1 Homework assignment No. 2, due on Thursday February 7, 2013

REMARK: Here, and throughout the course notes,

(i) "iff" means "if and only if", and the connective $\Longleftrightarrow$ *means "iff".*

(ii) "$\stackrel{\text{def}}{\Longleftrightarrow}$*" means "equivalent dy definition". That is, "*$\stackrel{\text{def}}{\Longleftrightarrow}$*" means 'the same as "*$\Longleftrightarrow$*". excepts that it adds the extra information that the equivalence is actually the definition of the sentence on the left of the connective. For example, "*x *is even* $\stackrel{\text{def}}{\Longleftrightarrow} (\exists k \in \mathbb{Z})x = 2k$*" says that not only is it true that "*x *is even* $\Longleftrightarrow (\exists k \in \mathbb{Z})x = 2k$*" but, in addition, this equivalence is the definition of what it means for* x *to be even. On the other hand, "*x *is even* $\Longleftrightarrow x - 1$ *is odd" is a true statement, but it's not the definition of what it means for* x *to be even, so it would not be appropriate to write "*x *is even* $\stackrel{\text{def}}{\Longleftrightarrow} x - 1$ *is odd".*

NOTE: Remember that "$A \Longleftrightarrow B$" is true if and only if either A and B are both true or A and B are both false. **This has nothing to do with whether A and B are "related" to each other.** For example, some students wrote in their homework that the sentence

(*) The Eiffel Tower is in Paris if and only if the chemical symbol for Helium is H.

is "false because the location of the Eiffel tower has nothing to do with the chemical symbol for Helium." **This is not a valid argument!** It so happens that Sentence (*) is indeed false, but the reason is totally different: The sentence "The Eiffel Tower is in Paris" is true, but the sentence "the chemical symbol for Helium is H" is false, because the chemical symbol for Helium is He, not H. So, if we represent Sentence (*) as "$A \Longleftrightarrow B$", then A is true and B is false, so "$A \Longleftrightarrow B$" is false.

To see this even more clearly, consider the sentence

(**) The Eiffel Tower is in Paris if and only if the chemical symbol for Helium is He.

Here again our student could say that "the location of the Eiffel tower has nothing to do with the chemical symbol for Helium", but this does not imply that the sentence is false. On the contrary,

- The sentence "The Eiffel Tower is in Paris" is true,

- the sentence "the chemical symbol for Helium is He" is also true,
- so, if we represent Sentence (**) as "$A \Longleftrightarrow B$", then A is true and B is also true, so "$A \Longleftrightarrow B$" is true.

Here is another example, Consider the sentence

(#) The Eiffel Tower is in London if and only if the chemical symbol for Helium is H.

What do you think? Well, let us look at the truth values of the compoents of our sentence, as before:

- The sentence "The Eiffel Tower is in London" is false,
- the sentence "the chemical symbol for Helium is H" is also false,
- so, if we represent Sentence (#) as "$A \Longleftrightarrow B$", then A is false and B is also false, so "$A \Longleftrightarrow B$" is true.

(1) Book, pages 25-26-27:, Problem 1 (non-starred[1] items); introduce symbols for all predicates; for example, to do Part (a), you should write more or less this:

> Let "$B(x)$" stand for "x is beautiful", and "$P(x)$" stand for "x is a precious stone". Then the given sentence says that $\sim (\forall x)(P(x) \Longrightarrow B(x))$,

[1] Throughout this list of problems, and in all future lists of homework problems in this course, "starred" means "with a full star"; items with a non-full star count as non-starred.

Please do **not** write

Let $B(x) = x$ is beautiful.

This is an atrocity for at least two reasons:

- the equal sign "=" is used to connect *mathematical objects*, such as numbers or functions or sets. It's not used to connect sentences. (In other words, "=" is **not** a logical connective.)
- the text

  Let $B(x) = x$ is beautiful.

  is unreadable, because the reader doesn't know, and cannot possibly know, whether you mean to say that $B(x)$ is supposed to be equal to x and the sentence "$B(x) = x$ is supposed to be beautiful, or $B(x)$ is supposed to be "x is beautiful".

