LIMITS OF HIGHLY OSCILLATORY CONTROLS AND THE APPROXIMATION OF GENERAL PATHS BY ADMISSIBLE TRAJECTORIES*

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Abstract
We describe sufficient conditions, extending earlier work by Kurzweil and Jarnik, for a sequence of inputs $u^j = (u_1^j, \ldots, u_m^j) \in L^1([0,T], \mathbb{R}^m)$ to be such that, for every m-tuple $(f_1, \ldots, f_m)$ of smooth vector fields, the trajectories of $\dot{x}(t) = \sum_{k=1}^m u_k^j(t) f_k(x(t))$ converge to those of an "extended system" $\dot{x}(t) = \sum_{k=1}^r v_k(t) f_k(x(t))$, where the new vector fields $f_{m+1}, \ldots, f_r$ are Lie brackets of the original f_k's. Using these conditions, we can solve the inverse problem: given a trajectory γ of the extended system, find trajectories of the original system that converge to γ. This is done by means of a universal construction that only involves a knowledge of the v's. These results can be applied to solve the problem of approximate tracking for a controllable system without drift.

1. Introduction

In this paper we report a number of results on the relation between the solutions of an equation

$$\dot{x}(t) = \sum_{k=1}^{m} u_k(t) f_k(x(t)) \ , \tag{1}$$

where $f_1, \ldots, f_m$ are smooth vector fields on a smooth manifold M, and a "Lie bracket extension" of (1), given by

$$\dot{x}(t) = \sum_{k=1}^{r} v_k(t) f_k(x(t)) \ , \tag{2}$$

where the first m vector fields $f_1, \ldots, f_m$ are the same as in (1), and the new admissible directions of motion $f_{m+1}, \ldots, f_r$ are Lie brackets of the f_i, $i \in \{1, \ldots, m\}$. We will concentrate on describing the main results. The proofs, which are quite technical, will be given in [12] and [13].

The extended equation (2) clearly gives rise to more trajectories than (1). So, if we want to construct paths that satisfy some extra conditions (e.g. if we specify points p and q and want to find a path from p to q), it is easier to do it for (2) than for (1). In particular, in the extreme case when the system (1) satisfies the Lie algebra rank condition‡ (LARC), one can choose an extension such that the set of admissible directions at each point is simply the set of all possible directions. Then every smooth curve is a trajectory of (2). If we could somehow produce, for any given trajectory γ of (2), a sequence $\{\gamma_j\}$ of trajectories of (1) that converges to γ as $j \to \infty$, then we would have solved (for systems with the LARC) the problem of "approximate tracking," i.e. of approximating any given curve by admissible trajectories. Suppose that, in addition, M is such that it is easy to find a smooth curve in M that joins any two given points. In that case we will have solved the problem of producing an admissible trajectory that approximately steers p to q. This leads to an alternative approach to the Motion Planning Problem for nonholonomic systems studied, e.g., by Brockett, Sastry, Hauser, Murray, Li, Lafferriere and Sussmann (cf. [2], [4], [6], [7], [8], [9]).

For systems with the LARC it was proved by Haynes and Hermes in [3] that every trajectory of (2) can be uniformly approximated by trajectories of (1). However, their proof does not easily lead to an explicit constructive procedure for producing such a sequence. In this paper we describe a different approach, that starts with a systematic investigation of the limiting behavior of trajectories of (1), and of sufficient conditions for a sequence of such trajectories to converge to a trajectory of (2). These conditions are such that one can then attack in various ways the inverse problem of finding a sequence of trajectories of (1) approximating a given trajectory of (2).

The key point is to reformulate the problem in terms of *convergence of inputs*, rather than of trajectories. Suppose that we have a sequence of inputs $u^j = (u_1^j, \ldots, u_m^j) \in L^1([0,T], \mathbb{R}^m)$. Fix an initial condition $x(0)$. Then if the u^j converge in some sense that can be made technically precise (cf. below) to an input u^∞, it follows that the corresponding trajectories x^j converge to the trajectory x^∞ of u^∞. However, the technical definition of convergence that is required to make this true involves in particular the condition that the u^j stay bounded in L^1 norm as $j \to \infty$. If this fails, then the x^j may fail to converge to x^∞. It turns out, however, that under much weaker conditions it is still true that the x^j converge to a trajectory x^∞ of (2), which may no longer be a trajectory of (1).

For a simple illustration of this phenomenon, consider the system

$$\dot{x}_1 = u_1 \ , \ \dot{x}_2 = u_2 \ , \ \dot{x}_3 = u_2 x_1 \ ,$$

with initial condition $(0,0,0)$. If $u^j(t) = \sqrt{j}(\cos jt, \sin jt)$, then the u^j "converge to 0" in the sense that their indefinite integrals $U^j(t) = \int_0^t u^j(s)\, ds$ converge to 0 uniformly. However, a simple calculation shows that $x^j(t)$ converges uniformly to $x^\infty(t) = (0, 0, \frac{t}{2})$. On the other hand, the

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‡The system (1) satisfies the *Lie algebra rank condition* if for every $x \in M$ we have $\Lambda(x) = T_x M$, where Λ is the Lie algebra of vector fields generated by the f_k, $\Lambda(x) = \{X(x) : X \in \Lambda\}$, and $T_x M$ is the tangent space of M at x.

Lie bracket $f_3 = [f_1, f_2]$ has components $(0,0,1)$, and the limiting trajectory x^∞ turns out to satisfy $\dot{x} = \frac{1}{2} f_3(x)$.

