7}$$

From this it follows immediately that

$$||D\mathcal{E}_p(\eta)^\dagger z||^2 = \langle z, G_p(\eta)\cdot z\rangle = \sum_{i=1}^{m}\int_0^1 \varphi_{i,p,z,\eta}(t)^2dt. \tag{3.8}$$

We define

$$\varphi_{p,z,\eta}(t) \stackrel{\text{def}}{=} \Big(\varphi_{1,p,z,\eta}(t)\,,\,\ldots\,,\,\varphi_{m,p,z,\eta}(t)\Big)^\dagger\,,$$

$$||\varphi_{p,z,\eta}||^2_{L^2([a,b])} \stackrel{\text{def}}{=} \int_a^b\Big(\sum_{i=1}^{m}\varphi_{i,p,z,\eta}(t)^2\Big)dt\,, \tag{3.9}$$

if $[a,b]$ is an arbitrary subinterval of $[0,1]$.

As will become clear later, the crucial step in the proofs of our main results will be the derivation of appropriate lower bounds for the norms

$||\varphi_{p,z,\eta}||_{L^2([a,b])}$ defined in (3.9). One way to get these bounds is to study the time evolution of the switching functions, by looking at their time derivatives. To do this, we introduce some more notation. For $B \in V(M)$, $p \in M$, $\eta \in H$, $z \in T^*_{\mathcal{E}_p(\eta)}M$, we define $\varphi_{B,p,z,\eta}(t) = H_B(\xi_{p,\eta}(t), \lambda_{p,z,\eta}(t))$, so $\varphi_{B,p,z,\eta}$ is the momentum function H_B evaluated along the trajectory $(\xi_{p,\eta}, \lambda_{p,z,\eta})$ of $T^*\Sigma$. Then the adjoint equation implies that

$$\dot{\varphi}_{B,p,z,\eta}(t) = \sum_{j=1}^{m} \eta_j(t)\varphi_{[f_j,B],p,z,\eta}(t) \quad \text{a.e. in } [0,1]. \tag{3.10}$$

If B is an iterated Lie bracket of the f_i's of degree greater than 1, then φ_B will be called a *derived switching function* (DSF). Equation (3.10) expresses the time derivative of each switching function or derived switching function as a linear combination of higher-order derived switching functions. The LARC implies that for any compact subset K of M there exists a finite collection $\mathcal{F}_K$ of brackets such that, on K, every iterated bracket B can be expressed as a linear combination $\sum_{B' \in \mathcal{F}_K} \alpha_{B,B'} B'$ with smooth coefficients of the vector fields in $\mathcal{F}_K$. The NEC implies that for every $C > 0$ and every compact subset K of M there exists a compact subset K_C of M such that $\xi_{p,\eta}$ is entirely contained in K_C whenever $p \in K$ and $||\eta||_H \le C$. (Indeed, the NEC implies that the map $(p,t,\eta) \to \xi_{p,\eta}(t)$ is defined on $M \times [0,1] \times H$. Moreover, this map is continuous if H is given the weak topology. Since $K \times [0,1] \times \{\eta \in H : ||\eta||_H \le C\}$ is compact in $M \times [0,1] \times H^{weak}$, the conclusion follows.)

Combining these two observations and (3.10), we have

PROPOSITION 3.1. *For every compact subset K of M and every $C > 0$ there exists a finite set $\mathcal{F}$ of iterated brackets such that for every iterated bracket B there are smooth functions $\beta_{j,B,B'}$, for $j = 1,\dots,m$, $B' \in \mathcal{F}$, with the property that*

$$\dot{\varphi}_{B,p,z,\eta}(t) = \sum_{j=1}^{m} \sum_{B' \in \mathcal{F}} \eta_j(t)\beta_{j,B,B'}(\xi_{p,\eta}(t))\varphi_{B',p,z,\eta}(t) \ \text{a.e.}, \tag{3.11}$$

for every $p \in K$, $z \in T^*_{\mathcal{E}_p(\eta)}M$, $\eta \in H$ such that $||\eta|| \le C$. □

In particular, we can always take $\mathcal{F}$ to contain the zeroth-order brackets f_i, and then Proposition (3.11) says that the switching functions are part of a finite set of functions that satisfy a first-order linear homogeneous system of ordinary differential equations.

Let us now explain how the continuation method can be applied to the MPP for a system Σ that satisfies our hypotheses.

Fix p in M. Then $\mathcal{E}_p : H \to M$ is defined as before. The procedure is divided into three steps.

STEP 1 Determine the singular set $S_{\Sigma,p}$, and the singular value set $\mathcal{E}_p(S_{\Sigma,p})$.

The determination of $S_{\Sigma,p}$ can be done as follows: let us consider a nontrivial abnormal extremal $\xi_{p,\eta}$, arising from a control $\eta \in H$. The fact that $\xi_{p,\eta}$ is an abnormal extremal says that there exists a $z \in T^*_{\mathcal{E}_p(\eta)}M \setminus \{0\}$ such that the switching functions $\varphi_{i,p,z,\eta}$ vanish identically. One can then compute explicitly a system of differential equations of the kind described in Proposition (3.11). Careful analysis of this system, using the fact that $\varphi_{i,p,z,\eta} \equiv 0$ for $i = 1, \ldots, m$, may then lead to an explicit characterization of the sets $S_{\Sigma,p}$ and $\mathcal{E}_p(S_{\Sigma,p})$.

For a simple example, consider the *contact case*, i.e. suppose $n = 3$, $m = 2$, and $f_1(x), f_2(x), f_3(x)$ span T_xM for every $x \in M$, where we let $f_3 = [f_1, f_2]$. Let $\varphi_{3,p,z,\eta} = \varphi_{f_3,p,z,\eta}$. Then the system of Proposition (3.11) is, simply,

$$\dot{\varphi}_{1,p,z,\eta} = -\eta_2\varphi_{3,p,z,\eta}\,, \quad \dot{\varphi}_{2,p,z,\eta} = \eta_1\varphi_{3,p,z,\eta}\,, \quad \dot{\varphi}_{3,p,z,\eta} = \sum_{j=1}^{3} \eta_j\alpha_j\varphi_{j,p,z,\eta}\,,$$

where the α_j are smooth functions on M. If $\varphi_{1,p,z,\eta} \equiv \varphi_{2,p,z,\eta} \equiv 0$, then $\varphi_{3,p,z,\eta}$ never vanishes, because $\lambda_{p,z,\eta}(t) \neq 0$. So $\eta_1 \equiv \eta_2 \equiv 0$. Therefore $S_{\Sigma,p} = \{0\}$.

STEP 2 Fix $q \notin \mathcal{E}_p(S_{\Sigma,p})$. Find a control $\bar{\eta}$ such that $\bar{\eta} \notin S_{\Sigma,p}$. Let $\bar{q} = \mathcal{E}_p(\bar{\eta})$. Find, if possible, a C^1 arc $\pi : [0,1] \to M$ such that

$$\pi(0) = \bar{q}, \quad \pi(1) = q, \quad \pi([0,1]) \cap \overline{\mathcal{E}_p(S_{\Sigma,p})} = \emptyset.$$

This can be done, for example, if the set $M \setminus \overline{\mathcal{E}_p(S_{\Sigma,p})} = \emptyset$ is connected, and both $\bar{q}$ and q belong to it.

Clearly, the trivial (or zero) control $\bar{0}$ is in $S_{\Sigma,p}$. It turns out that when the SBGC holds S reduces to $\{0\}$ (cf. [19]; we just saw a special case of this in our analysis of the contact case). Unfortunately, for general nonholonomic problems the singular set $S_{\Sigma,p}$ is nontrivial.

STEP 3 By means of the PLE, lift the path π to a path Π in H. This requires the global existence of the solution of the PLE with the initial condition $\Pi(0) = \bar{\eta}$.

The desired Π is an absolutely continuous curve $\Pi : [0,1] \to H$ such that

$$\mathcal{E}_p(\Pi(s)) = \pi(s) \quad \text{for } s \in [0,1] \quad \text{and} \quad \Pi(0) = \bar{\eta}. \tag{3.12}$$

Equation (3.12) implies

$$\begin{cases} D\mathcal{E}_p(\Pi(s)) \cdot \frac{d\Pi}{ds}(s) = \frac{d\pi}{ds}(s) \ \text{ a.e. in } [0,1]\,, \\ \Pi(0) = \bar{\eta}\,. \end{cases} \tag{3.13}$$

Conversely, if (3.13) is satisfied in local coordinates, then it is easy to see that the derivative of $\mathcal{E}_p(\Pi(s)) - \pi(s)$ vanishes, so (3.12) holds. (We only talk about (3.13) being satisfied in local coordinates, because we do not want to demand *a priori* that the first equation of (3.13) be an equality of tangent vectors, since this would require that we already know that $\mathcal{E}_p(\Pi(s)) = \pi(s)$, because $D\mathcal{E}_p(\Pi(s)) \cdot \frac{d\Pi}{ds}(s)$ is a tangent vector at $\mathcal{E}_p(\Pi(s))$ and $\frac{d\pi}{ds}(s)$ is a tangent vector at $\pi(s)$. The main point here is that, *even if we do not know that $D\mathcal{E}_p(\Pi(s)) \cdot \frac{d\Pi}{ds}(s)$ and $\frac{d\pi}{ds}(s)$ are tangent vectors at the same point*, the fact that they are equal in coordinates implies that (3.12) holds.)

To solve (3.13) we proceed as follows. For $\eta \in H \setminus S_{\Sigma,p}$, let $P_p(\eta)$ denote the Moore-Penrose pseudo-inverse of $D\mathcal{E}_p(\eta)$, defined by

$$P_p(\eta) \stackrel{def}{=} D\mathcal{E}_p(\eta)^\dagger G_p(\eta)^{-1}, \tag{3.14}$$

where $G_p(\eta) \stackrel{def}{=} D\mathcal{E}_p(\eta) D\mathcal{E}_p(\eta)^\dagger : T^*_{\mathcal{E}_p(\eta)}M \to T_{\mathcal{E}_p(\eta)}M$ is the controllability Grammian defined above. Clearly, $P_p(\eta)$ is a linear map from $T_{\mathcal{E}_p(\eta)}M$ to H, such that $D\mathcal{E}_p(\eta) \circ P_p(\eta)$ is the identity map of $T_{\mathcal{E}_p(\eta)}M$.

Notice that the Hilbert space structure of the control space H plays a crucial role here, because $G_p(\eta)$ is well defined thanks to the fact that H is naturally identified with its dual, as explained earlier.

We now work with the initial value problem

$$\begin{cases} \frac{d\Pi}{ds}(s) = P_p(\Pi(s)) \cdot V(\mathcal{E}_p(\Pi(s)), s) \quad \text{a.e. in } [0,1], \\ \Pi(0) = \bar{\eta}, \end{cases} \tag{3.15}$$

where $(x,s) \to V(x,s)$ is a continuous map from $M \times [0,1]$ to TM such that each map $x \to V(x,s)$ is a smooth vector field on M whose first-order partial derivatives with respect to x are jointly continuous in x and s, and $V(\pi(s), s) = \frac{d\pi}{ds}(s)$. (The existence of such a V is trivial.)

It is easy to see that P_p is Lipschitz continuous, so the differential equation of (3.15) has local existence and uniqueness of solutions. Moreover, it is clear that if Π is any solution of (3.15), and we define $\theta(s) = \mathcal{E}_p(\Pi(s))$, then $\dot\theta(s) = V(\theta(s), s)$ and $\theta(0) = \bar{q}$. Since $\dot\pi(s) = V(\pi(s), s)$ and $\pi(0) = \bar{q}$, we have $\theta \equiv \pi$ (because the O.D.E. $\dot x = V(x,s)$ obviously has uniqueness of solutions), so Π is a lift of π.

