LOCAL CONTROLLABILITY AND MOTION PLANNING FOR SOME CLASSES OF SYSTEMS WITH DRIFT

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Abstract

We explain how some recent results on the design of controls for nonoholomic systems without drift can be extended to some classes of sytems with drift. In particular, we show that a dynamic extension of a driftless system that satisfies the Lie algebra rank condition necessary satisfies algebraic sufficient conditions for small-time local controllability at all its equilibrium points.

1. Introduction.

Many mechanical systems can be modelled as control systems of the form

$$\Sigma : \quad \dot{x} = f_0(x) + u_1 f_1(x) + \ldots + u_m f_m(x) \quad , \quad (1)$$

where the f_i are smooth vector fields on a smooth connected manifold M. The *motion planning problem* (MPP) is the problem of designing reasonable algorithms that, for any given pair (p, q) of points in M (or in some subset S of M), produce a control that steers p to q. Various such designs have been proposed in work by Brockett [2], Hauser-Sastry-Kokotovic [6], Murray-Sastry [13], [14], Sastry-Li [15], Fernandes-Gurvits-Li [4], Lafferriere [10], Lafferriere-Sussmann [11], [12], Sussmann-Liu [21], [22], [23] (based on earlier ideas of Haynes and Hermes [5] and Kurzweil-Jarnik [7], [8], [9] and the product expansion of [18]), and others, for the *driftless* case, i.e. when $f_0 \equiv 0$. The purpose of this note is to indicate how such designs are possible for certain classes of systems with drift including, in particular, a number of examples recently considered in the literature by Bloch, McClamroch and Reyhanoglu (cf., e.g., [1]). The systems to be considered here are characterized by the fact that they have a "large" manifold of equilibrium points, i.e. of points x such that $f_0(x) = 0$.

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Let E be the set of equilibrium points of Σ. We say that Σ has an *algebraic small-time local controllability (ASTLC) property* if at every point of E it satisfies one of the Lie algebraic sufficient conditions for small-time local controllability given in [19]. (We write "an" rather than "the" ASTLC because, to be completely precise, this is not a mathematical definition but a recipe for generating such definitions: for each algebraic sufficient condition for local controllability we get a different ASTLC property. One particular example of this will be described below.)

The proof that the conditions of [19] are sufficient is actually constructive, so one can use it to effectively construct controls that steer a given p to a given q provided that p or q is in E and p and q are close enough. If in addition

(a) one knows how to steer any given $p \in M$ to some point of E, and how to reach any q in M from some point of E,

(b) E is connected,

then we can actually join any given $p \in M$ to any given q.

Mathematically, it may appear that a system is very unlikely to have the above properties in such a way that the above results are useful. Indeed, for a "generic" drift f_0 the set E is discrete, so our connectedness assumption means that there is only one equilibrium $\bar{p}$. In this case, Assumption (a) essentially amounts to supposing that we know how to drive every point to p and how to reach every point from p, so we would in fact be assuming that MPP has already been solved. Our main point is that *the class of systems that have the above properties, though highly non-generic, contains many systems that occur in the control of mechanical systems.* An important example of such systems are the ones that arise as *dynamic* *extensions of controllable real analytic driftless systems*, as will be explained below.

2. Controllability of dynamic extensions.

Proposition 1 *Let $f_1, \ldots, f_m$ be C^∞ vector fields on the C^∞ manifold M. Assume that the control-linear driftless system*

$$\Sigma: \quad \dot{x} = u_1 f_1(x) + \ldots + u_m f_m(x) \ , \qquad (2)$$

is completely controllable. Then its dynamic extension

$$\Sigma_e: \begin{cases} \dot{x} &= y_1 f_1(x) + \ldots + y_m f_m(x) \ , \\ \dot{y}_1 &= u_1 \ , \\ & \vdots \\ \dot{y}_m &= u_m \ , \end{cases} \qquad (3)$$

with state space $M \times \mathbb{R}^m$, is completely controllable as well.