The sufficient condition for convergence generalizes the well known *classical theorem* concerning the limit as $j \to \infty$ of the solutions x^j of an initial value problem

$$\dot{x}(t) = \sum_{k=1}^{m} u_k^j(t) f_k(x(t)), \ x(0) = x_0, \tag{3}$$

where $\{u^j\}$ is a sequence of functions $u^j = (u_1^j, \ldots, u_m^j)$ in $L^1([0,T], \mathbb{R}^m)$. This theorem says that the x^j converge uniformly on $[0,T]$ to the solution x^∞ of

$$\dot{x}(t) = \sum_{k=1}^{m} u_k^\infty(t) f_k(x(t)), \ x(0) = x_0 \tag{4}$$

provided that

(c1) the indefinite integrals $U^j(t) = \int_0^t u^j(s)\,ds$ converge uniformly on $[0,T]$ to $U^\infty(t) = \int_0^t u^\infty(s)\,ds$,

(c2) the functions u^j are uniformly bounded in L^1.

Our generalization to this is as follows. Instead of **(c1)** we will assume that all the iterated integrals

$$U_I^j(t) \stackrel{\text{def}}{=} \int_0^t \cdots \int_0^{t_{k-1}} u_{i_1}^j(t_1) u_{i_2}^j(t_2) \ldots u_{i_k}^j(t_k) dt_k \ldots dt_1$$

converge uniformly, as $j \to \infty$, to the indefinite integrals H_I of certain L^1 functions h_I, for all multiindices $I = (i_1, \ldots, i_k) \in \{1, \ldots, m\}^k$, $k = 1, 2, \ldots$, such that $|I| \stackrel{\text{def}}{=} k \leq r$. (In that case, we will say that the u^j *converge to* $h = \{h_I\}_{|I| \leq r}$ *in the r-th order iterated integrals sense* or, for short, that they *ii*(r)*-converge* to h.)

Instead of **(c2)**, we will assume an *r-th order boundedness condition* $bc(r)$, that states that certain sequences of functions $\tilde{u}_I^j$, associated to the multiindices I, can be chosen to be bounded in L^1 norm for $|I| = r$. The conclusion is that the trajectories associated to u^j of (1) converge uniformly to that of an extended system of the form (2).

2. The convergence theorem.

In order to state the convergence theorem, it is convenient to introduce some algebraic machinery. We follow the notations and definitions of [11].

As in [11], let $\mathbf{X} = \{X_1, X_2, \ldots, X_m\}$ be a finite sequence of objects, that will be called *indeterminates*. Let $A(\mathbf{X})$ be the free associative algebra in $X_1, \ldots, X_m$ and let $L(\mathbf{X})$ be the Lie subalgebra of $A(\mathbf{X})$ generated by $X_1, \ldots, X_m$, so $L(\mathbf{X})$ can be identified with the free Lie algebra in $X_1, \ldots, X_m$.

Also, let $\hat{A}(\mathbf{X})$, $\hat{L}(\mathbf{X})$ be, respectively, the algebra of *formal noncommutative power series* in the X_i, and the Lie algebra of *Lie series* in the X_i. Let $\hat{G}(\mathbf{X})$ be the set of exponentials of elements of $\hat{L}(\mathbf{X})$. Then the Campbell-Hausdorff Formula (cf., e.g., [1]) implies that $\hat{G}(\mathbf{X})$ is a group under the operation of multiplication in $\hat{A}(\mathbf{X})$. The elements of $\hat{G}(\mathbf{X})$ are the *exponential Lie series*.

A basis of $A(\mathbf{X})$ consists of the monomials X_I defined by

$$X_I \stackrel{\text{def}}{=} X_{i_1} X_{i_2} \cdots X_{i_k} \tag{5}$$

for all multiindices I of all lengths $|I| \geq 0$. (The only index of length 0 is the empty index $\emptyset$, and then $X_\emptyset \stackrel{\text{def}}{=} 1$.) So every element of $A(\mathbf{X})$ is a finite linear combination of the X_I. It turns out to be more convenient to label the coefficients so that every $S \in A(\mathbf{X})$ is written as $\sum_I S_I X_{\tilde{I}}$, where $\tilde{I}$ is I run backwards, i.e. $\tilde{I} = (i_k, \ldots, i_1)$ if $I = (i_1, \ldots, i_k)$. We adopt a similar convention for $\hat{A}(\mathbf{X})$, so the elements of $\hat{A}(\mathbf{X})$ are arbitrary (not necessarily finite) sums $\sum_I S_I X_{\tilde{I}}$.

An *extended input* is a function $\mathbf{v}$ on an interval $[0,T]$ with values in $\hat{L}(\mathbf{X})$, which is *integrable*, in the sense that, if we write $\mathbf{v}(t) = \sum_I v_I(t) X_{\tilde{I}}$, then all the functions v_I are integrable. An ordinary input $u = (u_1, \ldots, u_m)$ can be regarded as an extended input, by identifying it with the function $\mathbf{u}(t) = u_1(t) X_1 + \cdots + u_m(t) X_m$.

To each extended input $\mathbf{v}$ we associate its *Chen-Fliess series* $S_\mathbf{v}$, defined as the $\hat{A}(\mathbf{X})$-valued function S on $[0,T]$ that solves the initial value problem

$$\dot{S}(t) = S(t)\mathbf{v}(t), \ S(0) = 1. \tag{6}$$

Equation (6) will be called the *formal equation* associated to $\mathbf{v}$. The Chen-Fliess series $S_\mathbf{v}$ will also be called the *formal trajectory* of $\mathbf{v}$.