It follows that the continuation method will yield a global lift if we establish the global existence of the solution of (3.15). This requires that we estimate the growth as $||\eta||$ gets large of the time-varying vector field $(\eta, s) \to P_p(\eta) \cdot V(\mathcal{E}_p(\eta), s)$, defined on $H \setminus S_{\Sigma,p}$.

The following statement describes precisely the bounds we need:

PROPOSITION 3.2. *Let M, $\mathbf{f}$, p, Σ be as before, and assume M is endowed with a Riemannian metric, with distance function d. Let K be a closed subset of M such that*

(a) $\alpha \stackrel{\text{def}}{=} d(K, \overline{\mathcal{E}_p(S_{\Sigma,p})}) > 0$;
(b) *there exists a* $C_K > 0$ *such that for every* $\eta \in H$ *for which* $\mathcal{E}_p(\eta) \in K$,

$$\|P_p(\eta)\| \leq C_K \|\eta\|_H. \tag{3.16}$$

Then for every path $\pi : [0,1] \to K$ *of class* C^1 *and every control* $\bar{\eta} \in H$ *such that* $\mathcal{E}_p(\bar{\eta}) = \pi(0)$ *the solution of the initial value problem (3.15) exists globally on the interval* $[0,1]$.

The proof of Proposition 3.2 is quite easy: we let Π be the maximal solution of (3.15), and assume that the domain of Π is $[0, a[$, with $0 < a \leq 1$. Then $\mathcal{E}_p(\Pi(s)) = \pi(s)$ for $0 \leq s < a$, and therefore $\mathcal{E}_p(\Pi(s)) \in K$. So $\|P_p(\Pi(s))\| \leq C_K \|\Pi(s)\|_H$. Then Gronwall's inequality, together with the fact that $V(\pi(s), s)$ is bounded, imply that Π is bounded on $[0, a[$, and then the differential equation implies that $\dot{\Pi}$ is bounded, so $\lim_{s \uparrow a} \Pi(s)$ exists, and Π can be extended to $[0, a]$, contradicting the maximality of Π. □

To conclude, we rewrite the basic estimate (3.16) in terms of the switching functions. Using the Riemannian metric on M, we can identify the tangent and cotangent spaces to M at each point. If $z \in T^*_{\mathcal{E}_p(\eta)} M$, and $\eta \notin S_{\Sigma,p}$, then

$$\|P_p(\eta) \cdot z\|^2 = \langle P_p(\eta) \cdot z, P_p(\eta) \cdot z \rangle = \langle z, G_p(\eta)^{-1} \cdot z \rangle .$$

So

$$\|P_p(\eta)\|^2 = \max \left\{ \langle z, G_p(\eta)^{-1} \cdot z \rangle : z \in T^*_{\mathcal{E}_p(\eta)} M \, , \, \|z\| = 1 \right\} . \tag{3.17}$$

The Grammian $G_p(\eta)$ is a linear map from $T^*_{\mathcal{E}_p(\eta)} M$ to $T_{\mathcal{E}_p(\eta)} M$, i.e. from $T^*_{\mathcal{E}_p(\eta)} M$ to $T^*_{\mathcal{E}_p(\eta)} M$, using the identification of the tangent and cotangent spaces. We have

$$\langle z, G_p(\eta) \cdot z' \rangle = \langle z, D\mathcal{E}_p(\eta) D\mathcal{E}_p(\eta)^\dagger z' \rangle = \langle D\mathcal{E}_p(\eta)^\dagger z, D\mathcal{E}_p(\eta)^\dagger z' \rangle ,$$

from which it is clear that $G_p(\eta)$, regarded as a map from the inner product space $T^*_{\mathcal{E}_p(\eta)} M$ to itself, is symmetric and positive definite. It then follows that the right-hand side of (3.17) is the largest eigenvalue $\lambda_{\max}(G_p(\eta)^{-1})$ of $G_p(\eta)^{-1}$, which is equal to the inverse of the smallest eigenvalue $\lambda_{\min}(G_p(\eta))$ of $G_p(\eta)$. On the other hand, $\lambda_{\min}(G_p(\eta))$ is the minimum of $\langle z, G_p(\eta) \cdot z \rangle$, ranging over the z's of norm one. So, using (3.8) and (3.17), we get

$$\|P_p(\eta)\| = \left(\min \left\{ \langle z, G_p(\eta) \cdot z \rangle : z \in T^*_{\mathcal{E}_p(\eta)} M \, , \, \|z\| = 1 \right\} \right)^{-1/2}$$

$$= \max \left\{ \left(\|\varphi_{p,z,\eta}\|_{L^2([0,1])} \right)^{-1} : z \in T^*_{\mathcal{E}_p(\eta)} M, \|z\| = 1 \right\} . \tag{3.18}$$

Hence Estimate (3.16) will follow if we find a constant $c_K > 0$ such that

$$\text{(CRIT)}\qquad \boxed{\begin{array}{l}\|\eta\|_H\|\varphi_{p,z,\eta}\|_{L^2([0,1])}\geq c_K\|z\| \\ \text{whenever } \eta\in H\,,\ \mathcal{E}_p(\eta)\in K\,,\ z\in T^*_{\mathcal{E}_p(\eta)}M.\end{array}}$$

So (CRIT) is the critical estimate that will have to be proved to establish global existence of solutions of the PLE.

4. Estimates for line integrals. Let $\Sigma = (M, \mathbf{f})$ be a system with an n-dimensional state space and m-dimensional inputs. If k is a nonnegative integer, we use $C_b^k(\Sigma)$ to denote the space of all functions $\psi : M \to \mathbb{R}$ that are of class C^k and such that the functions $L_{f_{i_1}} L_{f_{i_2}} \ldots L_{f_{i_r}} \psi$ are bounded for all $r \in \{0, 1, \ldots, k\}$ and all multiindices $(i_1, i_2, \ldots, i_r) \in \{1, 2, \ldots, m\}^r$. We define

$$\|\psi\|_{C_b^k(\Sigma)} \stackrel{\text{def}}{=} \sum_{r=0}^{k} \sum_{i_1,\ldots,i_r} \|L_{f_{i_1}} L_{f_{i_2}} \ldots L_{f_{i_r}} \psi\|_{sup}\,, \tag{4.1}$$

where, for a bounded function $\psi : M \to \mathbb{R}$, $\|\psi\|_{sup} \stackrel{\text{def}}{=} \sup\{|\psi(x)| : x \in M\}$.

Then $\|\psi\|_{C_b^k(\Sigma)}$ can also be defined recursively by letting

$$\|\psi\|_{C_b^0(\Sigma)} = \|\psi\|_{sup}\,, \tag{4.2}$$

$$\|\psi\|_{C_b^k(\Sigma)} = \|\psi\|_{sup} + \sum_{i=1}^{m} \|L_{f_i}\psi\|_{C_b^{k-1}(\Sigma)} \quad \text{for} \quad k > 0\,. \tag{4.3}$$

It is clear that $C_b^k(\Sigma)$ is an algebra of functions. (That is, $C_b^k(\Sigma)$ is a linear space and is closed under multiplication.) Moreover, $\|\cdot\|_{C_b^k(\Sigma)}$ is a norm on $C_b^k(\Sigma)$, and satisfies the inequality

$$\|\psi_1\psi_2\|_{C_b^k(\Sigma)} \leq 2^k \|\psi_1\|_{C_b^k(\Sigma)} . \|\psi_2\|_{C_b^k(\Sigma)} \text{ if } \psi_1, \psi_2 \in C_b^k(\Sigma)\,. \tag{4.4}$$

(The proof of (4.4) is an easy induction, using (4.2) and (4.3). Indeed,

$$\begin{aligned}
\|\psi_1\psi_2\|_{C_b^k(\Sigma)} = \;& \|\psi_1\psi_2\|_{sup} + \sum_{i=1}^{m} \|L_{f_i}(\psi_1\psi_2)\|_{C_b^{k-1}(\Sigma)} \\
= \;& \|\psi_1\psi_2\|_{sup} + \sum_{i=1}^{m} \|(L_{f_i}\psi_1).\psi_2) + \psi_1.(L_{f_i}\psi_2)\|_{C_b^{k-1}(\Sigma)} \\
\leq \;& \|\psi_1\|_{sup}.\|\psi_2\|_{sup} + 2^{k-1} \sum_{i=1}^{m} \|L_{f_i}\psi_1\|_{C_b^{k-1}(\Sigma)} \|\psi_2\|_{C_b^{k-1}(\Sigma)} \\
& +2^{k-1} \sum_{i=1}^{m} \|\psi_1\|_{C_b^{k-1}(\Sigma)} . \|L_{f_i}\psi_2)\|_{C_b^{k-1}(\Sigma)} \\
\leq \;& \|\psi_1\|_{sup}.\|\psi_2\|_{C_b^k(\Sigma)} + \|\psi_1\|_{C_b^k(\Sigma)} . \|\psi_2\|_{sup} \\
& +2^{k-1} \Big(\sum_{i=1}^{m} \|L_{f_i}\psi_1\|_{C_b^{k-1}(\Sigma)} \Big) \|\psi_2\|_{C_b^k(\Sigma)}
\end{aligned}$$

$$\begin{aligned}
& +2^{k-1}\|\psi_1\|_{C_b^k(\Sigma)}.\Big(\sum_{i=1}^m \|L_{f_i}\psi_2\|_{C_b^{k-1}(\Sigma)}\Big) \\
\leq \;& 2^{k-1}\Big(\|\psi_1\|_{sup} + \sum_{i=1}^m \|L_{f_i}\psi_1\|_{C_b^{k-1}(\Sigma)}\Big).\|\psi_2\|_{C_b^k(\Sigma)} \\
& +2^{k-1}\|\psi_1\|_{C_b^k(\Sigma)}.\Big(\|\psi_2\|_{sup} + \sum_{i=1}^m \|L_{f_i}\psi_2\|_{C_b^{k-1}(\Sigma)}\Big) \\
= \;& 2^k\|\psi_1\|_{C_b^k(\Sigma)}.\|\psi_2\|_{C_b^k(\Sigma)},
\end{aligned}$$

so (4.4) holds.)

If $\mu : \mathbb{R} \to \mathbb{R}$ is a function, $\psi \in C_b^k(\Sigma)$, and $\rho > 0$, we can define a function $\alpha_{\mu,\psi,\rho} : M \to \mathbb{R}$ by letting

$$\alpha_{\mu,\psi,\rho}(x) \stackrel{\text{def}}{=} \mu\Big(\frac{\psi(x)}{\rho}\Big). \tag{4.5}$$

Proposition 4.1. *Suppose $\mu : \mathbb{R} \to \mathbb{R}$ is of class C^{k+1}, and is bounded together with all the derivatives $\mu', \mu'', \ldots, \mu^{(k+1)}$. Assume also that $\mu(0) = \ldots = \mu^{(k)}(0) = 0$. Then $\alpha_{\mu,\psi,\rho} \in C_b^k(\Sigma)$ whenever $\psi \in C_b^k(\Sigma)$, and there exists a constant $C_k(\mu)$, depending only on μ and k, such that*

$$\|\alpha_{\mu,\psi,\rho}\|_{C_b^k(\Sigma)} \leq C_k(\mu)\rho^{-1}(1+\rho^{-k})\,\|\psi\|_{C_b^k(\Sigma)} \tag{4.6}$$

for all $\psi \in C_b^k(\Sigma)$ and all $\rho > 0$.

Proof. We use induction on k. If $k = 0$ then the conclusion is trivial, since

$$|\alpha_{\mu,\psi,\rho}(x)| \leq \frac{1}{\rho}\|\mu'\|_{sup}.\|\psi\|_{sup},$$

so we can take $C_0(\mu) = \frac{1}{2}\|\mu'\|_{sup}$.