Proof. It suffices to show that, if p, q are two arbitrary points of M, and α, β are arbitrary in $\mathbb{R}^m$, then p can be steered to q by means of a smooth open-loop control $\eta : [0,T] \to \mathbb{R}^m$ such that $\eta(0) = \alpha$, $\eta(T) = \beta$. This follows from general results on reachability by means of nice controls, described, e.g. in [20].) However, the proof in our case is particularly simple, as we now show.

Let $n = \dim M$. Complete controllability of Σ implies that q belongs to the *orbit* (cf. [16]) of p with respect to the vector field system $\mathcal{F} = \{f_1, \ldots, f_m\}$, and we can write $q = \Phi(\bar{t}_1, \ldots, \bar{t}_r)$, where Φ is a smooth map from a neighborhood Ω of $(\bar{t}_1, \ldots, \bar{t}_r)$ in $\mathbb{R}^r_+$ into M, such that (a) Φ arises from a piecewise constant control by varying the switching times and keeping the initial state fixed at p, and (b) the differential of Φ has rank n at $(\bar{t}_1, \ldots, \bar{t}_r)$. (Here $\mathbb{R}^r_+$ is the set of $(t_1, \ldots, t_r)$ such that all the t_j are > 0.) The precise meaning of (a) is as follows: we pick points $\hat{u}_1, \ldots, \hat{u}_r$ in $\mathbb{R}^m$, and let $\eta_{t_1,\ldots,t_r}$ be the control whose value is $\hat{u}_1$ during time t_1, then $\hat{u}_2$ during time t_2, etc., and finally $\hat{u}_r$ during time t_r. Then we let $\Phi(t_1, \ldots, t_r)$ be the point to which $\eta_{t_1,\ldots,t_r}$ steers p.

Since we may always concatenate with a zero control (because Σ has no drift), we may assume that all the $\eta_{t_1,\ldots,t_r}$ are defined on a fixed interval $[0,T]$. Next we approximate the controls $\eta_{t_1,\ldots,t_r}$ in $L^1([0,T], \mathbb{R}^m)$ by smooth controls whose values at 0 and T are α, β, respectively. For instance, we may let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\varphi(t) = 1$ for $|t| \leq \frac{1}{2}$ and $\varphi \equiv 0$ outside the interval $[-1,1]$, and then we may define $\tilde{\eta}^\varepsilon_{t_1,\ldots,t_r}(t)$ to be the product of $\eta_{t_1,\ldots,t_r}(t)$ times $\prod_{j=0}^r (1 - \varphi(\frac{t-\tau_j}{\varepsilon}))$, where the τ_j are the switching times of $\eta_{t_1,\ldots,t_r}(t)$, including the endpoints (that is: $\tau_0 = 0$, $\tau_1 = t_1$, $\tau_2 = t_1 + t_2$, $\ldots$, $\tau_r = T$. We then let

$$\eta^\varepsilon_{t_1,\ldots,t_r}(t) = \left(1 - \phi\Big(\frac{t}{\varepsilon}\Big) - \phi\Big(\frac{t-T}{\varepsilon}\Big)\right)\tilde{\eta}^\varepsilon_{t_1,\ldots,t_r}(t)$$

$$+ \phi\Big(\frac{t}{\varepsilon}\Big)\alpha + \phi\Big(\frac{t-T}{\varepsilon}\Big)\beta \, ,$$