The extended input $\mathbf{v}$ can be computed from $S_\mathbf{v}$ using standard algebraic tools. If $S_\mathbf{v}(t) = 1 + \sum_{|I|>0} H_I(t) X_{\tilde{I}}$, then we get $\mathbf{v}(t) = \sum_I v_I(t) X_{\tilde{I}}$, where the v_I are given by

$$v_I(t) = \sum_k (-1)^k \sum_{J_0 J_1 \cdots J_k = I} h_{J_0} H_{J_1} \ldots H_{J_k}. \tag{7}$$

The inner summation is over all ways of expressing the multiindex I as a concatenation $J_0 J_1 \cdots J_k$ of $k+1$ indices, and $h_I = \dot{H}_I$. Next we use the *a priori* knowledge that $\mathbf{v}(t)$ is a Lie series. This implies that $\mu(\mathbf{v}(t)) = \mathbf{v}(t)$, where μ is the projection map from $\hat{A}(\mathbf{X})$ to $\hat{L}(\mathbf{X})$ that sends each monomial $X_{i_1} \cdots X_{i_k}$ to $\frac{1}{k}[X_{i_1}, [X_{i_2}, \cdots, [X_{i_{k-1}}, X_{i_k}] \ldots]]$ (cf. [1]). So, if we write $[X_I] \stackrel{\text{def}}{=} [\cdots[[X_{i_1}, I_{i_2}], \cdots, I_{i_k}]$, we have

$$\mathbf{v}(t) = \sum_I \frac{(-1)^{|I|-1}}{|I|} v_I(t) [X_I], \tag{8}$$

where the v_I are given by (7).

In the particular case when $\mathbf{v}$ is actually an ordinary input $\mathbf{u}(t) = u_1(t) X_1 + \ldots + u_m(t) X_m$, the Chen-Fliess series $S_\mathbf{u}$ is given by the formula

$$S_\mathbf{u}(t) = 1 + \sum_{|I|>0} U_I(t) X_{\tilde{I}} \tag{9}$$

where, if $I = (i_1, \ldots, i_k)$, then U_I is the iterated integral defined by

$$U_I(t) \stackrel{\text{def}}{=} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} u_{i_1}(t_1) \ldots u_{i_k}(t_k) dt_k \ldots dt_1.$$

A sequence of extended inputs $\mathbf{v}^j$ *FT-converges to* $\mathbf{v}^\infty$ if the corresponding formal trajectories $S_{\mathbf{v}^j}$ converge uniformly to $S_{\mathbf{v}^\infty}$.

Hence, if $\{\mathbf{u}^j\}$ is a sequence of ordinary inputs, and $\mathbf{v}$ is an extended input such that

$$S_\mathbf{v}(t) = 1 + \sum_{|I|>0} H_I(t) X_{\tilde{I}}, \tag{10}$$

the $\{\mathbf{u}^j\}$ FT-converge to $\mathbf{v}$ if and only if the functions U_I^j converge uniformly to H_I for each I. Using Formulas (7) and (8), we can explicitly compute the limiting extended input $\mathbf{v}$ in terms of the limits H_I of the U_I^j.

Our next task is to define the functions $\tilde{u}_I^j$ referred to in §1. The general definition of the $\tilde{u}_I^j$ can be given easily in terms of the *chronological product* operation $(\alpha,\beta) \to \alpha * \beta$ from $L^1([0,T]) \times L^1([0,T])$ to $L^1([0,T])$, defined by $(\alpha * \beta)(t) = \mathcal{I}\alpha(t)\beta(t)$ where $\mathcal{I}\alpha(t) = \int_0^t \alpha(s)\,ds$. We have

$$\tilde{u}^j_{i_1 i_2 \ldots i_k} \stackrel{\text{def}}{=}$$

$$\sum_{r=1}^{k} (-1)^{r+1} (\ldots (v^j_{i_1 i_2 \ldots i_{k-r}} * u^j_{i_{k-r+1}}) * \ldots * u^j_{i_k}) \,, \quad (11)$$

where the v_I^j are L^1 functions. (The $\tilde{u}_I^j$ depend on the choice of the v_I^j.) Formula (11) then implies the recursive relation

$$\tilde{u}^j_{i_1 i_2 \ldots i_{k+1}} = (v^j_{i_1 i_2 \ldots i_k} - \tilde{u}^j_{i_1 i_2 \ldots i_k}) * u^j_{i_{k+1}} \,. \quad (12)$$

We let $A^r(\mathbf{X})$ denote the free nilpotent associative algebra of step r+1, so $A^r(\mathbf{X})$ is defined like $\hat{A}(\mathbf{X})$, except that now all the monomials X_I with $|I| > r$ are set equal to zero. Then $A^r(\mathbf{X})$ can be thought of as the quotient of $A(\mathbf{X})$ or $\hat{A}(\mathbf{X})$ modulo the two-sided ideal of all sums that do not contain any monomials of degree $\le r$. The canonical projection $\mathbf{T}_r$ from $\hat{A}(\mathbf{X})$ to $A^r(\mathbf{X})$ is the operator that assigns to a series S the series $\mathbf{T}_r(S)$ obtained from S by deleting all the terms of degree $> r$. We let $L^r(\mathbf{X})$ be the Lie subalgebra of $A^r(\mathbf{X})$ generated by the X_i, and define $G^r(\mathbf{X})$ to be the subset of $A^r(\mathbf{X})$ consisting of all the exponentials of elements of $L^r(\mathbf{X})$.