Now suppose $k > 0$, and assume that the conclusion holds for $k - 1$. Then, if $i \in \{1, \ldots, m\}$, we have

$$L_{f_i}\alpha_{\mu,\psi,\rho}(x) = \frac{1}{\rho}\mu'\Big(\frac{\psi(x)}{\rho}\Big).(L_{f_i}\psi)(x) = \frac{1}{\rho}\alpha_{\mu',\psi,\rho}(x).(L_{f_i}\psi)(x),$$

so

$$L_{f_i}\alpha_{\mu,\psi,\rho} = \frac{1}{\rho}\alpha_{\mu',\psi,\rho}.L_{f_i}\psi.$$

The inductive hypothesis implies that

$$\|\alpha_{\mu',\psi,\rho}\|_{C_b^{k-1}(\Sigma)} \leq C_{k-1}(\mu')\rho^{-1}(1+\rho^{1-k})\|\psi\|_{C_b^{k-1}(\Sigma)}.$$

Therefore

$$\|L_{f_i}\alpha_{\mu,\psi,\rho}\|_{C_b^{k-1}(\Sigma)} \le 2^{k-1}C_{k-1}(\mu')\rho^{-2}(1+\rho^{1-k})\|L_{f_i}\psi\|_{C_b^{k-1}(\Sigma)}\,.$$

So

$$\begin{aligned}
\|\alpha_{\mu,\psi,\rho}\|_{C_b^k(\Sigma)} &= \|\alpha_{\mu,\psi,\rho}\|_{sup} + \sum_{i=1}^{m}\|L_{f_i}\alpha_{\mu,\psi,\rho}\|_{C_b^{k-1}(\Sigma)} \\
&\le \rho^{-1}\|\mu'\|_{sup}.\|\psi\|_{sup} \\
&\quad +2^{k-1}C_{k-1}(\mu')\rho^{-2}(1+\rho^{1-k})\sum_{i=1}^{m}\|L_{f_i}\psi\|_{C_b^{k-1}(\Sigma)} \\
&\le \Big(\|\mu'\|_{sup}+2^{k-1}C_{k-1}(\mu')\Big).\frac{1}{\rho}\Big(1+\frac{1}{\rho}+\frac{1}{\rho^k}\Big).\|\psi\|_{C_b^k(\Sigma)} \\
&\le 2\Big(\|\mu'\|_{sup}+2^{k-1}C_{k-1}(\mu')\Big).\frac{1}{\rho}\Big(1+\frac{1}{\rho^k}\Big).\|\psi\|_{C_b^k(\Sigma)} \\
&= \frac{C_k(\mu)}{\rho}\Big(1+\frac{1}{\rho^k}\Big).\|\psi\|_{C_b^k(\Sigma)}\,,
\end{aligned}$$

where we have used $\rho^{-1} \le 1+\rho^{-k}$ —which is valid since $k \ge 1$— and defined

$$C_k(\mu) = 2\Big(\|\mu'\|_{sup}+2^{k-1}C_{k-1}(\mu')\Big)\,.$$

This completes the proof of Proposition 4.1. $\square$

If $\omega : M \to \mathbb{R}^m$ is a continuous map, then we can think of ω as an object analogous to a differential 1-form, and define the integral $\int_\gamma \omega$ of ω along a trajectory-control pair $\gamma = (\xi,\eta)$ of Σ, by letting

$$\int_\gamma \omega \overset{\text{def}}{=} \int_{a(\gamma)}^{b(\gamma)} \eta(t)^\dagger\omega(\xi(t))\,dt\,. \tag{4.7}$$

Suppose $\zeta : M \to \mathbb{R}$ is continuous and nonnegative. We define

$$\|\gamma\|_\zeta \overset{\text{def}}{=} \int_{a(\gamma)}^{b(\gamma)} \zeta(\xi(t)).\|\eta(t)\|\,dt\,. \tag{4.8}$$

Now suppose that k is a nonnegative integer. We define a subspace $\mathcal{A}_k(\Sigma,\zeta)$ of $C^0(M,\mathbb{R})^m$ by letting $\omega = (\omega_1,\ldots,\omega_m)^\dagger \in \mathcal{A}_k(\Sigma,\zeta)$ if there exists a continuous monotonically nondecreasing function $\Theta : [0,\infty[\to[0,\infty[$ such that $\Theta(0) = 0$ and $\Theta(s) > 0$ whenever $s > 0$, having the property that the inequality

$$\Big|\int_\gamma \beta.\omega\Big| \le \|\beta\|_{C_b^k(\Sigma)}\Theta(\|\gamma\|_\zeta) \tag{4.9}$$

holds for every $\beta \in C_b^k(\Sigma)$ and every $\gamma \in TCP(\Sigma)$.

It is clear that $\mathcal{A}_k(\Sigma,\zeta)$ is a linear space, and is closed under multiplication by members of $C_b^k(\Sigma)$. (That is, $\mathcal{A}_k(\Sigma,\zeta)$ is a submodule of the $C_b^k(\Sigma)$-module $C^0(M,\mathbb{R})^m$.)

More generally, a matrix-valued function $\Omega : M \to \mathbb{R}^{m\times\ell}$ will be said to be in $\mathcal{A}_k(\Sigma,\zeta)^\ell$ if the columns $\omega_1,\ldots,\omega_\ell$ of Ω are in $\mathcal{A}_k(\Sigma,\zeta)$. It is easy to see that $\Omega \in \mathcal{A}_k(\Sigma,\zeta)^\ell$ if and only if $\Omega.v \in \mathcal{A}_k(\Sigma,\zeta)$ for every constant vector $v \in \mathbb{R}^\ell$, in which case $\Omega.V \in \mathcal{A}_k(\Sigma,\zeta)$ for every vector-valued function $V : M \to \mathbb{R}^\ell$ all whose components belong to $C_b^k(\Sigma)$.

We say that a function $\psi \in C^0(M,\mathbb{R})$ is in the space $\mathcal{B}_k(\Sigma,\zeta)$ if the $m\times m$ diagonal-matrix-valued function $\psi.I_m$ —where I_m is the $m\times m$ identity matrix— is in $\mathcal{A}_k(\Sigma,\zeta)^m$. Equivalently, $\psi \in \mathcal{B}_k(\Sigma,\zeta)$ iff $\psi.\omega \in \mathcal{A}_k(\Sigma,\zeta)$ for all $\omega \in C_b^k(\Sigma)^m$.

THEOREM 4.1. *Let M be a smooth manifold, and let $\mathbf{f}$ be an m-tuple of smooth vector fields on M. Assume that $\psi \in \mathcal{B}_k(\Sigma,\zeta) \cap C_b^{k+1}(\Sigma)$. Then $\nabla_{\mathbf{f}}\psi \in \mathcal{A}_{k+1}(\Sigma,\zeta)$.*

Proof. Assume $\psi \in \mathcal{B}_k(\Sigma,\zeta)$. Let $(e_1,\ldots,e_m)$ be the canonical basis of $\mathbb{R}^m$, so $e_1,\ldots,e_m$ are the columns of I_m. From the hypothesis that $\psi \in \mathcal{B}_k(\Sigma,\zeta)$ it follows that $\psi.e_i \in \mathcal{A}_k(\Sigma,\zeta)$ for each i, so we may choose monotonically nondecreasing continuous functions $\Theta_i : [0,\infty[\to [0,\infty[$ such that $\Theta_i(0) = 0$, $\Theta_i(s) > 0$ when $s > 0$, and

$$\left|\int_\gamma \psi.\beta.e_i\right| \le \|\beta\|_{C_b^k(\Sigma)}\Theta_i(\|\gamma\|_\zeta) \tag{4.10}$$

for every $\gamma = (\xi,\eta) \in TCP(\Sigma)$ and every $\beta \in C_b^k(\Sigma)$.

Choose a function $\mu : \mathbb{R} \to \mathbb{R}$ of class C^∞ such that μ and all its derivatives of all orders vanish at 0, $\mu(s) = 1$ for $|s| \ge 1$, and $0 \le \mu(s) \le 1$ for all $s \in \mathbb{R}$. Define $\nu(s) = \int_0^1 \mu'(rs)dr$, so $\mu(s) = s.\nu(s)$, and ν is also of class C^∞ and vanishes at 0 together with all its derivatives. Then, for each positive constant ρ, define

$$\hat{\mu}_\rho(x) = \mu\Big(\frac{\psi(x)}{\rho}\Big)\,,\quad \hat{\nu}_\rho(x) = \nu\Big(\frac{\psi(x)}{\rho}\Big)\,,$$

so

$$\hat{\mu}_\rho(x) = \frac{\psi(x)}{\rho}\hat{\nu}_\rho(x)\,.$$

Then $\hat{\mu}_\rho$, $\hat{\nu}_\rho$ are the functions $\alpha_{\mu,\psi,\rho}$, $\alpha_{\nu,\psi,\rho}$. Since μ, ν, and all the derivatives $\mu^{(j)}$, $\nu^{(j)}$ are obviously bounded and vanish at 0, we can apply Proposition 4.1 and conclude that

$$\|\hat{\mu}_\rho\|_{C_b^k(\Sigma)} \le C_k(\mu)\rho^{-1}(1+\rho^{-k})\|\psi\|_{C_b^k(\Sigma)}$$

and

$$\|\hat{\nu}_\rho\|_{C_b^k(\Sigma)} \leq C_k(\nu)\rho^{-1}(1+\rho^{-k})\|\psi\|_{C_b^k(\Sigma)}.$$

Now let $\beta \in C_b^{k+1}(\Sigma)$, and let $\gamma \in TCP(\Sigma)$. Let

$$I = \int_\gamma \beta.\nabla_{\mathbf{f}}\psi\,.$$

Fix a number ρ such that $0 < \rho \leq 1$. We can then write

$$I = I_1 + I_2\,, \tag{4.11}$$

where

$$I_1 = \int_\gamma \hat{\mu}_\rho.\beta.\nabla_{\mathbf{f}}\psi\,,$$
$$I_2 = \int_\gamma (1-\hat{\mu}_\rho).\beta.\nabla_{\mathbf{f}}\psi\,.$$

Then

$$I_1 = \rho^{-1}\int_\gamma \hat{\nu}_\rho.\psi.\beta.\nabla_{\mathbf{f}}\psi = \rho^{-1}\int_\gamma \psi.\omega_\rho\,,$$

were $\omega_\rho = \hat{\nu}_\rho.\beta.\nabla_{\mathbf{f}}\psi$. Clearly, $\omega_\rho = \sum_{i=1}^m (L_{f_i}\psi).\beta.\hat{\nu}_\rho.e_i$, so

$$\begin{aligned}
|I_1| &\leq \rho^{-1}\,.\sum_{i=1}^m \Theta_i(\|\gamma\|_\varsigma)\,.\,\|(L_{f_i}\psi)\,.\,\beta\,.\,\hat{\nu}_\rho\|_{C_b^k(\Sigma)} \\
&\leq \rho^{-1}\,.\,2^{2k}\Theta(\|\gamma\|_\varsigma)\,.\,\|\beta\|_{C_b^k(\Sigma)}\,.\,\|\hat{\nu}_\rho\|_{C_b^k(\Sigma)}\,.\sum_{i=1}^m \|L_{f_i}\psi\|_{C_b^k(\Sigma)} \\
&\leq \rho^{-2}(1+\rho^{-k})\,.\,2^{2k}C_k(\nu)\,.\,\Theta(\|\gamma\|_\varsigma)\,.\,\|\beta\|_{C_b^k(\Sigma)}\,.\,\|\psi\|_{C_b^k(\Sigma)}\,.\,\|\psi\|_{C_b^{k+1}(\Sigma)}\,,
\end{aligned}$$

where we let $\Theta(s) = \max(\Theta_1(s),\ldots,\Theta_m(s))$. Therefore (using the fact that $\rho \leq 1$), we find

$$|I_1| \leq \frac{C_1}{\rho^{k+2}}\,.\,\Theta(\|\gamma\|_\varsigma)\,.\,\|\beta\|_{C_b^k(\Sigma)}\,, \tag{4.12}$$

where the constant C_1 and the function Θ do not depend on β, γ or ρ.