and define $\Phi^\varepsilon(t_1, \ldots, t_r)$ to be the point to which $\eta^\varepsilon_{t_1,\ldots,t_r}$ steers p. Then $\Phi^\varepsilon(t_1, \ldots, t_r)$ is continuous as function of $t_1, \ldots, t_r$ and ε. For $\varepsilon = 0$, the map Φ^ε has rank n at $(\bar{t}_1, \ldots, \bar{t}_r)$. Then it follows by a standard topological argument that, for $\varepsilon > 0$ but near 0, the image of Φ^ε covers a neighborhood of q. (By the Implicit Function Theorem, we can choose an n-dimensional submanifold W of R^r_+, containing $(\bar{t}_1, \ldots, \bar{t}_r)$, which is mapped diffeomorphically by Φ to a ball B centered at q. Let $\rho : B \to W$ be the inverse map. Let $\theta^\varepsilon = \Phi^\varepsilon \circ \rho$. Then θ^ε converges uniformly, as $\varepsilon \to 0$, to the identity map of B.) In particular, q itself is in the image of Φ^ε, and this proves our conclusion. ∎

Remark 1 In the real analytic case, a controllable system (1) necessarily satisfies the well known *Lie algebra rank condition* (LARC): if Λ is the Lie algebra of vector fields generated by $f_0, \ldots, f_m$, then $\Lambda(x)$ must be the full tangent space at x for each $x \in M$. Moreover, if $f_0 = 0$ and the state space is connected, then this condition is sufficient as well as necessary for controllability. Proposition 1 therefore implies that, if Σ is driftless real analytic and satisfies the LARC, then Σ_e has the LARC as well. We now show that this can be established directly by an algebraic argument, without using real analyticity.

Write $\xi = (x,y)$ for the state variable of Σ_e. Then Σ_e has the form $\dot{\xi} = F_0(\xi) + u_1 F_1(\xi) + \ldots + u_m F_m(\xi)$, where $F_0 = \sum_{i=1}^m y_i f_i(x)$ (regarded, in an obvious way, as a vector field on (x,y) space) and $F_i = \frac{\partial}{\partial y_i}$ for $i > 0$. Notice that the F_i for $i > 0$ commute with each other. It is then easy to compute all the brackets of degree 2: $[F_i, F_0] = f_i$ (so that $[F_0, F_i] = -f_i$) for $i > 0$, and all the other degree 2 brackets vanish. If we bracket again with F_0 and then with F_j, we get $[F_j,[F_0,[F_i,F_0]]] = [f_j,f_i]$. Continuing in this way we can show that every bracket of the f_i's, $i \in \{1,\ldots,m\}$, can be obtained as a bracket of the F_i's, $i \in \{0,\ldots,m\}$. Since the f_i satisfy the Lie algebra rank condition, we conclude that the Lie algebra Λ_e generated by the F_i is such that, at each point $\xi = (x,y)$, $\Lambda_e(x)$ contains all the x-directions. The vector fields F_i themselves give us all the y-directions, and we can then conclude that $\Lambda_e(\xi)$ is the full tangent space at ξ. So Σ_e satisfies the Lie algebra rank condition, as stated. ▮

Remark 2 The proof of Prop. 1 was based on the existence of the map Φ, which follows from the general properties of the orbits described in [16]. Using the deeper results of [17] (i.e. that, for a completely controllable system, every point is "normally reachable" from every point), the existence of the map Φ also follows for systems with drift, so the statement of Prop. 1 is also true for systems with drift. ▮

3. Local controllability of dynamic extensions.

Proposition 2 *Let $f_1,\ldots,f_m$ be C^∞ vector fields on the C^∞ manifold M. Assume that the control-linear driftless system 2 satisfies the Lie algebra rank condition. Then its dynamic extension Σ_e satisfies an algebraic small-time local controllability property at each of its equilibrium points.*