The concepts of extended input, formal trajectory, etc., have truncated analogues. Let us say that an extended input $\mathbf{v}$ has *order* $\le r$ if its values are linear combinations of brackets of degree $\le r$. In that case we can regard $\mathbf{v}$ as $L^r(\mathbf{X})$-valued rather than $\hat{L}(\mathbf{X})$-valued, and we can associate to it its *r-th order truncated formal trajectory*, defined to be the $A^r(\mathbf{X})-$valued solution of

$$\dot{S}(t) = S(t)\mathbf{v}(t)\,, \;\; S(0) = 1\,. \quad (13)$$

Equation (13) is the *r-th order truncated formal equation.*

We will say that a sequence $\{u^j\}$ of ordinary inputs *ii*(r)-*converges* to an extended input $\mathbf{v}$ of order $\le r$, with Chen-Fliess series $S_{\mathbf{v}} = \sum_I H_I X_{\tilde{I}}$, if the iterated integrals $U_I^j(t)$ converge uniformly to the functions $H_I(t)$ for all multi-indices I such that $|I| \le r$. We can still define the $\tilde{u}_I^j$ for $|I| \le r$ by (11). We will say that $\{u^j\}$ satisfies Condition $bc(r)$ with respect to an extended input $\mathbf{v}$ if there exist sequences $\{v_I^j\}$, for $|I| \le r$, such that (a) the L^1 norms of the v_I^j for $|I| \le r$ and those of the $\tilde{u}_I^j$ for $|I| = r$ are uniformly bounded, (b) $\mathcal{I}v_I^j \to \mathcal{I}v_I$ uniformly as $j \to \infty$ for $|I| \le r$.

With these definitions we are ready to state the convergence theorem, whose proof will be given in [12].

Theorem 1 *Let $\{u^j\}_{j=1}^{\infty}$ be a sequence of input functions belonging to $L^1([0,T],\mathbb{R}^m)$. Let r be a positive integer. Assume that*

(c1(r)) *$\{u^j\}$ ii(r)-converges to an extended input $\mathbf{v}$ of order $\le r$;*

(c2(r)) *the sequence $\{u^j\}$ satisfies the boundedness condition $bc(r)$ with respect to $\mathbf{v}$.*

Assume that $f_1, f_2, \ldots, f_m$ are vector fields of class C^{r-1} on a manifold M of class C^r. Assume also that the initial value problem

$$\dot{x}(t) = \sum_{|I| \le r} \frac{(-1)^{|I|-1}}{|I|} v_I(t)\,[f_I](x)\,,\;\; x(0) = \bar{x} \quad (14)$$

has a unique solution $t \to x^\infty(t)$ which is defined on the interval $[0,T]$. (Here we write $[f_I] \stackrel{\text{def}}{=} [\cdots[[f_{i_1}, f_{i_2}], \cdots, f_{i_k}]$.) Let $\{\bar{x}^j\}$ be a sequence of points of M that converges to $\bar{x}$. Then

I. *for sufficiently large j, there exists a solution $t \to x^j(t)$, defined on the whole interval $[0,T]$, of the initial value problem arising from (1) with inputs u^j and initial condition $x(0) = \bar{x}^j$,*

II. *the $x^j(t)$ converge to $x^\infty(t)$ as $j \to \infty$, uniformly with respect to t for $t \in [0,T]$.*

3. Approximate tracking: a simple example.

The preceding theorem can be applied to the problem of approximating a given trajectory of the extended system, as explained in §1. To show how this is done, consider first the simple case of a two-input system in $\mathbb{R}^3$, i.e. a system

$$\dot{x} = u_1(t)f_1(x) + u_2(t)f_2(x)\,, \quad (15)$$

where f_1, f_2 are smooth vector fields in $\mathbb{R}^3$. Assume that for every $x \in \mathbb{R}^3$ the vectors $f_1(x), f_2(x), [f_1,f_2](x)$ span $\mathbb{R}^3$. For any given pair of points p, q, choose a smooth path $[0,1] \ni t \to \gamma(t) \in \mathbb{R}^3$ going from p to q. Then by the span condition there exist smooth functions $w_1(t), w_2(t), w_3(t)$ on $[0,1]$ such that, for $t \in [0,1]$,

$$\dot{\gamma}(t) = w_1(t)f_1(\gamma(t)) + w_2(t)f_2(\gamma(t)) + w_3(t)[f_1,f_2](\gamma(t))\,.$$

So γ is a solution of the initial value problem

$$\begin{aligned} \dot{x}(t) &= w_1(t)f_1(x(t)) + w_2(t)f_2(x(t)) \\ &\quad + w_3(t)[f_1,f_2](x(t)) \\ x(0) &= p\,. \end{aligned} \quad (16)$$

If we now let $u_1^j(t) = w_1(t) + j^{\frac{1}{2}} w_3(t)\sin jt$, $u_2^j(t) = w_2(t) - 2j^{\frac{1}{2}}\cos jt$, then it is easily checked that the conditions of Theorem 1 are satisfied with $r = 2$ and

$$\begin{aligned} H_1(t) &= W_1(t), \quad H_2(t) = W_2(t), \\ H_{1,2}(t) &= \int_0^t w_1(s)W_2(s)ds - W_3(t), \\ H_{2,1}(t) &= \int_0^t w_2(s)W_1(s)ds + W_3(t)\,. \end{aligned}$$

(Here the W_i are the integrals of the w_i. We have omitted the formulas for H_{11} and H_{22}.) Then (7) gives $v_1 = w_1$, $v_2 = w_2$, $v_{12} = -w_3$, $v_{21} = w_3$. The limiting extended input $\mathbf{v}$ can then be computed from (8), and turns out to be $w_1(t)X_1 + w_2(t)X_2 + w_3(t)[X_1, X_2]$. So the solutions $x^j(t)$ of the initial value problem

$$\dot{x}(t) = u_1^j(t)f_1(x) + u_2^j(t)f_2(x)\,,\ x(0) = p$$

converge uniformly to the solution of (16), i.e. to $\gamma(t)$. So we have solved the problem of approximately tracking a given path γ for this system.

4. Approximate tracking: the general algorithm.

We now describe the algorithm for approximating an arbitrary smooth r-th order extended input by ordinary inputs. We will limit ourselves to the presentation of the algorithm. The proof that it actually works is quite long, and will be given in [13].