To estimate I_2, we let $a = a(\gamma)$, $b = b(\gamma)$, and observe that

$$I_2 = \int_a^b \Big(1-\hat{\mu}_\rho(\xi(t))\Big)\,.\,\beta\Big(\xi(t)\Big)\,.\,\frac{d}{dt}\Big(\psi(\xi(t))\Big)\,dt\,. \tag{4.13}$$

We then use integration by parts to write

$$I_2 = I_{2,1} + I_{2,2} + I_{2,3}\,,$$

where

$$
\begin{aligned}
I_{2,1} &= \ \Big(1-\hat{\mu}_\rho(\xi(b))\Big)\,.\,\beta\Big(\xi(b)\Big)\,.\,\psi\Big(\xi(b)\Big)\,,\\
I_{2,2} &= -\Big(1-\hat{\mu}_\rho(\xi(a))\Big)\,.\,\beta\Big(\xi(a)\Big)\,.\,\psi\Big(\xi(a)\Big)\,,\\
I_{2,3} &= -\int_a^b \frac{d}{dt}\left(\Big(1-\hat{\mu}_\rho(\xi(t))\Big)\,.\,\beta\Big(\xi(t)\Big)\right).\,\psi\Big(\xi(t)\Big)\,dt\,.
\end{aligned}
$$

Since $0 \le \hat{\mu}_\rho(x) \le 1$ for all x, and $1-\hat{\mu}_\rho(x) = 0$ unless $|\psi(x)| < \rho$, the bound

$$
|I_{2,1}| + |I_{2,2}| \le 2\rho\,.\,\|\beta\|_{sup} \tag{4.14}
$$

holds.

We now estimate $I_{2,3}$. For this purpose, we observe that

$$
I_{2,3} = -\int_\gamma \psi.\tilde{\omega}_\rho\,,
$$

where $\tilde{\omega}_\rho \stackrel{\text{def}}{=} \nabla_{\mathbf{f}}\Big((1-\hat{\mu}_\rho)\beta\Big)$. Then

$$
I_{2,3} = -\sum_{i=1}^m \int_\gamma L_{f_i}\Big((1-\hat{\mu}_\rho)\,.\,\beta\Big)\,.\,\psi\,.\,e_i\,,
$$

so

$$
\begin{aligned}
|I_{2,3}| &\le \sum_{i=1}^m \Theta_i(\|\gamma\|_\varsigma)\,.\,\Big\|L_{f_i}\Big((1-\hat{\mu}_\rho)\,.\,\beta\Big)\Big\|_{C_b^k(\Sigma)}\\
&\le \Theta(\|\gamma\|_\varsigma)\,.\,\sum_{i=1}^m \Big\|L_{f_i}\Big((1-\hat{\mu}_\rho)\,.\,\beta\Big)\Big\|_{C_b^k(\Sigma)}\\
&\le \Theta(\|\gamma\|_\varsigma)\,.\,\|(1-\hat{\mu}_\rho)\,.\,\beta\|_{C_b^{k+1}(\Sigma)}\\
&\le \Theta(\|\gamma\|_\varsigma)\,.\,\|1-\hat{\mu}_\rho\|_{C_b^{k+1}(\Sigma)}\,.\,\|\beta\|_{C_b^{k+1}(\Sigma)}\\
&\le \Theta(\|\gamma\|_\varsigma)\,.\,\Big(1+\|\hat{\mu}_\rho\|_{C_b^{k+1}(\Sigma)}\Big)\,.\,\|\beta\|_{C_b^{k+1}(\Sigma)}\\
&\le \Theta(\|\gamma\|_\varsigma)\,.\,\left(1+C_{k+1}(\mu)\Big(\frac{1}{\rho}+\frac{1}{\rho^{k+1}}\Big)\,.\,\|\psi\|_{C_b^{k+1}(\Sigma)}\right).\,\|\beta\|_{C_b^{k+1}(\Sigma)}\,.
\end{aligned}
$$

Then (using $\rho \le 1$) we get

$$
|I_{2,3}| \le \frac{C_2}{\rho^{k+2}}\Theta(\|\gamma\|_\varsigma)\,.\,\|\beta\|_{C_b^{k+1}(\Sigma)}\,, \tag{4.15}
$$

where the constant C_2 does not depend on β, γ or ρ.

If we combine the bounds (4.12), (4.14) and (4.15), use $\rho \leq 1$ again, as well as the inequalities $||\beta||_{sup} \leq ||\beta||_{C_b^{k+1}(\Sigma)}$ and $||\beta||_{C_b^k(\Sigma)} \leq ||\beta||_{C_b^{k+1}(\Sigma)}$, we find

$$|I| \leq \left(2\rho + \frac{C}{\rho^{k+2}} \cdot \Theta(||\gamma||_\varsigma)\right) \cdot ||\beta||_{C_b^{k+1}(\Sigma)}, \tag{4.16}$$

where $C = C_1 + C_2$.

Now assume that $0 < |I| \leq 4||\beta||_{C_b^{k+1}(\Sigma)}$. Choose

$$\rho = \frac{|I|}{4||\beta||_{C_b^{k+1}(\Sigma)}},$$

so $0 < \rho \leq 1$. Then (4.16) implies

$$|I| \leq \frac{|I|}{2} + \frac{C}{\rho^{k+2}}\Theta(||\gamma||_\varsigma) \cdot ||\beta||_{C_b^{k+1}(\Sigma)},$$

so

$$|I| \leq \frac{2C}{\rho^{k+2}}\Theta(||\gamma||_\varsigma) \cdot ||\beta||_{C_b^{k+1}(\Sigma)},$$

and then

$$\frac{|I|^{k+3}}{4^{k+2}||\beta||^{k+2}_{C_b^{k+1}(\Sigma)}} = \rho^{k+2}|I| \leq 2C\Theta(||\gamma||_\varsigma) \cdot ||\beta||_{C_b^{k+1}(\Sigma)},$$

from which it follows that

$$|I| \leq \tilde{\Theta}(||\gamma||_\varsigma) \cdot ||\beta||_{C_b^{k+1}(\Sigma)}, \tag{4.17}$$

where

$$\tilde{\Theta}(s) = \left(2^{2k+5}C\Theta(s)\right)^{\frac{1}{k+3}}.$$

Now suppose that $|I| > 4||\beta||_{C_b^{k+1}(\Sigma)}$. In this case, we choose $\rho = 1$, and get

$$\begin{aligned} |I| &\leq 2||\beta||_{C_b^{k+1}(\Sigma)} + C\Theta(||\gamma||_\varsigma) \cdot ||\beta||_{C_b^{k+1}(\Sigma)} \\ &\leq \frac{|I|}{2} + C\Theta(||\gamma||_\varsigma) \cdot ||\beta||_{C_b^{k+1}(\Sigma)}, \end{aligned}$$

so

$$|I| \leq 2C\Theta(||\gamma||_\varsigma) \cdot ||\beta||_{C_b^{k+1}(\Sigma)}. \tag{4.18}$$

If we let

$$\hat{\Theta}(s) = \max(\tilde{\Theta}(s), 2C\Theta(s))\,,$$

then we have shown that the inequality

$$|I| \leq \hat{\Theta}(\|\gamma\|_{\zeta}) \,.\, \|\beta\|_{C_b^{k+1}(\Sigma)} \tag{4.19}$$

holds whenever $|I| > 0$. Clearly, (4.19) also holds when $|I| = 0$, so we have established (4.19) in all possible cases.

Since the function $\hat{\Theta}$ does not depend on the choice of β or γ, we have proved that $\nabla_{\mathbf{f}}\psi \in \mathcal{A}_{k+1}(\Sigma, \zeta)$, as desired. $\square$

We now discuss two applications of Theorem 4.1. In both cases, the theorem will be applied to the lifted Hamiltonian system $T^*\Sigma = (T^*M, \mathbf{F})$ obtained from a smooth system $\Sigma = (M, \mathbf{f})$, where M is a smooth n-dimensional manifold, $\mathbf{f} = (f_1, \ldots, f_m)$ is an m-tuple of smooth vector fields on M, $\mathbf{F} = (F_1, \ldots, F_m)$, and $F_i = T^*f_i$ for $i = 1, \ldots, m$. Then the space of input values for $T^*\Sigma$ is $\mathbb{R}^m$.

We define the momentum functions $\varphi_i : T^*M \to \mathbb{R}$, for $i = 1, \ldots, m$, by letting $\varphi_i(x, z) = z.f_i(x)$ for $x \in M$, $z \in T_x^*M$. Then, for $i = 1, \ldots, m$, F_i is the Hamilton vector field arising from the function φ_i.

We also define

$$\psi_{ij}(x, z) = z.[f_i, f_j](x)\,, \tag{4.20}$$

so $\psi_{ij} = \{\varphi_i, \varphi_j\}$, where "$\{\cdot,\cdot\}$" denotes Poisson bracket. Then

$$\nabla_{\mathbf{F}}\varphi_j = \begin{pmatrix} \psi_{1j} \\ \vdots \\ \psi_{mj} \end{pmatrix} . \tag{4.21}$$

We let Ψ, Φ denote, respectively, the $m \times m$ matrix $(\psi_{ij})_{1 \leq i,j \leq m}$ and the m-dimensional column vector $(\varphi_1, \ldots, \varphi_m)^\dagger$. Then the columns of Ψ are $\nabla_{\mathbf{F}}\varphi_1, \ldots, \nabla_{\mathbf{F}}\varphi_m$.

Next, pick

$$\zeta = \sqrt{\varphi_1^2 + \ldots + \varphi_m^2}\,. \tag{4.22}$$

Let Ω be any open subset of T^*M. Let $T^*\Sigma\lceil\Omega$ be the restriction of the system $T^*\Sigma$ to Ω.

Then, if $\Gamma = (\Xi, \eta)$ is a trajectory-control pair of $T^*\Sigma\lceil\Omega$, ω belongs to $C_b^0(T^*\Sigma\lceil\Omega)^m$, and $\beta \in C_b^0(T^*\Sigma\lceil\Omega)$, we have

$$\Big|\int_\Gamma \beta \,.\, \varphi_i \,.\, \omega\Big| = \Big|\int_{a(\gamma)}^{b(\gamma)} \beta(\Xi(t)) \,.\, \varphi_i(\Xi(t)) . \Big(\eta(t)^\dagger \,.\, \omega(\Xi(t))\Big)\, dt\Big|$$

$$\leq \|\beta\|_{sup} \,.\, \|\omega\|_{sup} \,.\, \int_{a(\gamma)}^{b(\gamma)} \zeta(\Xi(t)) \,.\, \|\eta(t)\|\, dt$$
$$= \|\beta\|_{sup} \,.\, \Theta(\|\Gamma\|_\zeta)\,,$$

where $\Theta(s) = s.\|\omega\|_{sup}$.

It follows that $\boxed{\varphi_i \in \mathcal{B}_0(T^*\Sigma\lceil\Omega,\zeta) \text{ for } i = 1,\ldots,m}$. If in addition the φ_i and the ψ_{ij} are bounded on Ω, then the φ_i belong to $C_b^1(T^*\Sigma\lceil\Omega)$, so Theorem 4.1 implies that the vector-valued maps $\nabla_{\mathbf{F}}\varphi_i$ are in $\mathcal{A}_1(T^*\Sigma\lceil\Omega,\zeta)$, i.e. that the matrix Ψ is in $\mathcal{A}_1(T^*\Sigma\lceil\Omega,\zeta)^m$. It then follows that

PROPOSITION 4.2. *If the φ_i and the ψ_{ij} are bounded on Ω, and $\omega_1,\ldots,\omega_m, V$ belong to $C_b^1(T^*\Sigma\lceil\Omega)^m$, then $\varphi_1\omega_1 + \ldots + \varphi_m\omega_m + \Psi.V$ belongs to $\mathcal{A}_1(T^*\Sigma\lceil\Omega,\zeta)$.* □

Now let $\varphi_0(x,z)$ denote the determinant of $\Psi(x,z)$. Define

$$\mu = \varphi_0^2 + \varphi_1^2 + \ldots + \varphi_m^2\,.$$

We remark that:

1. *The SBGC is equivalent to the condition that μ never vanishes on $T^*M\setminus\{0\}$.*
2. *$\mu \in C^\infty$, and $\mu \in C_b^{k-1}(T^*\Sigma\lceil\Omega)$ whenever $\varphi_1,\ldots,\varphi_m$ belong to $C_b^k(T^*\Sigma\lceil\Omega)$.*

If Ω is an open subset of $T^*M\setminus\{0\}$, we will say that Σ *satisfies the SBGC uniformly on Ω* if

(U1) *the functions φ_i, $i = 1,\ldots,m$, belong to $C_b^2(T^*\Sigma\lceil\Omega)$,*
(U2) *μ is bounded below in Ω by a strictly positive constant.*

(Notice that, if Σ satisfies the SBGC, then it does so uniformly on Ω whenever Ω is a relatively compact subset of $T^*M\setminus\{0\}$.)