Proof. Let $\mathcal{M}_k$ denote the set of all smooth vector fields V that are of the form $V(\xi) = \sum_j \varphi_j(y)X_j(x)$, where the X_j are smooth vector fields and the φ_j are homogeneous polynomials in y of degree k. It is clear that the bracket of $V_1 \in \mathcal{M}_{k_1}$ and $V_2 \in \mathcal{M}_{k_2}$ is in $\mathcal{M}_{k_1+k_2}$. Moreover, $F_0 \in \mathcal{M}_1$, and therefore $V \in \mathcal{M}_k$ implies $[F_0,V] \in \mathcal{M}_{k+1}$. On the other hand, one easily verifies that, if $i > 0$, then $V \in \mathcal{M}_k$ implies $[F_i,V] \in \mathcal{M}_{k-1}$. So, although the operations of bracketing with F_0 and the other F_i have different effects on the degree k of V, one raising it and the other one lowering it, both have the same effect on the *parity* of k. More precisely, if we let $\mathcal{M}_{\rm odd}$ be the sum of all the $\mathcal{M}_k$ with k odd, and define $\mathcal{M}_{\rm even}$ similarly, then bracketing with any of the F_i, $i = 0,1,\ldots,m$ maps $\mathcal{M}_{\rm odd}$ to $\mathcal{M}_{\rm even}$ and $\mathcal{M}_{\rm even}$ to $\mathcal{M}_{\rm odd}$. Since all the brackets of degree 2 of the F_i are in $\mathcal{M}_0$, we conclude that all the brackets of degree $k \geq 2$ are in $\mathcal{M}_{\rm odd}$ if k is odd, and in $\mathcal{M}_{\rm even}$ if k is even. In particular, all the brackets of the F_i of odd degree vanish on E.

We now use the algebraic conditions of [19]. The transformations σ_i, $i = 1,\ldots,m$, that change the sign of the control u_i (i.e. that map each bracket of the F_j to the one obtained by changing F_i to $-F_i$) are symmetries of the system in the sense of [19]. Another symmetry is "control reversal," i.e. the map τ that sends each formal product $F_{i_1}F_{i_2}\ldots F_{i_m}$ to the reversed product $F_{i_m}\ldots F_{i_2}F_{i_1}$. It is clear that $\tau(F_i) = F_i$ for all i, and $\tau(AB) = \tau(B)\tau(A)$ for all formal products A, B. In particular, this imples that $\tau([A,B]) = [\tau(B),\tau(A)] = -[\tau(A),\tau(B)]$ From this it follows easily by induction that, if L is a homogeneous Lie bracket of degree k, then $\tau(L) = (-1)^{k+1}L$. If we now let G be the group of symmetries generated by τ and the σ_i, then G is finite. The algebraic condition of [19] says that every G-invariant linear combination of brackets must be "neutralized" at each equilibrium point p. Now, if Z is such a combination, then each homogeneous component Z_k of Z is mapped by τ to $(-1)^{k+1}(Z_k)$. If k is even, then the invariance of Z under τ implies that $Z_k = 0$. So Z is a sum of components Z for *odd* indices k. If k is odd and $k > 1$, we have shown that all brackets of degree k vanish at each equilibrium. So all the Z_k for $k > 1$ vanish on E. The component Z_1 has the form $a_0F_0 + a_1F_1 + \ldots + a_mF_m$. Invariance under the σ_i implies that $a_1 = \ldots = a_m = 0$, so Z_1 is in fact a multiple of F_0. Therefore Z_1 also vanishes at each equilibrium. So all the G-symmetric linear combinations of brackets are neutralized, and an algebraic sufficient condition for small-time local controllability is satisfied. ▮

Remark 3 It is clear that in the case studied in Proposition 2, the equilibrium set E is a "rather large" connected manifold, namely, the set of all pairs $(x,0)$. Moreover, Condition **(a)** of Section 1 holds trivially, because it is very easy to stop the motion (i.e. to drive y to zero) if one does not care at what point of E one ends up. ▮

