# ON THE STABILIZABILITY OF MULTIPLE INTEGRATORS BY MEANS OF BOUNDED FEEDBACK CONTROLS

Héctor J. Sussmann* Yudi Yang
Department of Mathematics
Rutgers University
New Brunswick, NJ 08903 USA

**Abstract**

It is known that a linear system $\dot{x} = Ax + Bu$ can be stabilized by means of a smooth bounded control if and only if it has no eigenvalues with positive real part, and all the uncontrollable modes have negative real part. Here we investigate, for single-input systems, the question whether such systems can be stabilized by means of a feedback $u = \sigma(h(x))$, where $h$ is linear and $\sigma(s)$ is a saturation function such as $\mathrm{sign}(s)\min(|s|, 1)$. A stabilizing feedback of this particular form exists if $A$ has no multiple eigenvalues, and also in some other special cases such as the double integrator. We show that for the multiple integrator of order $n$, with $n \geq 3$, no saturation of a linear feedback can be globally stabilizing.

## 1. Introduction

The problem of globally stabilizing a linear system $\dot{x} = Ax + Bu$ by means of a bounded smooth feedback has been considered in [1], where a proof was outlined of the fact that such a system admits a globally stabilizing bounded feedback $u = k(x)$ if and only if the matrix $A$ has no eigenvalues $z$ such that $\mathrm{Re}\, z > 0$, and all the uncontrollable eigenvalues have a strictly negative real part. The proof sketched in [1] involves a complicated recursive construction. One might think that global stabilization might be achievable by a much simpler approach, e.g. by taking a stabilizer of the form $u(x) = \sigma(h(x))$ where $\sigma$ is —assuming the input is scalar— a "saturation function" such as $\sigma(s) = -\mathrm{Tanh}(s)$ or $\sigma(s) = -\mathrm{sign}(s)\min(|s|, 1)$, and $h(x)$ is a *linear* feedback. It turns out that this is true in some cases, e.g. when the pure imaginary eigenvalues of $A$ are simple, or when the system under consideration is a double integrator $\dot{x} = y$, $\dot{y} = u$. (Cf. §3 below.)

The purpose of this note is to show that the "simple" approach cannot possibly work in general. We will consider the multiple integrator of order $n$, with $n \geq 3$, and show, for a piecewise linear saturation function of the form $\sigma(s) = -L\,\mathrm{sign}(s)\min(|s|/A, 1)$, where $A > 0$, $L > 0$, that *there does not exist a linear functional $h(x)$ such that the feedback law $u = \sigma(h(x))$ is globally stabilizing.*

The result is actually true for more general saturation functions. All one needs to assume is that $s\sigma(s) < 0$ for $s \neq 0$, $\sigma$ is bounded, and both limits $\lim_{s\to\pm\infty}\sigma(s)$ exist and nonzero. But we will only consider the particular saturation function described above because this considerably simplifies the proof, for two reasons. First, the fact that $\sigma$ is actually linear near zero implies that the linear feedback law $u = -Lh(x)$ must be stabilizing, and this has the consequence that the coefficients $a_j = \frac{\partial h}{\partial x_j}$ must be positive. Second, the fact that $|\sigma(s)| = L$ exactly for $|s| \geq A$ implies that, on the set $\{x : |h(x)| \geq A\}$, we can write exact formulas for the trajectories of the closed-loop system as polynomials in $t$.

## 2. The Proof

We consider an *$n$-th order integrator*

$$\begin{aligned} \dot{x}_1 &= x_2\ , \\ &\vdots \\ \dot{x}_{n-1} &= x_n\ , \\ \dot{x}_n &= u\ , \end{aligned}$$

where $n \geq 3$, and a feedback control of the form $u = \sigma(h(x))$, where

$$h(x) = a_1x_1 + a_2x_2 + \ldots + a_nx_n$$

is a linear function, and $\sigma : \mathbb{R} \to \mathbb{R}$ is a function of the form $\sigma(s) = -\frac{Ls}{A}$ for $-A \leq s \leq A$, $\sigma(s) = L$ for $s < -A$, and $\sigma(s) = -L$ for $s > A$. Here $A$ and $L$ are strictly positive constants. We will prove that no such feedback control law can be globally stabilizing. From now on we assume that the closed-loop system resulting from the above feedback makes the origin a globally asymptotically stable equilibrium, and we try to find a contradiction.