If Σ satisfies the SBGC uniformly on Ω, then φ_0 and μ belong to $C_b^1(T^*\Sigma\lceil\Omega)$. Since μ is bounded below by a positive constant, it follows easily that $\mu^{-1} \in C_b^1(T^*\Sigma\lceil\Omega)$, since $L_{F_i}(\mu^{-1}) = -\mu^{-2}L_{F_i}(\mu)$.

Now let $\omega \in C_b^1(T^*\Sigma\lceil\Omega)$ be arbitrary. Write

$$\omega = \sum_{i=0}^{m} \varphi_i\omega_i\,, \tag{4.23}$$

where

$$\omega_i = \frac{\varphi_i}{\mu}\omega\,.$$

Since the φ_i and μ^{-1} are in $C_b^1(T^*\Sigma\lceil\Omega)$, the vector functions ω_i are in $C_b^1(T^*\Sigma\lceil\Omega)^m$.

Use $\tilde{\Psi}$ to denote the matrix of complementary minors of Ψ, so that $\Psi\tilde{\Psi} = \tilde{\Psi}\Psi = \varphi_0 I_m$. Then $\varphi_0\omega_0 = \Psi\tilde{\Psi}\omega_0 = \Psi\hat{\omega}_0$, where $\hat{\omega}_0 = \tilde{\Psi}\omega_0$. So

$$\omega = \Psi.\hat{\omega}_0 + \sum_{i=1}^{m} \varphi_i\omega_i \,. \tag{4.24}$$

Clearly, $\hat{\omega}_0 \in C_b^1(T^*\Sigma\lceil\Omega)^m$, since the components of $\tilde{\Psi}$ are polynomials in the ψ_{ij}.

Using Proposition 4.2, we get

THEOREM 4.2. *Let Ω be any open subset of $T^*M \setminus \{0\}$ on which Σ satisfies the SBGC uniformly. Let ζ be defined by (4.22). Then every $\omega \in C_b^1(T^*\Sigma\lceil\Omega)^m$ is in $\mathcal{A}_1(T^*\Sigma\lceil\Omega,\zeta)$.* □

In particular, Theorem 4.2 implies

PROPOSITION 4.3. *$C_b^1(T^*\Sigma\lceil\Omega) \subseteq \mathcal{A}_1(T^*\Sigma\lceil\Omega,\zeta)$ whenever Σ satisfies the SBGC and Ω is a relatively compact subset of $T^*M \setminus \{0\}$.* □

We now show how to apply Theorem 4.1 to get estimates for some systems Σ such as the four-wheel driven car, that do not satisfy the SBGC.

Assume that $m = 2$. Let $\varphi_3(x,z) = z.[f_1, f_2](x)$, so

$$\Psi = \begin{pmatrix} 0 & \varphi_3 \\ -\varphi_3 & 0 \end{pmatrix}.$$

Suppose that φ_1 and φ_2 belong to $C_b^{k+1}(T^*\Sigma\lceil\Omega)$ for some nonnegative integer k. Since φ_1 and φ_2 belong to $\mathcal{B}_0(T^*\Sigma\lceil\Omega,\zeta)$, and $\mathcal{B}_0(T^*\Sigma\lceil\Omega,\zeta)$ is obviously a subset of $\mathcal{B}_k(T^*\Sigma\lceil\Omega,\zeta)$, Theorem 4.1 implies that $\nabla_{\mathbf{F}}\varphi_1$ and $\nabla_{\mathbf{F}}\varphi_2$ are in $\mathcal{A}_{k+1}(T^*\Sigma\lceil\Omega,\zeta)$. This implies that $\varphi_3 \,.\, \omega \in \mathcal{A}_{k+1}(T^*\Sigma\lceil\Omega,\zeta)$ for every $\omega = (\omega_1,\omega_2)^\dagger \in C_b^{k+1}(T^*\Sigma\lceil\Omega)^2$, since

$$\varphi_3 \,.\, \omega = \omega_1 \,.\, \nabla_{\mathbf{F}}\varphi_2 - \omega_2 \,.\, \nabla_{\mathbf{F}}\varphi_1 \,,$$

and $\mathcal{A}_{k+1}(T^*\Sigma\lceil\Omega,\zeta)$ is a $C_b^{k+1}(T^*\Sigma\lceil\Omega)$-module. But then φ_3 belongs to $\mathcal{B}_{k+1}(T^*\Sigma\lceil\Omega,\zeta)$. If φ_1 and φ_2 actually belong to $C_b^{k+3}(T^*\Sigma\lceil\Omega)$, then $\varphi_3 \in C_b^{k+2}(T^*\Sigma\lceil\Omega)$. This makes it possible to apply Theorem 4.1 again and conclude that $\nabla_{\mathbf{F}}\varphi_3 \in \mathcal{A}_{k+2}(T^*\Sigma\lceil\Omega,\zeta)$. Since $\mathcal{A}_{k+2}(T^*\Sigma\lceil\Omega,\zeta)$ is a $C_b^{k+2}(T^*\Sigma\lceil\Omega)$-module, every linear combination of $\nabla_{\mathbf{F}}\varphi_1$, $\nabla_{\mathbf{F}}\varphi_2$ and $\nabla_{\mathbf{F}}\varphi_3$ with coefficients in $C_b^{k+2}(T^*\Sigma\lceil\Omega)$ is again in $\mathcal{A}_{k+2}(T^*\Sigma\lceil\Omega,\zeta)$. Since φ_1, φ_2 and φ_3 belong to $\mathcal{B}_{k+2}(T^*\Sigma\lceil\Omega,\zeta)$, every map $\Omega \to \mathbb{R}^2$ of the form $\varphi_1\delta_1 + \varphi_2\delta_2 + \varphi_3\delta_3$, with the δ_i in $C_b^{k+2}(T^*\Sigma\lceil\Omega)^2$, is in $\mathcal{A}_{k+2}(T^*\Sigma\lceil\Omega,\zeta)$. So we have proved:

PROPOSITION 4.4. *Let $\Sigma = (M, \mathbf{f})$ be a smooth system with input dimension two. Let $\mathbf{f} = (f_1, f_2)$, define $f_3 = [f_1, f_2]$, and let φ_i, for $i = 1, 2, 3$, be the momentum function corresponding to f_i. Let k be a nonnegative integer, and let Ω be an open subset of $T^*M \setminus \{0\}$ having the property that φ_1 and φ_2 belong to $C_b^{k+3}(T^*\Sigma\lceil\Omega)$. Let*

$$\zeta(x, z) = \sqrt{\varphi_1(x, z)^2 + \varphi_2(x, z)^2}\,. \tag{4.25}$$

Let $\omega : \Omega \to \mathbb{R}^2$ be a map which is of the form

$$\omega = \varphi_1\delta_1 + \varphi_2\delta_2 + \varphi_3\delta_3 + \beta_1\nabla_{\mathbf{F}}\varphi_1 + \beta_2\nabla_{\mathbf{F}}\varphi_2 + \beta_3\nabla_{\mathbf{F}}\varphi_3 \tag{4.26}$$

with $\beta_1, \beta_2, \beta_3 \in C_b^{k+2}(T^*\Sigma\lceil\Omega)$ and $\delta_1, \delta_2, \delta_3 \in C_b^{k+2}(T^*\Sigma\lceil\Omega)^2$. Then ω belongs to $\mathcal{A}_{k+2}(T^*\Sigma\lceil\Omega, \zeta)$. $\square$

5. The critical estimate for the strongly bracket-generating case. In [21], the MPP was solved by using the following theorem (see also [3] for another proof):

THEOREM 5.1. *Let $\Sigma = (M, \mathbf{f})$ be a smooth driftless control-affine system with dynamical equation*

$$\dot{x} = \sum_{i=1}^{m} u_i f_i(x)\,,$$

whose state space M is a smooth connected n-dimensional manifold. Assume that M is endowed with a Riemannian metric, $\mathbf{f} = (f_1, \ldots, f_m)$ is an m-tuple of vector fields of class C^3, and Σ satisfies the LARC, the NEC and the SBGC. Let $p \in M$ and let $\mathcal{E}_p$ be the corresponding endpoint map. Then for every compact subset K of M such that $p \notin K$ there exists a constant $c_K > 0$ such that (CRIT) holds. $\square$

Theorem 5.1 is an easy corollary of the general estimate proved in Theorem 4.2. Indeed, if $K \subseteq M$ is compact, and $p \notin K$, then we can fix $\alpha > 0$ such that $d(p, K) > \alpha$ and $\{x : d(x, K) \leq \alpha\}$ is compact. We can then let J be the set of those $(x, z) \in T^*M$ such that $d(x, K) \leq \alpha$ and $\frac{1}{2} \leq \|z\| \leq \frac{3}{2}$. Then J is a compact subset of $T^*M \setminus \{0\}$. Let Ω denote the interior of J. Let $K_0 = \{(x, z) : x \in K\,,\ \|z\| = 1\}$. Then K_0 is compact and $K_0 \subseteq \Omega$. Let ψ be a smooth function on T^*M that vanishes on K_0 and is equal to 1 on $T^*M \setminus \Omega$. Let ζ be defined as in (4.22). Then the vector-valued map $\nabla_{\mathbf{F}}\psi$ is in $\mathcal{A}_1(T^*\Sigma\lceil\Omega, \zeta)$ by Proposition 4.3. Let $\Theta : [0, \infty[\to [0, \infty[$ be continuous, monotonically nondecreasing, and such that $\Theta(0) = 0$, $\Theta(s) > 0$ when $s > 0$, and

$$\left| \int_\Gamma \beta.\nabla_{\mathbf{F}}\psi \right| \leq \Theta(\|\Gamma\|_\zeta) \,.\, \|\beta\|_{C_b^1(T^*\Sigma\lceil\Omega)}$$

whenever $\Gamma \in TCP(T^*\Sigma\lceil\Omega)$ and $\beta \in C_b^1(T^*\Sigma\lceil\Omega)$.

Let $\eta \in H$, and let $\xi = \xi_{p,\eta}$, so $\xi : [0,1] \to M$ is a trajectory of Σ corresponding to η, and $\xi(0) = p$. Let $q = \mathcal{E}_p(\eta)$, and assume that $q \in K$. Let $z \in T_q^*M$ be such that $\|z\| = 1$. Let $\Xi = (\xi, \lambda)$, where λ is the solution of the adjoint equation along $\gamma = (\xi, \eta)$ with terminal condition $\lambda(1) = z$. Then $\Xi(1) = (q, z) \in K_0$, but $\Xi(0) \notin J$, because $p \notin K$. So there exists an $a \in [0, 1[$ such that $\Xi(a) \notin \Omega$ but $\Xi(t) \in \Omega$ for $t \in]a, 1]$. Then $\psi(\Xi(1)) = 0$ and $\psi(\Xi(a)) = 1$. If $t \in]a, 1]$ then, using $\beta = 1$, so that $\|\beta\|_{C_b^1(T^*\Sigma\lceil\Omega)} = 1$, we get

$$\begin{aligned} |\psi(\Xi(1)) - \psi(\Xi(t))| &= \Big| \int_t^1 \eta(s)^\dagger \nabla_{\mathbf{F}} \psi(\Xi(s)\, ds \Big| \\ &\le \Theta\Big(\int_t^1 \|\eta(s)\| \,.\, \zeta(\Xi(s))\, ds \Big) \\ &\le \Theta\Big(\|\eta\|_H \,.\, \|\varphi_{p,z,\eta}\|_{L^2([0,1])} \Big) \,, \end{aligned}$$

where the last inequality follows because Θ is nondecreasing.