Suppose $\mathbf{v}$ is an r-th order extended input with coefficients of class C^1. Our goal is to write an explicit formula for a sequence of ordinary inputs that converges to $\mathbf{v}$. The basic idea is to use highly oscillatory inputs, as was done in the example of §3. Roughly speaking, we want to "treat each bracket separately," i.e. to produce, for each of the terms that occur in $\mathbf{v}$, a sequence of ordinary inputs that converges to it. Although the input-to-trajectory map is nonlinear, it turns out that, if the frequencies associated to the various parts are independent (in a sense that will be made precise below), then a kind of "high frequency superposition principle" holds, i.e. the parts associated to the various brackets do not interfere with one another.

In order to carry out the above program, one has to take into account the fact that it is not obvious how to decompose an extended input into parts in a canonical way. The general expression for an extended input obtained in (8) is not suitable, because the brackets $[X_I]$ are not independent. What is needed is to write in an explicit way a *basis* of $\hat{L}(\mathbf{X})$. One way of doing that is by using a *P. Hall basis.* (Cf. [1]. For an explanation of the reason why this is the right kind of basis for our problem, cf. [10].)

We recall that a P. Hall basis of $L(\mathbf{X})$ is a set $\mathcal{B}$ of *formal brackets*, endowed with a total ordering $\prec$. (A formal bracket is a particular kind of word in the alphabet consisting of the symbols $X_1, X_2, \ldots, X_m$, the left and right square brackets, and the comma. The set $\mathcal{F}$ of formal brackets is the smallest set S of words that contains X_k, $k = 1, \ldots, m$, and is such that, whenever $w_1 \in \mathcal{F}, w_2 \in \mathcal{F}$, the word $[w_1, w_2]$ is in $\mathcal{F}$. Every formal bracket gives rise to a "real" bracket, i.e. a member of $L(\mathbf{X})$, but two formal brackets can be equal as members of $L(\mathbf{X})$, but different as formal brackets, as shown by the example $B_1 = [X_1, [X_1, X_2]]$, $B_2 = [[X_2, X_1], X_2]$. In particular, every formal bracket B of degree > 1 can be written in a unique way as $[B_1, B_2]$, where B_1, B_2 are formal brackets. Then B_1 is called the *left factor* of B.)

We use $\delta(B)$ to denote the degree of a formal bracket B, and let $\delta_k(B)$ be the degree of B in X_k, so $\delta_1(B) + \ldots + \delta_m(B) = \delta(B)$.

The precise requirements on the pair $(\mathcal{B}, \prec)$ are:

PH1. if $B, B' \in \mathcal{B}$ and $\delta(B) < \delta(B')$, then $B \prec B'$;

PH2. every X_k is in $\mathcal{B}$;

PH3. if B is a formal bracket and $\delta(B) > 1$, so that B can be written in a unique way as $[B_1, B_2]$, then $B \in \mathcal{B}$ if only if (i) $B_1 \in \mathcal{B}$, (ii) $B_2 \in \mathcal{B}$, (iii) $B_1 \prec B_2$ and (iv) either (iv.a) $\delta(B_2) = 1$ or (iv.b) $B_2 = [B_3, B_4]$ and $B_3 \preceq B_1$.

We will impose the additional requirement (which is not usually included in the definition of a P. Hall basis) that

PH4. $X_{k_1} \prec X_{k_2}$ iff $k_1 < k_2$.

We remark for future use that every $B \in \mathcal{B}$ such that $\delta(B) > 1$ can be written in a unique way as $\mathrm{ad}_{B_1}^{\kappa}(B_2)$, where $B_1 \prec B_2$ and either $\delta(B_2) = 1$ or the left factor B_3 of B_2 satisfies $B_3 \prec B_1$. (Here we are using the standard notation ad_B to denote the operator $Z \to [B, Z]$.)

From now on we fix a choice of P. Hall basis $(\mathcal{B}, \prec)$ of $L(\mathbf{X})$ for which **PH4** holds. We let $\mathcal{B}_n$ be the set of members of $\mathcal{B}$ that are of degree n.

We express our extended input $\mathbf{v}$ as a sum

$$\mathbf{v}(t) = \sum_{n=1}^{r} \sum_{B \in \mathcal{B}_n} v_B(t) B\ , \tag{17}$$

where the v_B are functions of class C^1.

We take our approximating sequence to be of the form

$$\begin{aligned} u_k^j(t) = \eta_{k,0}(t) \quad &+ \quad j^{\frac{1}{2}} \sum_{\omega \in \Omega(2,k)} \eta_{\omega,k}(t) e^{ij\omega t} \\ &+ \quad \sum_{n=3}^{r} j^{\frac{n-1}{n}} \sum_{\omega \in \Omega(n,k)} \eta_\omega(t) e^{ij\omega t}\ . \end{aligned} \tag{18}$$

Here

1. the $\Omega(n, k)$, for $(n, k) \in \{2, \ldots, r\} \times \{1, \ldots, m\}$, are pairwise disjoint finite subsets of $\mathbb{R} - \{0\}$,
2. each $\Omega(n, k)$ is symmetric, i.e. $\omega \in \Omega(n, k)$ implies $-\omega \in \Omega(n, k)$,
3. the $\eta_{k,0}$, $\eta_{\omega,k}$, η_ω are complex-valued functions of class C^1 on $[0, T]$,
4. the $\eta_{k,0}$ are in fact real-valued,
5. the identity $\eta_{-\omega} = \overline{\eta_\omega}$ holds for each $\omega \in \cup_{n=3}^{r} \cup_{k=1}^{m} \Omega(n, k)$, and the functions $\eta_{\omega,k}$ satisfy $\eta_{-\omega,k} = \overline{\eta_{\omega,k}}$ for $\omega \in \Omega(2, k)$.