First observe that, on a neighborhood of the origin, the closed-loop system is just linear, given by

$$\begin{aligned} \dot{x}_1 &= x_2\ , \\ &\vdots \\ \dot{x}_{n-1} &= x_n\ , \\ \dot{x}_n &= -k(a_1x_1 + a_2x_2 + \ldots + a_nx_n)\ , \end{aligned}$$

where $k = \frac{L}{A}$. So the matrix of this linear system must be Hurwitz, which implies that all $a_i$ are $> 0$. So from now on we shall assume that $a_i > 0$ for $1 \leq i \leq n$.

We let $H$ denote the hyperplane defined by $h(x) = 0$ For each $C > 0$ we consider within $H$ two regions $\mathcal{R}_-(C)$, $\mathcal{R}_+(C)$. We define $\mathcal{R}_-(C)$ to be the set of all those points $x \in H$ that satisfy $x_i \leq 0$ for $1 \leq i \leq n-1$, and $x_n \leq -C$. Similarly, $\mathcal{R}_+(C)$ is the set of those $x \in H$ such that $x_i \geq 0$ for $1 \leq i \leq n-1$, and $x_n \geq C$. We will prove our conclusion by studying the *first return map* $\Phi$ associated with $H$. Precisely, we will show that, if $C$ is sufficiently large, then the trajectory from

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any $x \in \mathcal{R}_-(C)$ returns to $H$ in finite time, and does so by hitting $H$ at a point in $\mathcal{R}_+(C)$ and that, similarly, a trajectory from $\mathcal{R}_+(C)$ returns to $\mathcal{R}_-(C)$. From this it follows easily that a trajectory that starts at some point of $\mathcal{R}(C) = \mathcal{R}_-(C) \cup \mathcal{R}_+(C)$ can never go to zero. We will only study the first return of trajectories from points in $\mathcal{R}_-(C)$.

It will be convenient to work instead with the hyperplane $H^A$ defined by $h(x) + A = 0$. We define the regions $\mathcal{R}^A(C)$, $\mathcal{R}^A_-(C)$, $\mathcal{R}^A_+(C)$ in a similar way.

As explained before, the constant $C$ will be required to be "sufficiently large." The precise meaning of this is that $C$ should satisfy the following inequalities:

$$\begin{aligned} a_{n-1}C &> a_{n-1} + a_n L + AL \ , \\ (C-1)\gamma &> L \ , \\ (C-1)[n(1-\delta)-2] &> 1 \ . \end{aligned}$$

Here $\delta > 0$ is chosen so that

$$n(1-\delta) > 2 \ , \quad \tfrac{n}{n-1}(1-\delta) > 1 \ .$$

Notice that it is possible to choose $\delta$ so as to satisfy the first inequality *because* $n \geq 3$.

The constant $\gamma$ will be chosen later, and at that point it will be clear that $\gamma$ only depends on the coefficients $a_j$ and $\delta$.

Our first task is to make sure that

(I) if we start with a point $\bar{x} \in \mathcal{R}_-(C)$, then the resulting trajectory $t \to x(t)$ reaches $\mathcal{R}^A_-(C-1)$ at some positive time,

and

(II) a trajectory starting at an $x \in \mathcal{R}^A_+(C)$ gets to $\mathcal{R}_+(C)$ at some later time.