Letting $t \downarrow a$, we find

$$\Theta\Big(\|\eta\|_H \,.\, \|\varphi_{p,z,\eta}\|_{L^2([0,1])} \Big) \ge 1 \,. \tag{5.1}$$

Let $\bar{s} \in]0, \infty[$ be such that $0 < \Theta(\bar{s}) < 1$. (The existence of $\bar{s}$ follows from the facts that $\Theta(0) = 0$, $\Theta(s) > 0$ when $s > 0$, and Θ is continuous.) Then (5.1) implies

$$\|\eta\|_H \,.\, \|\varphi_{p,z,\eta}\|_{L^2([0,1])} \ge \bar{s} \,. \tag{5.2}$$

Since (5.2) holds for all η such that $\mathcal{E}_p(\eta) \in K$ and all $z \in T^*_{\mathcal{E}_p(\eta)}M$ such that $\|z\| = 1$, and $\varphi_{p,rz,\eta}(t) = r\varphi_{p,z,\eta}(t)$ for all z, η, t, r, we conclude that (CRIT) holds with $c_K = \bar{s}$. $\square$

Theorem 5.1 implies that *every path π of class C^1 such that p does not belong to $\pi([0,1])$ can be lifted through $\mathcal{E}_p$ using the continuation method.*

Following the strategy described in Sections 1, 2 and 3, it is now easy to handle the MPP, i.e., to produce for any points $p, q \in M$ a control $\eta_{p,q}$ that steers p to q. We assume that $n > 1$, since the case $n = 1$ is trivial.

Pick two points $p, q \in M$ such that $p \neq q$. Consider the endpoint map $\mathcal{E}_p : H \to M$ corresponding to p. Since our control system satisfies the SBGC, the set $\mathcal{E}_p(S_{\Sigma,p})$ of singular values of $\mathcal{E}_p$ is just $\{p\}$. Pick $\bar{\eta} \in H$ such that $\bar{\eta} \not\equiv 0$. Then there exists a time $t_1 \in [0, 1]$ such that $p_1 \stackrel{\text{def}}{=} \gamma_{p,\bar{\eta}}(t_1) \neq p$. Define the control $\hat{\eta}$ by

$$\hat{\eta}(t) = \begin{cases} \bar{\eta}(t) & \text{if} \quad t \in [0, t_1]\,, \\ 0 & \text{if} \quad t \in (t_1, 1]\,. \end{cases}$$

Then $\mathcal{E}_p(\hat{\eta}) = p_1 \neq p$. Since $n > 1$, $\widetilde{M} = M \setminus p$ is path-connected, and we can choose a path $\pi : [0,1] \to \widetilde{M}$ which is of class C^1 and goes from p_1 to q. By Theorem 5.1, π can be lifted through $\mathcal{E}_p$ using the continuation method to a path $\Pi : [0,1] \to H$ such that $\Pi(0) = \hat{\eta}$, and we can take $\eta_{p,q} = \Pi(1)$.

A similar result applies to the MPPO, since the SBGC is still satisfied by the modified system $\widetilde{\Sigma}$, as explained in Section 2.

6. The front-wheel-driven car. We now analyze in detail a control system Σ that does not satisfy the SBGC, and is such that the singular-value set $\mathcal{E}_p(S_{\Sigma,p})$ is not just reduced to $\{p\}$.

We will study the application of the continuation method to a model for a car moving on a two-dimensional plane that one would like to bring from a given position to a desired one. (A typical special case of this may be the parking problem.)

The kinematic model is given by the driftless control system

$$\Sigma : \quad \begin{cases} \dot{x}_1 = u_1 \cos x_4 \cos x_3 \,, \\ \dot{x}_2 = u_1 \cos x_4 \sin x_3 \,, \\ \dot{x}_3 = u_1 \sin x_4 \,, \\ \dot{x}_4 = u_2 \,, \end{cases} \tag{6.1}$$

with two controls u_1, u_2, and state space equal to $\mathbb{R}^4$.

The physical meaning of the state variables x_i is as follows: x_1 and x_2 are the coordinates in the plane of the center of the rear-axle (whose length we normalize to be equal to 1); x_3 is the car orientation with respect to the x_1-axis, and x_4 indicates the steering angle, i.e., the angle between the front wheels and the rear-axle. The controls u_1, u_2 are, respectively, the forward velocity and the steering velocity of the car (see Figure 1).

The dynamical equations (6.1) describe a driftless control-affine system $\Sigma = (M, \mathbf{f})$, and can be rewritten in vector-field form as

$$\Sigma : \quad \dot{x} = u_1 f_1(x) + u_2 f_2(x),$$

where the manifold M is equal to $\mathbb{R}^4$, endowed with the Euclidean distance, and $\mathbf{f} = (f_1, f_2)$. The vector fields f_1 and $f_2 : \mathbb{R}^4$ —regarded as maps from $\mathbb{R}^4$ to $\mathbb{R}^4$— are defined by

$$f_1(x) = \begin{pmatrix} \cos x_4 \cos x_3 \\ \cos x_4 \sin x_3 \\ \sin x_4 \\ 0 \end{pmatrix}, \quad f_2(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

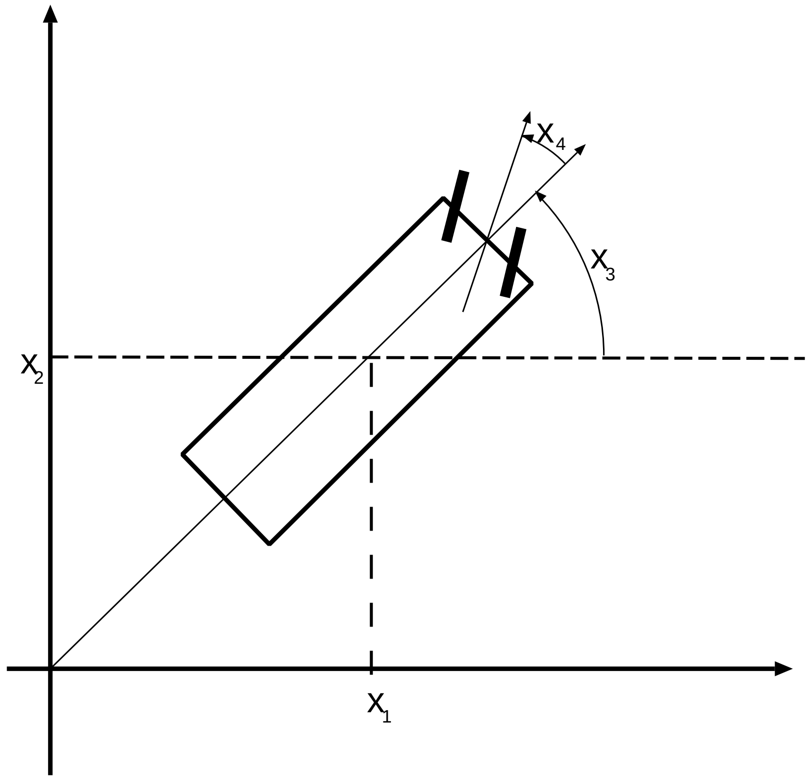

FIGURE 1.

Let

$$f_3 = [f_1, f_2] = \begin{pmatrix} \sin x_4 \cos x_3 \\ \sin x_4 \sin x_3 \\ -\cos x_4 \\ 0 \end{pmatrix}, \quad f_4 = [f_1, f_3] = \begin{pmatrix} -\sin x_3 \\ \cos x_3 \\ 0 \\ 0 \end{pmatrix}.$$

An easy computation then shows that

$$\text{(6.2)} \qquad \Big(\forall x \in \mathbb{R}^4\Big)\Big(\det(f_1(x), f_2(x), f_3(x), f_4(x)) = 1\Big).$$

Therefore Σ verifies the LARC at every point x of $\mathbb{R}^4$. In addition, the NEC holds, since f_1 and f_2 are bounded on $\mathbb{R}^4$.

On the other hand, Σ *does not satisfy the SBGC.* (Take, for example, $\theta = (1, 0)$. Then $\theta\mathbf{f} = f_1$, so

$$\text{span}\ (f_1, f_2, [\theta\mathbf{f}, f_1], [\theta\tilde{f}, f_2]) = \ \text{span}\ (f_1, f_2, f_3)\,,$$

which is 3-dimensional.) Therefore Theorem 5.1 does not apply in this situation.

We will prove the following two theorems (using H, as before, to denote the space $L^2([0,1],\mathbb{R}^2)$ of square-integrable open-loop controls):

THEOREM 6.1. *Let Σ be the system defined above. Let $p = (p_1,\dots,p_4)$ be a point in $\mathbb{R}^4$ and let $\mathcal{E}_p$ be the endpoint map associated to p. Then:*

(6.1.a) *The singular set $S_{\Sigma,p}$ is given by*

$$S_{\Sigma,p} = \{\eta = (\eta_1,\eta_2) \in H : \eta_1 = 0 \ \text{a.e. in } [0,1]\}\,, \tag{6.3}$$

and $\mathcal{E}_p(S_{\Sigma,p})$ is the line $p + \mathbb{R}v$, where v is the vector $(0,0,0,1)^\dagger$.

(6.1.b) *For every path $\pi : [0,1] \to \mathbb{R}^4$ of class C^1 such that $p \notin \pi([0,1])$ and $\pi_4(t) = p_4$ for all t, and every initial condition $\Pi(0) = \bar{\eta}$ with the property that $\mathcal{E}_p(\bar{\eta}) = \pi(0)$, the Path Lifting Equation has a solution $\Pi : [0,1] \to H$, i.e. π can be lifted through $\mathcal{E}_p$ using the continuation method.*

THEOREM 6.2. *Let $C \stackrel{\text{def}}{=} C_1 \times \mathbb{R}$, where C_1 is a closed subset of $\mathbb{R}^3$, and let $\psi : \mathbb{R}^4 \to \mathbb{R}$ be a C^∞ function such that $\psi = 0$ on C, ψ is independent of x_4, and $0 < \psi \le 1$ on $M \setminus C$. Let $\Sigma_{C,\psi}$ be the control system given by*

$$\Sigma_{C,\psi} = \left(\mathbb{R}^4 \setminus C, \hat{f}^\psi\right),$$

where $\hat{f}^\psi = (\hat{f_1}^\psi, \hat{f_2}^\psi) \stackrel{\text{def}}{=} (\psi f_1, \psi f_2)$. Let N be an arbitrary connected component of $M \setminus C$, and let $\Sigma_{N,\psi}$ denote the restriction of $\Sigma_{C,\psi}$ to N. Let $p = (p_1,\dots,p_4) \in N$. Then

(6.2.a) *The singular set $S_{\Sigma_{N,\psi},p}$ is given by*

$$S_{\Sigma_{N,\psi},p} = \{\eta = (\eta_1,\eta_2) \in H : \eta_1 = 0 \ \text{a.e. in } [0,1]\,,\} \tag{6.4}$$

and $\mathcal{E}_p(S_{\Sigma_{N,\psi},p}) = p + \mathbb{R}v$.

(6.2.b) *Conclusion (6.1.b) of Theorem 6.1 holds.*

Using Theorems 6.1 and 6.2, we can easily solve the MPP and the MPPO. We will only give a complete argument for the MPP since that same argument applies to the MPPO with minor modifications.