(Here $\bar{z}$ denotes the complex conjugate of z.) Notice that the various terms of (18) are in principle complex, but Conditions **4** and **5** imply that the u_k^j are real.

We now describe in detail how the sets $\Omega(n, k)$ are chosen. First, let us write $|F|$ to denote the number of elements of a finite set F. Call a finite set $F \subseteq \mathbb{R} - \{0\}$ *canceling* if the sum of all the members of F is equal to 0. Call F *properly noncanceling* (PNC) if no proper subset of F is canceling. Call F *minimally canceling* (MC) if the only linear combinations $\sum_{\omega \in F} a_\omega \omega$ that are equal to zero and have integer coefficients such that $\sum_{\omega \in F} |a_\omega| \leq |F|$ are those for which the a_ω are all equal (in which case, of course, they all have to be equal to 0, 1 or -1). It is clear that every MC set is PNC.

Let us call F *symmetrically minimally canceling* (SMC) if it is symmetric (i.e. $\omega \in F$ implies $-\omega \in F$) and contains an MC subset of cardinality $\frac{1}{2}|F|$. (In that case, it is easy to see that F has exactly two such subsets F_1, F_2, and these subsets are disjoint and satisfy $\omega \in F_1$ iff $-\omega \in F_2$.)

At this point we have to take care of an additional technical issue. The "nonlinear superposition principle" that was referred to above only makes it possible to separate brackets in $B \in \mathcal{B}$ *as long as not all their degrees $\delta_k(B)$ are equal.*

(So, for instance, the brackets $[X_2,[X_1,[X_1,[X_1,X_2]]]]$ and $[[X_1,X_2],[X_1,[X_1,X_2]]]$, both of which are in $\mathcal{B}$, cannot be separated.) This requires that we group brackets in $\mathcal{B}$ into classes, separate the classes rather than the individual brackets, and handle the problem of fitting the coefficients of the individual brackets by a different procedure.

Divide each $\mathcal{B}_n$ into equivalence classes by declaring two members B_1, B_2 of $\mathcal{B}_n$ to be equivalent if $\delta_k(B_1)=\delta_k(B_2)$ for $k=1,\ldots,m$. Let $\mathcal{E}_n$ be the set of equivalence classes. For each $E\in\mathcal{E}_n$, we can define $\delta_k(E)$, $\delta(E)$ to be $\delta_k(B)$, $\delta(B)$ for any $B\in E$.

For $n\geq 2$, $E\in\mathcal{E}_n$, $k\in\{1,\ldots,m\}$ choose $|E|$ sets $\Omega_{E,\rho,k}$, $\rho=1,\ldots,|E|$ of $\mathbb{R}-\{0\}$ such that

6. each $\Omega_{E,\rho,k}$, for $E\in\cup_{n=2}^{r}\mathcal{E}_n$, $\rho\in\{1,\ldots,|E|\}$, $k\in\{1,\ldots,m\}$ is a symmetric subset of $\mathbb{R}-\{0\}$ of cardinality $|\Omega_{E,\rho,k}|=2\delta_k(E)$.

About the sets $\Omega_{E,\rho,k}$ for $n\geq 3$, assume:

7. for each $E\in\cup_{n=3}^{r}\mathcal{E}_n$, $\rho\in\{1,\ldots,|E|\}$, the sets $\Omega_{E,\rho,k}$, $k\in\{1,\ldots,m\}$ are pairwise disjoint.

For $n=2$ we make a different assumption. We first observe that every $E\in\mathcal{E}_2$ consists of exactly one bracket, which is of the form $[X_{k_1},X_{k_2}]$ with $k_1<k_2$. (And, conversely, every such bracket gives rise to an $E\in\mathcal{E}_2$.) We take $\Omega_{E,\rho,k_1}=\Omega_{E,\rho,k_2}$, and we let $\Omega_{E,\rho,k}=\emptyset$ if $k\notin\{k_1,k_2\}$. So

8. for each $E\in\mathcal{E}_2$, $E=[X_{k_1},X_{k_2}]$, we have $\Omega_{E,1,k_1}=\Omega_{E,1,k_2}$.

We then let

$$\Omega_{E,\rho}=\bigcup_{k=1}^{m}\Omega_{E,\rho,k}\ , \tag{19}$$

so that $|\Omega_{E,\rho}|=2\delta(E)$ if $\delta(E)>2$, and $|\Omega_{E,\rho}|=2$ if $\delta(E)=2$. We assume

9. each $\Omega_{E,\rho}$, for $E\in\cup_{n=3}^{r}\mathcal{E}_n$, $\rho\in\{1,\ldots,|E|\}$, is symmetrically minimally canceling.

Next define $\Omega(n,k)$ by

$$\Omega(n,k)=\bigcup_{E\in\mathcal{E}_n}\bigcup_{\rho=1}^{|E|}\Omega_{E,\rho,k}\quad . \tag{20}$$

Since the sets $\Omega_{E,\rho}$ are SMC, each $\Omega_{E,\rho}$ has exactly two MC subsets, and these subsets are disjoint and have cardinality $\frac{1}{2}|\Omega_{E,\rho}|$. Let $Q_{E,\rho}$ be the set whose two elements are these two MC subsets of $\Omega_{E,\rho}$.

Let us call a finite collection $\{S_\lambda\}_{\lambda\in\Lambda}$ of finite subsets of $\mathbb{R}$ *independent* if the sets S_λ are pairwise disjoint and, whenever a linear combination $\sum_{\lambda\in\Lambda}\sum_{s\in S_\lambda}a_s s$ vanishes, and the a_s are integers, $\sum|a_s|\leq\sum_{\lambda\in\Lambda}|S_\lambda|$, it follows that each of the sums $\sum_{s\in S_\lambda}a_s s$ vanishes.