To prove (I) and (II), notice that the derivative $\dot{h}^*(t)$ of $h^*(t) = h(x(t))$ is given by

$$\dot{h}^*(t) = a_1 x_2(t) + \ldots + a_{n-1}x_n(t) + a_n \sigma(h^*(t)) \ .$$

If $x(0) \in \mathcal{R}_-(C)$, then $\dot{h}^*(0) \leq -a_{n-1}C + a_n L$, which is $< 0$ because $a_{n-1}C > a_n L$. We then see that $h^*(t) < 0$ on some interval of the form $(0, \hat{t})$. Choose $\hat{t}$ to be as large as possible. (So that in principle $\hat{t}$ could be infinite. We will establish below that it is finite.) Then on the interval $(0, \hat{t})$ the function $x_n(t)$ has a positive derivative and therefore is increasing. But $\dot{x}_n(t) \leq L$, so that on an interval of length $T$ it is not possible for $x_n(t)$ to increase by more than $LT$. If we let $\tilde{t} = 1/L$, we see that on the interval $I = [0, \tilde{t}]$ the function $x_n(t)$ satisfies $x_n(t) \leq 1 - C$. In particular (since $C > 1$), $x_n(t) < 0$ on $I$. Since $\dot{x}_{n-1} = x_n$, we see that $x_{n-1}$ is decreasing on $I$, so $x_{n-1}(t) \leq 0$ on $I$, because $x_{n-1}(0) \leq 0$. Continuing in this way, we find that all the functions $x_j(t)$, for $j = 1, 2, \ldots, n-1$ are decreasing on $I$, and all the $x_j(t)$, for $j = 2, 3, \ldots, n$ are negative on $I$. Moreover, since $\dot{h}^*(t) = a_1 x_2(t) + \ldots + a_{n-1}x_n(t) + a_n \sigma(h^*(t))$, we see that $\dot{h}^*(t) \leq a_{n-1}(1-C) + a_n L$. Since $a_{n-1}C > a_{n-1} + a_n L + AL$, we conclude that in fact $\dot{h}^*(t) \leq -AL$. So $h^*(t)$ is in fact decreasing on $I$, which means that $\hat{t} > \tilde{t}$. Moreover, the bound for $\dot{h}^*$, together with $h^*(0) = 0$, imply that $h^*(\tilde{t})) \leq -A$. So $h^*(t) = -A$ for some $t \in I$. Then all the $x_j(t)$ for $j = 2, \ldots, n$ are $\leq 0$, and $x_n(t) \leq 1 - C$. This implies that the point $x(t)$ is in $\mathcal{R}^A_-(C-1)$. So (I) is proved.

To prove (II) notice that, if we start a trajectory $t \to x(t)$ at a point $\bar{x}$ in $\mathcal{R}^A_+(C)$, then at that point $h^* = -A$, so that $\dot{x}_n(0) > 0$. Again, this implies that there is a maximal interval $J = (0, \hat{t})$ on which $h^* < 0$, and on that interval $x_n$ is increasing and therefore positive, and in fact $\geq C$. So $x_{n-1}$ is increasing and positive, and then the same is true for $x_{n-2}, \ldots, x_2$. Since $\dot{h}^*(t) = a_1 x_2(t) + \ldots + a_{n-1}x_n(t) + a_n \sigma(h^*(t))$, we see that on $J$ $\dot{h}^*(t) \geq a_{n-1}C$. If $J$ was infinite, then $h^*(t)$ would become arbitrarily large on $J$, and in particular it would become positive, contradicting the definition of $J$. So $J$ is finite, and this means that $h^*(\hat{t}) = 0$, while $x_n(\hat{t}) \geq C$ and $x_j(\hat{t}) \geq 0$ for $j = 2, \ldots, n-1$. Therefore $x(\hat{t}) \in \mathcal{R}_+(C)$, and (II) is proved.

In view of (I) and (II) above, our conclusion will follow if we prove:

(III) a trajectory starting at an $x \in \mathcal{R}^A_-(C-1)$ gets to $\mathcal{R}^A_+(C)$ at some later time.