To solve the MPP we have to find, for every point q in the state space, a control $\eta_{p,q}$ that steers p to q. Let $q = (q_1,\dots,q_4) \ne p$. Assume first that $p_i = q_i$ for $i = 1,2,3$. Then $\eta_{p,q} = (0,\eta_2)$, where $\eta_2(t) \equiv q_4 - p_4$, solves our problem. Next, we consider the case where $p_i \ne q_i$ for some $i \in \{1,2,3\}$. Suppose that we know a control $\tilde{u}$ that steers p to $\tilde{q} = (q_1,q_2,q_3,p_4)$. Then the concatenation of $\tilde{\eta}$ and $(0, q_4 - p_4)$ —properly reparametrized— will steer s to q. So it suffices to consider the case when $q_4 = p_4$. From now on, we assume that we are in this case, i.e. that $q_4 = p_4$.

Let $\bar\eta_0(t) = (1,0)$ for $t \in [0,1]$. Then the fourth coordinate $x_4(t)$ of $\xi_{p,\bar\eta_0}$ is constant, i.e. $x_4(t) \equiv p_4$. But, since $\dot x_1^2 + \dot x_2^2 + \dot x_3^2 = 1$, the car is moving. Hence, $\xi_{p,\bar\eta_0}(t_0) \neq p$ for some $t_0 \in [0,1]$.

Let $p_0 = \xi_{p,\bar\eta_0}(t_0)$, and let η_0 be given by

$$\eta_0(t) = \begin{cases} \bar\eta_0(t) & \text{if} \quad t \in [0,t_0]\,, \\ 0 & \text{if} \quad t \in (t_0,1]\,. \end{cases}$$

It is then possible to choose a C^1 path $\pi : [0,1] \to \mathbb{R}^4$ in the hyperplane $\{x \in \mathbb{R}^4 : x_4 = p_4\}$, such that $\pi(0) = p_0$ and $\pi(1) = q$. The path π can also be chosen in such a way that

$$(\forall t \in [0,1])\Big(\pi(t) \neq p_0 \text{ and } \pi_4(t) \equiv p_4\Big)\,. \tag{6.5}$$

By Theorem 6.1, Part (6.1.b), the Path Lifting Equation associated to π with initial condition η_0 admits a solution Π defined on $[0,1]$. We can then take $\eta_{p,q}$ to be equal to $\Pi(1)$.

We now prove Theorem 6.1. Pick $p = (p_1,\ldots,p_4)$ in $\mathbb{R}^4$, and consider the corresponding endpoint map $\mathcal{E}_p$. To characterize the singular set $S_{\Sigma,p}$ of $\mathcal{E}_p$, we analyze the linear differential system satisfied by the switching functions. For that purpose, we need to compute a few more Lie brackets of the f_i's:

$$[f_2,f_3] = f_1, \quad [f_1,f_4] = \begin{pmatrix} -\sin x_4 \cos x_3 \\ -\sin x_4 \sin x_3 \\ 0 \\ 0 \end{pmatrix}, \ [f_2,f_4] = 0\,,$$

and observe that

$$[f_1,f_4] = -\sin x_4 \cos x_4\, f_1 - \sin^2 x_4\, f_3\,.$$

Then for $\eta = (\eta_1,\eta_2) \in H$ and $z \in \mathbb{R}^4$, if we apply (3.10) for each vector field f_i, $1 \le i \le 4$, we get

$$\begin{cases} \dot\varphi_{1,z} = -\eta_2\,\varphi_{3,z}\,, \\ \dot\varphi_{2,z} = \quad \eta_1\,\varphi_{3,z}\,, \\ \dot\varphi_{3,z} = \quad \eta_1\,\varphi_{4,z} + \eta_2\,\varphi_{1,z}\,, \\ \dot\varphi_{4,z} = -\eta_1\, \sin x_4(\cos x_4 \varphi_{1,z} + \sin x_4 \varphi_{3,z})\,, \end{cases} \tag{6.6}$$

where

$$\varphi_{i,z} = \varphi_{f_i,p,z,\eta}\ , \quad i = 1,2,3,4\,. \tag{6.7}$$

Let

$$\varphi_z \stackrel{\text{def}}{=} (\varphi_{1,z},\, \varphi_{2,z},\, \varphi_{3,z},\, \varphi_{4,z})^\dagger\,.$$

The singular set $S_{\Sigma,p}$ of $\mathcal{E}_p$ is the set of those $\eta = (\eta_1, \eta_2)^\dagger \in H$ such that there exists a $z \neq 0$ in $\mathbb{R}^4$ for which $\varphi_{1,z} \equiv \varphi_{2,z} \equiv 0$. Since $z \neq 0$, (6.2) implies that $\varphi_z(1) \neq 0$. Suppose $\eta \neq (0,0)$ but $\eta \in S_{\Sigma,p}$. Then $\dot\varphi_{1,z} \equiv \dot\varphi_{2,z} \equiv 0$ and

$$\frac{d}{dt}(\varphi_{3,z}^2) = 2\varphi_{3,z}\dot\varphi_{3,z} = 2\varphi_{3,z}(\eta_1\varphi_{4,z} + \eta_2\varphi_{1,z}) = 2(\dot\varphi_{2,z}\varphi_{4,z} - \dot\varphi_{1,z}\varphi_{1,z}) = 0.$$

Then $\varphi_{3,z} \equiv \varphi_{3,z}(1)$ is constant, because $\varphi_{3,z}^2$ is constant, and $\varphi_{3,z}$ is continuous.

On the other hand, $(\eta_1^2 + \eta_2^2)\varphi_{3,z} = (-\eta_2\dot\varphi_{1,z} + \eta_1\dot\varphi_{2,z}) \equiv 0$. Since $\eta \neq (0,0)$ and $\varphi_{3,z}$ is constant, we conclude that $\varphi_{3,z} \equiv 0$. Then (6.6) implies $\eta_1\varphi_{4,z} \equiv 0$ and $\dot\varphi_{4,z} = 0$. So $\varphi_{4,z}$ is a nonzero constant, and then $\eta_1 = 0$ a.e. in $[0,1]$.

Conversely, if $\eta_1 = 0$ a.e. in $[0,1]$, then we can choose z such that $z.f_i(\xi_{p,\eta}(1)) = 0$ for $i = 1,2,3$, but $z.f_4(\xi_{p,\eta}(1)) \neq 0$. Then (6.6) implies that $\varphi_{2,z} \equiv 0$. Moreover, (6.6) also implies that $\dot\varphi_{1,z} = -\eta_2\,\varphi_{3,z}$ and $\dot\varphi_{3,z} = \eta_2\,\varphi_{1,z}$, from which it follows that $\varphi_{1,z} \equiv \varphi_{3,z} \equiv 0$. So $\eta \in S_{\Sigma,p}$. Thus we have established Part (6.1.a) of Theorem 6.1.

REMARK. Geometrically, the trajectories corresponding to $\eta \in S_{\Sigma,p}$ are the curves $t \to x(t)$ such that $x(t) = x_0 + \left(\int_0^t \eta_2(s)ds\right) v$. Along these curves, the body of the car does not move, but the front wheels are rotating with angular velocity $\eta_2(t)$. $\square$

We now prove (6.1.b). Let $q = (q_1, \ldots, q_4)$ be a point in $\mathbb{R}^4$ such that $p_i \neq q_i$ for some $i \in \{1,2,3\}$, and $p_4 = q_4$. Let $\pi : [0,1] \to \mathbb{R}^4$ be a path such that $\pi(t) \neq p$ for all $t \in [0,1]$, but $\pi_4(t) = p_4$ for all t. Then, for every η in H such that $\mathcal{E}_p(\eta) \in \pi([0,1])$, we know that $\mathcal{E}_p(\eta) \neq p$, but the fourth coordinate of $\mathcal{E}_p(\eta)$ is p_4. This means that, for some $i \in \{1,2,3\}$, the i-th coordinate of $\mathcal{E}_p(\eta)$ is not equal to p_i. Therefore the function η_1 does not vanish identically, so $\eta \notin S_{\Sigma,p}$.

Let

$$\begin{aligned}
&Q = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : (x_1, x_2, x_3, p_4) \in \pi([0,1])\}\,,\\
&\alpha = d(p, \pi([0,1]))\,,\\
&K = \{w = (x,z) \in \mathbb{R}^4 \times \mathbb{R}^4 : x \in Q\,,\ \|\varphi^{1,2,3,4}(x,z)\| = 1\}\,,\\
&J = \{w = (x,z)\ : d(x,Q) \le \frac{\alpha}{2}\,,\ \frac{1}{2} \le \|\varphi^{1,2,3,4}(x,z)\|^2 \le \frac{3}{2}\}\,,\\
&\Omega = \text{Interior}\,(J)\,,
\end{aligned}$$

where we use φ_i to denote H_{f_i} —so the φ_i are real-valued functions on T^*M, i.e. on $\mathbb{R}^4 \times \mathbb{R}^4$— and write $\varphi^{1,2}$, $\varphi^{1,2,3}$, $\varphi^{1,2,3,4}$ for the columnn vectors $(\varphi_1,\varphi_2)^\dagger$, $(\varphi_1,\varphi_2,\varphi_3)^\dagger$, $(\varphi_1,\varphi_2,\varphi_3,\varphi_4)^\dagger$, respectively. In particular,

$$\|\varphi^{1,2}\|^2=\varphi_1^2+\varphi_2^2 \quad \text{and} \quad \|\varphi^{1,2,3,4}\|^2=\varphi_1^2+\varphi_2^2+\varphi_3^2+\varphi_4^2 .$$

Then $\alpha>0$, Q is a closed subset of $\mathbb{R}^4$, and K, J are closed subsets of $\mathbb{R}^8$.

If $F_i=T^*f_i$ for $i=1,2$, $\mathbf{F}=(F_1,F_2)$, and $\mathbf{f}=(f_1,f_2)$, we have

$$\nabla_{\mathbf{F}}\varphi_1 = \begin{pmatrix} 0 \\ -\varphi_3 \end{pmatrix},\ \nabla_{\mathbf{F}}\varphi_2=\begin{pmatrix} \varphi_3 \\ 0 \end{pmatrix},\ \nabla_{\mathbf{F}}\varphi_3=\begin{pmatrix} \varphi_4 \\ \varphi_1 \end{pmatrix},$$

where, as explained before, we define $\nabla_{\mathbf{F}}\psi=\begin{pmatrix} L_{F_1}\psi \\ L_{F_2}\psi \end{pmatrix}$ for any function ψ.

The definition of J clearly implies that the four momentum functions φ_i, $i=1,2,3,4$, are bounded on J. Then (6.6) implies that the functions $L_{F_i}\varphi_j$, for $j=1,2,3,4$ and $i=1,2$, are bounded on J. Therefore the φ_j are in $C_b^1(T^*\Sigma\lceil\Omega)$. Since the components of the vector $\nabla_{\mathbf{F}}\varphi_3$ are just φ_1 and φ_4, both of which are in $C_b^1(T^*\Sigma\lceil\Omega)$, we can conclude that $\varphi_3\in C_b^2(T^*\Sigma\lceil\Omega)$. But then, since the components of the vectors $\nabla_{\mathbf{F}}\varphi_j$, for $j=1,2$, are just $\pm\varphi_3$ and 0, and $\varphi_3\in C_b^2(T^*\Sigma\lceil\Omega)$, we can conclude that φ_1 and φ_2 are in $C_b^3(T^*\Sigma\lceil\Omega)$.

On the other hand, the functions $x\to\sin x_4\cos x_4$ and $x\to\sin^2 x_4$ are clearly in $C_b^\infty(\Sigma)$. Therefore the components of $\nabla_{\mathbf{F}}\varphi_4$ are in $C_b^2(T^*\Sigma\lceil\Omega)$, so $\varphi_4\in C_b^3(T^*\Sigma\lceil\Omega)$. Then the components of $\nabla_{\mathbf{F}}\varphi_3$ are in $C_b^2(T^*\Sigma\lceil\Omega)$, so $\varphi_3\in C_b^3(T^*\Sigma\lceil\Omega)$.