We then require:

10. the $\Omega_{E,\rho}$, as (E,ρ) ranges over all pairs such that $E\in\cup_{n=2}^{r}\mathcal{E}_n$, $\rho\in\{1,\ldots,|E|\}$, are independent.

Finally, the last condition requires several new definitions. To each bracket $B\in\mathcal{B}$ let us associate a function $\hat{g}_B$ of $\delta(B)$ variables by means of the following recursive definition. First, to each X_k we associate the function $\hat{g}_{X_k}(\omega)=1$. Next, assume that the $\hat{g}_B$ have been defined for all B of degree $<n$, and let $B\in\mathcal{B}$ have degree n. We then write the unique decomposition $B=\mathrm{ad}_{B_1}^{\kappa}(B_2)$, where $B_1,B_2\in\mathcal{B}$ and the left factor of B_2 is not equal to B_1. Let n_1, n_2 be the degrees of B_1, B_2, so that $\kappa n_1+n_2=n$. We then let

$$\hat{g}_B(\omega_1,\ldots,\omega_n)=\frac{1}{\kappa!}\hat{g}_{B_2}(\omega_{\kappa n_1+1},\omega_{\kappa n_1+2},\ldots,\omega_n)$$

$$\times\prod_{q=1}^{\kappa}\left(\frac{\hat{g}_{B_1}(\omega_{(q-1)n_1+1},\omega_{(q-1)n_1+2},\ldots,\omega_{qn_1})}{\omega_{(q-1)n_1+1}+\omega_{(q-1)n_1+2}+\ldots+\omega_{qn_1}}\right). \tag{21}$$

(For example, to the bracket

$$B=[[[X_1,X_2],[X_1,[X_1,X_2]]]]\ , \tag{22}$$

we associate the function of 5 variables

$$\hat{g}(\omega_1,\ldots,\omega_5)=\frac{1}{2}\frac{1}{\omega_1\omega_3\omega_4(\omega_1+\omega_2)}\ .) \tag{23}$$

Notice that $\hat{g}_B(\omega_1,\ldots,\omega_n)$ is well defined if the set $\{\omega_1,\ldots,\omega_n\}$ is PNC, and in particular if $\{\omega_1,\ldots,\omega_n\}$ is minimally canceling.

Next we have to define a function g_B by a symmetrization procedure. To each B associate the sequence Σ_B of indeterminates obtained by just deleting all the brackets and commas, so that, for instance, to the B of (22) we associate the sequence

$$\Sigma_B=X_1X_2X_1X_1X_2\,.$$

We then define a map $\theta_B:\{1,\ldots,\delta(B)\}\to\{1,\ldots,m\}$ by letting $\theta_B(s)=k$ if the indeterminate in the s-th place of Σ_B is X_k. (For instance, for the B of (22) we have $\theta_B(1)=1$, $\theta_B(2)=2$, $\theta_B(3)=1$, $\theta_B(4)=1$, $\theta_B(5)=2$.) Next let P_B be the set of all permutations of the set $\{1,2,\ldots,\delta(B)\}$ that map the set $\{s:\theta_B(s)=k\}$ to the interval $\mathcal{I}_{B,k}=\{\delta_1(B)+\ldots+\delta_{k-1}(B)+1,\ldots,\delta_1(B)+\ldots+\delta_k(B)\}$ for each $k\in\{1,\ldots,m\}$. (For instance, the permutation

$$\begin{pmatrix}1&2&3&4&5\\1&4&2&3&5\end{pmatrix} \tag{24}$$

is in P_B for the particular B of (22).) We then define $g_B(\omega_1,\ldots,\omega_{\delta(B)})$ by

$$g_B(\omega_1,\ldots,\omega_{\delta(B)})=\sum_{\pi\in P_B}\hat{g}_B(\omega_{\pi(1)},\ldots,\omega_{\pi(\delta(B))})\ . \tag{25}$$

Notice that the resulting function g_B is symmetric under permutation of the variables in each of the intervals $\mathcal{I}_{B,k}$. So, in particular, $g_B(S_1,\ldots,S_m)$ is well defined if S_1, …, S_m are just pairwise disjoint *sets* of real numbers such that $|S_k|=\delta_k(E)$ and the union $S=S_1\cup\ldots\cup S_m$ is PNC. In particular, the numbers

$$\xi_{B,\rho}^{F}=g_B\big(F\cap\Omega_{E,\rho,1},F\cap\Omega_{E,\rho,2},\ldots,F\cap\Omega_{E,\rho,m}\big) \tag{26}$$

are well defined if $F\in Q_{E,\rho}$.

In the special case when $\delta(E)=2$, each E is of the form $\{[X_{k_1},X_{k_2}]\}$, where $k_1<k_2$. In that case, the set $\Omega_{E,1}=\Omega_{E,1,k_1}=\Omega_{E,1,k_2}$ consists of a frequency $\omega_{k_1,k_2}>0$ together with its negative.