Remark 4 The class of systems discussed here contains many systems that are "truly nonlinear," in that, for instance, local stabilization by means of a continuous feedback is not possible. Indeed, the well known topological obstruction to stabilizability discovered by Brockett (cf. [3]) is clearly preserved under dynamic extension, i.e. if Σ does not satisfy the Brockett necessary condition at x, then Σ_e does not satisfy the condition at $(x, 0)$. One can then take any driftless system where Brockett's condition yields nonstabilizability, and construct by dynamical extension a similar example involving a system with drift which is locally controllable but not locally stabilizable. In particular, this means that for any such system *the first-order sufficient conditions for local controllability cannot possibly be satisfied*, because those conditions imply local stabilizability as well. (Indeed, the first-order conditions at a point $p \in E$ assert that the brackets $\mathrm{ad}_{F_0}^k(F_i)(p)$, for all $i \in \{1, \ldots, m\}$ and all $k = 1, 2, \ldots$, span the tangent space at p, which is precisely the condition that the linearization of Σ is controllable at p.) So in these cases local controllability really requires the use of higher-order "nonlinear" brackets. ∎

4. Homogeneous systems.

Local controllability is, of course, a local property, and the solutions of the MPP obtained by the above method are local as well. The local solutions can in principle be patched together to yield global ones, but it is interesting to know cases where one can directly obtain global solutions. A simple case where that works is that of *homogeneous systems.* For a control system of the form (1), define a *one-parameter invariance group of dilations* to be a pair $(\Delta, \tilde{\Delta})$, where $\Delta = \{\Delta_\rho\}_{0<\rho<\infty}$ is a smooth one-parameter group of diffeomorphisms of M, $\tilde{\Delta} = \{\tilde{\Delta}_\rho\}_{0<\rho<\infty}$ is a one-parameter group of linear invertible endomorphisms of of the control space, and the following properties hold:

(I) If $t \to x(t)$ is a trajectory of Σ corresponding to a control $t \to u(t)$, then $t \to \Delta_\rho(x(t))$ is a trajectory corresponding to the control $t \to \tilde{\Delta}_\rho(u(t))$.

(D) There is a compact subset K of M such that, if J is an arbitrary compact subset of M, then there is a $\rho > 0$ such that $J \subseteq \Delta_\rho(K)$.

Property **(I)** is the invariance condition. Property **(D)** is the "dilation" requirement: every compact set, no matter how large, can be covered by "dilating" some fixed compact set. "One-parameter group" is meant in the multiplicative sense, i.e. $\Delta_{\rho_1\rho_2} = \Delta_{\rho_1}\Delta_{\rho_2}$, etc.

A system will be called *homogeneous* if it admits a one-parameter invariance group of dilations as defined above.

Example 1 The system $\dot{x} = u$, $\dot{y} = v$, $\dot{z} = vx$, with state space $\mathbb{R}^3$, is homogeneous with respect to the dilations $\Delta_\rho(x, y, z) = (\rho x, \rho y, \rho^2 z)$, with $\tilde{\Delta}_\rho(u, v) = (\rho u, \rho v)$.

Example 2 The system $\dot{x} = u\cos\theta$, $\dot{y} = u\sin\theta$, $\dot{\theta} = v$, with state space $\mathbb{R}^2 \times S^1$, is homogeneous with respect to the dilations $\Delta_\rho(x, y, \theta) = (\rho x, \rho y, \theta)$, with $\tilde{\Delta}_\rho(u, v) = (\rho u, v)$.

Notice that in Example 2 the dilation condition **(D)** is satisfied even though Δ does not "dilate" in the θ direction. Intuitively, this happens because in the θ direction the state space already is compact. More formally, we may take K to be the set $\{(x, y, \theta) : |x| \leq \varepsilon\,, |y| \leq \varepsilon\}$, *with no restrictions on* θ.

It is then easy to prove:

Proposition 3 *If a system of the form (1) is homogeneous with respect to a group of dilations* $(\Delta, \tilde{\Delta})$, *then its dynamic extension is homogeneous with respect to* $\Delta_e = (\Delta, \tilde{\Delta})$ *and* $\tilde{\Delta}_e = \tilde{\Delta}$. ∎

For a homogeneous system, the controls used for the local solutions of the MPP can be extended by homogeneity to get global solutions as well.

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