We can then apply Proposition 4.4, with $k=0$, and conclude that, if $\beta_1,\beta_2,\beta_3\in C_b^2(T^*\Sigma\lceil\Omega)$ and $\delta_1,\delta_2,\delta_3\in C_b^2(T^*\Sigma\lceil\Omega)^2$, then the linear combination

$$\omega=\varphi_1\delta_1+\varphi_2\delta_2+\varphi_3\delta_3+\beta_1\nabla_{\mathbf{F}}\varphi_1+\beta_2\nabla_{\mathbf{F}}\varphi_2+\beta_3\nabla_{\mathbf{F}}\varphi_3 \tag{6.8}$$

belongs to $\mathcal{A}_2(T^*\Sigma\lceil\Omega,\zeta)$.

Now let $\sigma:\mathbb{R}\to\mathbb{R}$ be a smooth function such that $\sigma(s)=1$ when $|s|\le\frac{1}{16}$, $\sigma(s)=0$ when $|s|\ge\frac{1}{8}$, and $0\le\sigma(s)\le1$ for all s. Define

$$\hat\sigma(w)=\sigma\Big(\varphi_1(w)^2+\varphi_2(w)^2+\varphi_3(w)^2\Big).$$

The function φ_4^2 belongs to $C_b^2(T^*\Sigma\lceil\Omega)$ and is bounded below by a positive constant on the open set

$$S_1=\{w\in\Omega:\varphi_1^2(w)+\varphi_2^2(w)+\varphi_3^2(w)<\frac{1}{4}\},$$

so $\varphi_4^{-2}\hat\sigma$ is smooth on $\Omega\cap S_1$ and belongs to $C_b^2(T^*\Sigma\lceil(\Omega\cap S_1))$. Since $\varphi_4^{-2}\hat\sigma$ vanishes on

$$S_2=\{w\in\Omega:\varphi_1^2(w)+\varphi_2^2(w)+\varphi_3^2(w)>\frac{1}{8}\},$$

and $S_1 \cup S_2 = T^*M$, it follows that $\varphi_4^{-2}\hat{\sigma}$ is smooth on Ω and belongs to $C_b^2(T^*\Sigma\lceil\Omega)$.

On the other hand, we can write $1-\sigma(s) = s.\tau(s)$, where τ is smooth and bounded together with all its derivatives. Then, if $w \in T^*M$, we have

$$1-\hat{\sigma}(w) = \varphi_1(w)\hat{\tau}_1(w) + \varphi_2(w)\hat{\tau}_2(w) + \varphi_3(w)\hat{\tau}_3(w)\,,$$

where $\hat{\tau}_i(w) = \varphi_i(w)\tau\Big(\varphi_1(w)^2 + \varphi_2(w)^2 + \varphi_3(w)^2\Big)$.

Then

$$\begin{aligned}\begin{pmatrix}1\\0\end{pmatrix} &= \hat{\sigma}\begin{pmatrix}1\\0\end{pmatrix} + (1-\hat{\sigma})\begin{pmatrix}1\\0\end{pmatrix}\\ &= \varphi_4(\varphi_4^{-2}\hat{\sigma})\begin{pmatrix}\varphi_4\\0\end{pmatrix} + (\varphi_1\hat{\tau}_1 + \varphi_2\hat{\tau}_2 + \varphi_3\hat{\tau}_3)\begin{pmatrix}1\\0\end{pmatrix},\end{aligned}$$

so

$$\begin{pmatrix}1\\0\end{pmatrix} = \beta\,.\,\nabla_{\mathbf{F}}\varphi_3 + \varphi_1\delta_1 + \varphi_2\delta_2 + \varphi_3\delta_3\,,$$

where $\beta = \varphi_4(\varphi_4^{-2}\hat{\sigma})$, $\delta_1 = \begin{pmatrix}\hat{\tau}_1\\-\beta\end{pmatrix}$, $\delta_2 = \begin{pmatrix}\hat{\tau}_2\\0\end{pmatrix}$, and $\delta_3 = \begin{pmatrix}\hat{\tau}_3\\0\end{pmatrix}$.

It is clear that $\beta \in C_b^2(T^*\Sigma\lceil\Omega)$ and $\delta_1, \delta_2, \delta_3 \in C_b^2(T^*\Sigma\lceil\Omega)^2$. So the vector $\begin{pmatrix}1\\0\end{pmatrix}$ belongs to $\mathcal{A}_2(T^*\Sigma\lceil\Omega,\zeta)$. If ψ is any of the functions x_1, x_2, x_3, φ_2, φ_4, then $\nabla_{\mathbf{F}}\psi$ is a multiple of $\begin{pmatrix}1\\0\end{pmatrix}$ with coefficients in $C_b^2(T^*\Sigma\lceil\Omega)$, so $\nabla_{\mathbf{F}}\psi \in \mathcal{A}_2(T^*\Sigma\lceil\Omega,\zeta)$. We also know that $\nabla_{\mathbf{F}}\varphi_1$ and $\nabla_{\mathbf{F}}\varphi_3$ are in $\mathcal{A}_2(T^*\Sigma\lceil\Omega,\zeta)$. So we have in fact shown that

PROPOSITION 6.1. *If ψ is any of the seven functions x_1, x_2, x_3, φ_1, φ_2, φ_3, φ_4, then $\nabla_{\mathbf{F}}\psi \in \mathcal{A}_2(T^*\Sigma\lceil\Omega,\zeta)$.* □

We are now ready to prove Theorem 6.1. We will apply Propositions 3.2 and 6.1. Assume that the control η steers p to a point of $\pi([0,1])$. Pick a covector z in $\mathbb{R}^4$ (i.e. in $T^*_{\xi_{p,\eta}(1)}M$) such that

$$||\varphi^{1,2,3,4}(\xi_{p,\eta}(1), z)|| = 1\,. \tag{6.9}$$

Then the point $\bar{w} = (\xi_{p,\eta}(1), z)$ belongs to K. Let $\Xi : [0,1] \to \mathbb{R}^4 \times \mathbb{R}^4$ be the trajectory of $T^*\Sigma$ for the control η such that $\Xi(1) = \bar{w}$, and write

$$\begin{aligned}\Xi(t) &= (x(t), z(t))\,,\\ x(t) &= (x_1(t), x_2(t), x_3(t), x_4(t))\,,\\ \tilde{x}(t) &= (x_1(t), x_2(t), x_3(t), p_4)\,,\end{aligned}$$

for $0 \le t \le 1$.

Then there exists a maximal subinterval $[t_0, 1]$ of $[0, 1]$ for which the set $\Xi(]t_0, 1])$ is entirely contained in Ω. Since $\Xi(0) \notin J$ (because $\xi_{p,z}(0) = p$, and $d(p, \pi([0, 1]) \ge \alpha)$ and $\Xi(1) \in K$, the point $\Xi(t_0)$ must belong to ∂J. It follows that one of the inequalities

$$d(\tilde{x}(t_0), \pi([0, 1])) \ge \frac{\alpha}{2},$$
$$\|\varphi^{1,2,3,4}(\xi(t_0))\|^2 \le \frac{1}{2},$$
$$\|\varphi^{1,2,3,4}(\xi(t_0))\|^2 \ge \frac{3}{2},$$

holds.

If the first inequality holds, then $\sum_{i=1}^{3}(x_i(t_0) - x_i(1))^2 \ge \frac{\alpha^2}{4}$, so $|x_i(t_0)-x_i(1)| \ge \frac{\alpha}{\sqrt{12}}$ for some $i \in \{1, 2, 3\}$. If the second or third inequality holds, it follows that $\sum_{i=1}^{4}\left|\varphi_i(\xi(t_0))^2 - \varphi_i(\xi(1))^2\right| \ge \frac{1}{2}$. This implies that $|\varphi_i(\xi(t_0))^2 - \varphi_i(\xi(1))^2| \ge \frac{1}{8}$ for some i, and then $|\varphi_i(\xi(t_0) - \varphi_i(\xi(1))| \ge \frac{1}{20}$, because $|\varphi_i(\xi(t_0)) + \varphi_i(\xi(1))| \le \frac{5}{2}$, since $|\varphi_i(\xi(1))| \le 1$ and $|\varphi_i(\xi(t_0))|$ is bounded by $\sqrt{\frac{3}{2}}$, which is $\le \frac{3}{2}$.

Proposition 6.1 tells us that for every $\psi \in \Psi$ there exists a function $\Theta_\psi : [0, \infty[\to [0, \infty[$ which is continuous, monotonically nondecreasing, and such that $\Theta_\psi(0) = 0$, $\Theta_\psi(s) > 0$ when $s > 0$, and

$$\left|\int_\Gamma \beta.\nabla_{\mathbf{F}}\psi\right| \le \Theta_\psi(\|\Gamma\|_\zeta) \cdot \|\beta\|_{C_b^2(T^*\Sigma\lceil\Omega)} \tag{6.10}$$

whenever $\Gamma \in TCP(T^*\Sigma\lceil\Omega)$ and $\beta \in C_b^2(T^*\Sigma\lceil\Omega)$.

Let

$$\bar{\alpha} = \min\left(\frac{\alpha}{\sqrt{12}}, \frac{1}{20}\right).$$

Let

$$\Theta(s) = \max\{\Theta_\psi(s) : \psi \in \Psi\}.$$

Also, let $\Gamma = (\Xi, \eta)$. Then, using (6.10) with $\beta = 1$, we get

$$\begin{aligned} |\psi(\Xi(1)) - \psi(\Xi(t_0))| &= \int_{t_0}^{1} \eta(t)^\dagger \nabla_{\mathbf{F}}\psi(\Xi(t))\,dt \\ &\le \Theta\Big(\int_{t_0}^{1} \|\eta(t)\|.\sqrt{\varphi_1(\Xi(t))^2 + \varphi_2(\Xi(t))^2}\,dt\Big) \\ &\le \Theta\Big(\|\eta\|_H \cdot \|\varphi_{p,z,\eta}\|_{L^2([0,1])}\Big) \end{aligned}$$

for all $\psi \in \Psi$. Choose ψ such that $|\psi(\Xi(1)) - \psi(\Xi(t_0))| \geq \bar{\alpha}$. Then

$$\Theta\Big(\|\eta\|_H \cdot \|\varphi_{p,z,\eta}\|_{L^2([0,1])}\Big) \geq \bar{\alpha}\,. \tag{6.11}$$

Let $\bar{s} \in]0, \infty[$ be such that $0 < \Theta(\bar{s}) < \bar{\alpha}$. (The existence of $\bar{s}$ follows from the facts that $\Theta(0) = 0$, $\Theta(s) > 0$ when $s > 0$, and Θ is continuous.) Then (6.11) implies

$$\|\eta\|_H \cdot \|\varphi_{p,z,\eta}\|_{L^2([0,1])} \geq \bar{s}\,. \tag{6.12}$$

Since (6.12) holds for all η such that $q = \mathcal{E}_p(\eta) \in Q$ and all $z \in T^*_{\mathcal{E}_p(\eta)}M$ such that $\|\varphi^{1,2,3,4}(q,z)\| = 1$, and $\|\varphi^{1,2,3,4}(q,rz)\| = |r|\|\varphi^{1,2,3,4}(q,z)\|$ for all q, z, r, we conclude that (CRIT) holds, with $c_Q = \bar{s}$. $\square$

The proof of Theorem 6.2 is exactly like that of Theorem 6.1, except that we work with the system $\widetilde{\Sigma}$ obtained from Σ by means of the construction described in Section 2. We omit the details.

7. Conclusion. The preceding analysis shows that it is possible to prove estimates for line integrals that go beyond those needed to handle systems that satisfy the SBGC. This suggests that the rigorous justification of the continuation method may be possible not only when the SBGC holds, but in many other cases. It would be desirable to achieve a better understanding of the precise conditions under which this can be done, but at the moment it is not clear what this conditions should be.

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