¿From now on, we write $K = C - 1$ and assume that $t \to x(t)$ is a trajectory such that $x(0) = \bar{x} \in \mathcal{R}^A_-(K)$. As before, we let $h^*(t) = h(x(t))$. Then $h^*(0) = -A$, and $\dot{h}^*(0) = a_1\bar{x}_2 + \ldots + a_{n-1}\bar{x}_n + a_n L \leq -a_{n-1}K + a_n L < 0$ (because $a_{n-1}C > a_{n-1} + a_n L$). So there is a maximal interval $I = (0, \hat{t})$ on which $h^* < -A$. Our goal is to prove that $\hat{t}$ is finite and $x(\hat{t}) \in \mathcal{R}^A_+(C)$. Since $h^*(\hat{t})$ must be equal to $-A$, what we need to prove is the inequalities $x_n(\hat{t}) \geq C$, $x_{n-1}(\hat{t}) \geq 0$, …, $x_2(t) \geq 0$. On $I$ we have $\dot{x}_n(t) = L$, so by successive integrations we get the explicit formulas

$$\begin{aligned} x_n(t) &= \bar{x}_n + Lt \ , \\ x_{n-1}(t) &= \bar{x}_{n-1} + \bar{x}_n t + L\tfrac{t^2}{2} \ , \\ &\ \vdots \\ x_1(t) &= \bar{x}_1 + \bar{x}_2 t + \ldots + \bar{x}_n \tfrac{t^{n-1}}{(n-1)!} + L\tfrac{t^n}{n!} \ . \end{aligned}$$

If we multiply the above equations by $a_n$, $a_{n-1}$, $\cdots$, $a_1$ and then add and use the fact that $a_1\bar{x}_1 + \ldots + a_n\bar{x}_n = h(\bar{x}) = -A$, we get $h^*(t) + A = tf(t)$, where

$$f(t) = Lq(t) + \bar{x}_2 p_2(t) + \ldots + \bar{x}_n p_n(t) \ ,$$

and the polynomials $q(t)$, $p_j(t)$ are given by

$$\begin{aligned} q(t) &= \tfrac{a_1}{n!}t^{n-1} + \tfrac{a_2}{(n-1)!}t^{n-2} + a_n \ , \\ p_n(t) &= \tfrac{a_1}{(n-1)!}t^{n-2} + \tfrac{a_2}{(n-2)!}t^{n-3} + a_{n-1} \ , \\ &\ \vdots \\ p_3(t) &= \tfrac{a_1}{2}t + a_2 \ , \\ p_2(t) &= a_1 \ . \end{aligned}$$

Notice that, on $I$, $h^*(t)$ is a polynomial in $t$ of degree $n$, whose leading coefficient is positive. If $I$ was infinite it would follow that $h^*(t) > 0$ for large enough $t$, contradicting the fact that $h^* < -A$ on $I$. So $I$ is finite, and of course $h^*(\hat{t}) = -A$. So all we need is to prove that $x_n(\hat{t}) \geq C$, $x_{n-1}(\hat{t}) \geq 0$, …, $x_2(\hat{t}) \geq 0$.

Observe that the polynomials $q$ and $p_j$ are strictly positive for $t \geq 0$. Moreover, we have

$$\frac{p_k(t)}{q(t)} = \frac{n!}{(k-1)!} t^{k-n-1}(1+\rho_k(t)) ,$$

where $\rho_k(t) \to 0$ as $t \to \infty$. We choose $T > 0$ such that $|\rho_k(t)| < \delta$ for all $t \geq T$ and all $k = 2, \ldots, n$. We now choose $\gamma$ as follows: we let $\gamma > 0$ be such that

$$\frac{p_n(t)}{q(t)} \geq \gamma \quad \text{for} \quad 0 \leq t \leq T .$$

We can then write, for $t \in I$,

$$h^*(t) + A = Ltq(t)\left\{1 + \frac{\bar{x}_n p_n(t)}{Lq(t)} + \ldots + \frac{\bar{x}_2 p_2(t)}{Lq(t)}\right\} .$$

All the terms $\frac{\bar{x}_j p_j(t)}{Lq(t)}$ are negative, and the one corresponding to $j = n$ has an absolute value bounded below by $K\gamma/L$ whenever $t \in [0, T]$. Since $K\gamma > L$, we conclude that $t^{-1}(h^*(t) + A)$ is negative and bounded away from 0 on $I \cap [0, T]$. If $\hat{t}$ was $\leq T$, it would follow that $h^*(t) + A$ is negative and bounded away from 0 for $t$ near $\hat{t}$, $t < \hat{t}$, and so $h^*(\hat{t}) + A < 0$, which is a contradiction. Hence $\hat{t} > T$. This implies in particular that $|\rho_k(\hat{t})| < \delta$ for $k = 2, \ldots, n$.