We then have the following

Theorem 2 *Let m, r be positive integers. Let $\{u^j\}_{j=1}^{\infty}$ be a sequence of input functions defined by (18), where the functions $\eta_{0,k}$ η_ω and the sets $\Omega(n,k)$, $\Omega_{E,\rho}$, $\Omega_{E,\rho,k}$ satisfy* **1**, …, **10***. Then the sequence $\{u^j\}$ (a) ii(r)-converges to the extended input*

$$u^\infty(t) = \sum_{k=1}^{m} \eta_{0,k}(t) X_k + \sum_{1\le k_1<k_2\le m} \hat{\zeta}_{k_1,k_2}[X_{k_1}, X_{k_2}]$$

$$+\sum_{n=3}^{r}\sum_{E\in\mathcal{E}_n}\sum_{B\in E}\sum_{\rho=1}^{|E|}\Big(\sum_{F\in Q_{E,\rho}} i^{1-|E|}\xi^F_{B,\rho}\prod_{\omega\in F}\eta_\omega(t)\Big)B\,, \quad (27)$$

where

$$\hat{\zeta}_{k_1,k_2} = \frac{\eta_{\omega_{k_1,k_2},k_1}\eta_{-\omega_{k_1,k_2},k_2} - \eta_{-\omega_{k_1,k_2},k_1}\eta_{\omega_{k_1,k_2},k_2}}{i\omega_{k_1,k_2}}\,,$$

and (b) satisfies the boundedness condition $bc(r)$ with respect to u^∞.

It is now easy to see how to construct the η_ω so that the limiting input has a desired value. Fix an E and a ρ.

CH1. Assume first that $\delta(E)$ is odd and $\delta(E) > 1$. Then the numbers $\xi^F_{B,\rho}$, for the two members of $Q_{E,\rho}$, are equal. Call their common value $\xi_{B,\rho}$. Also, $i^{1-\delta(E)} = \pm 1$. Let $\hat{\xi}_{B,\rho} = \xi_{B,\rho}$. Pick all the η_ω for $\omega \in \Omega_{E,\rho}$ to be equal to 1, except for one pair $\{\omega, -\omega\}$ of members of $\Omega_{E,\rho}$, for which we pick $\eta_\omega = \eta_{-\omega} = \frac{i^{1-\delta(E)}}{2}\zeta_{E,\rho}$, where $\zeta_{E,\rho} \in C^1([0,T], \mathbb{R})$.

CH2. Assume that $\delta(E)$ is even and $\delta(E) > 2$. Then the numbers $\xi^F_{B,\rho}$, for the two members of $Q_{E,\rho}$, are negatives of each other. Pick one of the two members of $Q_{E,\rho}$ and call it $F_{E,\rho}$. Also, $i^{1-\delta(E)} = \pm i$. Let $\hat{\xi}_{B,\rho} = \xi^F_{B,\rho}$. Pick all the η_ω for $\omega \in \Omega_{E,\rho}$ to be equal to 1, except for one pair $\{\omega, -\omega\}$ of members of $\Omega_{E,\rho}$, such that $\omega \in F_{E,\rho}$. Then pick $\eta_\omega = -\frac{i^{1-\delta(E)}}{2}\zeta_{E,\rho}$, where $\zeta_{E,\rho} \in C^1([0,T], \mathbb{R})$, and let $\eta_{-\omega} = -\eta_\omega = \overline{\eta_\omega}$.

CH3. Assume that $\delta(E) = 2$. Let $E = \{[X_{k_1}, X_{k_2}]\}$, $k_1 < k_2$. We pick $\eta_{\omega_{k_1,k_2},k_2} = \eta_{-\omega_{k_1,k_2},k_2} = 1$, and $\eta_{\omega_{k_1,k_2},k_1} = \eta_{-\omega_{k_1,k_2},k_1} = \frac{i\omega_{k_1,k_2}}{2}\zeta_{k_1,k_2}$, where $\zeta_{k_1,k_2} \in C^1([0,T], \mathbb{R})$. If $k_1 \ne k \ne k_2$, we let $\eta_{\omega_{k_1,k_2},k} = \eta_{-\omega_{k_1,k_2},k} = 0$.

CH4. Choose $\zeta_{0,k} \in C^1([0,T], \mathbb{R})$.

With these choices, we have

$$u^\infty(t) = \sum_{k=1}^{m} \zeta_{0,k}(t) X_k + \sum_{1\le k_1<k_2\le m} \zeta_{k_1,k_2}[X_{k_1}, X_{k_2}]$$

$$+\sum_{n=3}^{r}\sum_{E\in\mathcal{E}_n}\sum_{B\in E}\sum_{\rho=1}^{|E|}\Big(\hat{\xi}_{B,\rho}\zeta_{E,\rho}(t)\Big)B\,. \quad (28)$$

In order to get u^∞ to be equal to $\mathbf{v}$, we need to choose

$$\zeta_{0,k} = v_{X_k}\,, \quad \zeta_{k_1,k_2} = v_{[X_{k_1},X_{k_2}]}\,, \quad (29)$$

and to let the $\zeta_{E,\rho}$ be solutions of

$$\sum_{\rho=1}^{|E|}\hat{\xi}_{B,\rho}\zeta_{E,\rho}(t) = v_B(t) \text{ for each } E \in \mathcal{E}_n,\ n > 2\,. \quad (30)$$

The possibility of solving (30) is guaranteed if the frequency sets $\Omega_{E,\rho,k}$ are chosen so that

11. the square matrix $\{\hat{\xi}_{B,\rho}\}_{B\in E,\,1\le\rho\le|E|}$ is invertible for each $E \in \cup_{n=3}^{r}\mathcal{E}_n$.

It turns out that

A. *it is always possible to choose frequencies so that Conditions* **1,2,6,7,8,9,10,11** *hold.*

B. *if all the frequencies are chosen so that Conditions* **1,2,6,7,8,9,10,11** *hold, and if the η's are chosen according to* **CH1, CH2, CH3, CH4** *(so that, in particular,* **3,4,5** *hold), where the ζ's satisfy (29) and (30), then the u^j satisfy $bc(r)$ and $ii(r)$-converge to* $\mathbf{v}$.

This completes the description of the algorithm. Its justification requires, of course, that we prove Theorem 2 and Statement **A**. The proofs are quite technical, and will be given in [13].

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