The point $\hat{t}$ satisfies $h^*(\hat{t}) = -A$, and therefore

$$\frac{\bar{x}_n p_n(\hat{t})}{Lq(\hat{t})} + \ldots + \frac{\bar{x}_2 p_2(\hat{t})}{Lq(\hat{t})} = -1 . \qquad (*)$$

Since all the terms in the left-hand side are negative, their absolute values have to be smaller than 1. In particular, we get the bound $|p_n(\hat{t})/q(\hat{t})| \leq L/|\bar{x}_n|$. On the other hand, $|p_n(\hat{t})/q(\hat{t})| = \frac{n}{\hat{t}}(1 + \rho_n(\hat{t}))$, and $|\rho_n(\hat{t})| < \delta$. Therefore $|p_n(\hat{t})/q(\hat{t})| \geq \frac{n}{\hat{t}}(1 - \delta)$, and

$$\hat{t} \geq \frac{n|\bar{x}_n|(1-\delta)}{L} .$$

The formula $x_n(\hat{t}) = \bar{x}_n + L\hat{t}$ then gives $\hat{x}_n(\hat{t}) \geq \bar{x}_n + n|\bar{x}_n|(1-\delta)$, so that $x_n(\hat{t}) \geq (n(1-\delta) - 1)|\bar{x}_n|$.

Now, the inequality $(C-1)[n(1-\delta)-2] > 1$ says that $1/K < n(1-\delta)-2$, so that $1+(1/K) < n(1-\delta)-1$, i.e. $C/K < n(1-\delta) - 1$. We then get $x_n(\hat{t}) \geq C|\bar{x}_n|/K$. Since $|\bar{x}_n| \geq K$, we conclude that $x_n(\hat{t}) \geq C$.

Notice that this step was possible because we were able to choose $\delta$ and $C$ such that $(C-1)[n(1-\delta)-2] > 1$, and it is here that the assumption that $n \geq 3$ has been used.

We now establish the inequalities $x_j(\hat{t}) \geq 0$ for $j = 2, \ldots, n-1$. In Equation $(*)$, rewrite each quotient $p_k(\hat{t})/q(\hat{t})$ as $\frac{n!}{(k-1)!}\hat{t}^{k-n-1}(1 + \rho_k(\hat{t}))$, multiply both sides by $L\hat{t}^{n-1}$, divide by $n!$ and add the negative of the right-hand side to both sides. The result is

$$\frac{L\hat{t}^{n-1}}{n!} + \sum_{k=2}^{n} \bar{x}_k \frac{\hat{t}^{k-2}}{(k-1)!}(1 + \rho_k(\hat{t})) = 0 . \qquad (\#)$$

We compare this with the formula for $x_2(\hat{t})$:

$$x_2(\hat{t}) = \frac{L\hat{t}^{n-1}}{(n-1)!} + \sum_{k=2}^{n} \bar{x}_k \frac{\hat{t}^{k-2}}{(k-2)!} .$$

In Equation (#), all the terms other than the first one are negative, so we can rewrite (#) as

$$\sum_{k=2}^{n} |\bar{x}_k| \frac{\hat{t}^{k-2}}{(k-1)!}(1 + \rho_k(\hat{t})) = \frac{L\hat{t}^{n-1}}{n!} . \qquad (\#\#)$$

Since $\frac{n}{n-1}(1 - \delta) > 1$ and $|\rho_k(\hat{t})| < \delta$, we have

$$|\bar{x}_k| \frac{\hat{t}^{k-2}}{(k-2)!} \leq n|\bar{x}_k| \frac{\hat{t}^{k-2}}{(k-1)!}(1 + \rho_k(\hat{t})) , \quad \text{for } k = 2, \ldots, n.$$

If we sum over $k$ and use (##), we get

$$\sum_{k=2}^{n} |\bar{x}_k| \frac{\hat{t}^{k-2}}{(k-2)!} \leq n \frac{L\hat{t}^{n-1}}{n!} = \frac{L\hat{t}^{n-1}}{(n-1)!} . \qquad (+)$$

Then the expression for $x_2(\hat{t})$ implies that $x_2(\hat{t}) \geq 0$.

The inequality in (+) remains true if we drop the term for $k = 2$ and divide through by $\hat{t}$. Applying this and noting that $\frac{1}{(k-3)!} \leq \frac{n-1}{(k-2)!}$, we can obtain

$$\sum_{k=3}^{n} |\bar{x}_k| \frac{\hat{t}^{k-3}}{(k-3)!} \leq \frac{L\hat{t}^{n-2}}{(n-2)!} .$$

Then again, the expression for $x_3(\hat{t})$ implies that $x_3(\hat{t}) \geq 0$. Continuing in this way, we show that $x_k(\hat{t}) \geq 0$ for all $k \in \{2, 3, \ldots, n\}$, and our proof is complete.

## 3. Some special cases

As we indicated in §1, in some simple cases the system $\dot{x} = Ax + Bu$ can be stabilized by a saturated linear feedback. Without loss of generality, let us suppose that all the eigenvalues of $A$ have zero real part and $(A, B)$ is controllable.

**Example 3.1** Assume that all the eigenvalues of $A$ are simple. Then there exists an invertible matrix $T$ such that $T^{-1}AT$ is skew-symmetric. Therefore the feedback $u = \sigma((T^{-1}B)^\dagger T^{-1}x)$ will stabilize the system $\dot{x} = Ax + Bu$, if $\sigma(s)$ is any saturation function (with the property that $s\sigma(s) < 0$ whenever $s \neq 0$). □

**Example 3.2** Consider the double integrator

$$\begin{aligned} \dot{x} &= y , \\ \dot{y} &= u . \end{aligned}$$

Let $\sigma : \mathbb{R} \to \mathbb{R}$ be any bounded strictly increasing continuous function such that $\sigma(0) = 0$. Define $\Phi(s) = \int_0^s \sigma(t)\, dt$. Then the feedback law $u = -\sigma(x+y)$ stabilizes the system and $V(x, y) = \Phi(x) + \Phi(x + y) + y^2$ is a strict Lyapunov function of the resulting closed-loop system. □

**Example 3.3** Consider the system

$$\begin{aligned} \dot{x} &= y , \\ \dot{y} &= u , \\ \dot{z} &= \alpha w , \\ \dot{w} &= -\alpha z + u , \end{aligned}$$

where $\alpha > 0$ is a constant. Notice that now zero is a double eigenvalue and there is in addition a pair of imaginary eigenvalues.

In this case, define $\sigma$ and $\Phi$ be as the same as in Example 3.2. Let $u = -\sigma(x+y+z)$, $V = \Phi(x+y+z) + \frac{1}{2}y^2 + \frac{\alpha}{2}z^2 + \frac{\alpha}{2}w^2$. Then a simple calculation shows that $\dot{V} = -\sigma(x+y+z) \leq 0$. Clearly, equality holds if and only if $x + y + z \equiv 0$. So, if $\dot{V} \equiv 0$ along a trajectory, then the second equation of the system implies that $\dot{y} \equiv 0$. Differentiating twice, we get $y + \alpha w \equiv 0$, $-\alpha^2 z \equiv 0$. Thus $z \equiv 0$. Differentiating the equation $x + y \equiv 0$, we then get $y \equiv 0$. So $x \equiv 0$ and $w \equiv 0$ as well. Once again, global asymptotic stability follows from LaSalle's Invariance Principle. □

## References

[1] Sontag, E.D. and H. J. Sussmann, "Nonlinear output feedback design for linear systems with saturating controls," *Proc. IEEE Conf. Decision and Control, Honolulu, Dec. 1990,* IEEE Pub, 1990: 3414-3416.