When I write

Let $B(x)$ stand for "x is beautiful".

there is no ambigüity. Thanks to the quotation marks, it is very clear that the thing "$B(x)$" is supposed to stand for is the whole sentence.

(2) Book, pages 25-26-27:, Problem 2 (non-starred items), To give you an idea of what you have to do, here is the solution to Part (a). In Problem 1, we have introduced the notations "$B(x)$" and "$P(x)$" to stand for "x is beautiful" and "x is a precious stone", respectively[2]. The statement

[2]The word "respectively" is used very often in mathematical writing, so you should learn how to use it and then use it. Here are two examples of the use of "respectively",

for which we are asked to write a denial is "$\sim (\forall x)(B(x) \Longrightarrow P(x))$". So a denial is "$(\forall x)(B(x) \Longrightarrow P(x))$", which translates into English as "All precious stones are beautiful".

(3) Book, pages 25-26-27: Problem 3, parts (b), (c), (d), (e). You should write a complete definition, as in the following examples of solutions of Part (a):

> **DEFINITION.** Let x be an integer. Then
> x is **even** iff $(\exists k \in \mathbb{Z})x = 2k$.

or, if you prefer to have more symbols,

> **DEFINITION.** Let $x \in \mathbb{Z}$. Then
> x is **even** $\stackrel{\text{def}}{\Longleftrightarrow} (\exists k \in \mathbb{Z})x = 2k$

or, if you like symbols a lot,

> **DEFINITION.** $(\forall x \in \mathbb{Z})(x \text{ is } \textbf{even} \Longleftrightarrow (\exists k \in \mathbb{Z})x = 2k)$,

or, if you are so crazy about symbols that you want *absolutely everything* to be in symbols:

> **DEFINITION.** Let "$E(x)$" stand for "x is even". Then
> $$(\forall x \in \mathbb{Z})(E(x) \stackrel{\text{def}}{\Longleftrightarrow} (\exists k \in \mathbb{Z})x = 2k)\,.$$

Alternatively, you could introduce a symbol (say, the symbol $\mathbb{E}$) for the set of all even integers, and then write

> **DEFINITION.** Let $\mathbb{E}$ denote the set of all even integers. Then
> $$(\forall x \in \mathbb{Z})(x \in \mathbb{E} \stackrel{\text{def}}{\Longleftrightarrow} (\exists k \in \mathbb{Z})x = 2k)\,.$$

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from which it ought to become clear to you how to use it. Example 1: The first, second, and third powers of 3 are 3, 9 and 27, respectively. Example 2: We will use the letters a,b,c and d to stand for Bob, Carol, Ted, and Alice, respectively.

(4) Prove, *using the definitions given in the book* (Page xv) , that a natural number n is composite iff there exist natural numbers a, b such that $a < n$, $b < n$, and $n = ab$. In other words, you have to prove that

$$(\forall n \in \mathbb{N})(n \quad \text{is} \quad \text{composite} \Longleftrightarrow (\exists a, b \in \mathbb{N})(a < n \wedge b < n \wedge n = ab)\,).$$

Here is the way your proof should start:

We want to prove that

$$(\forall n \in \mathbb{N})(n \quad \text{is} \quad \text{composite} \Longleftrightarrow (\exists a, b \in \mathbb{N})(a < n \wedge b < n \wedge n = ab)\,)$$

Let n be an arbitrary natural number[3].
We want to prove that

$$n \quad \text{is} \quad \text{composite} \Longleftrightarrow (\exists a, b \in \mathbb{N})(a < n \wedge b < n \wedge n = ab)\,.$$

We are going to prove first that

$$n \quad \text{is} \quad \text{composite} \Longrightarrow (\exists a, b \in \mathbb{N})(a < n \wedge b < n \wedge n = ab)\,, \tag{1}$$

and then that

$$(\exists a, b \in \mathbb{N})(a < n \wedge b < n \wedge n = ab) \Longrightarrow \quad n \quad \text{is} \quad \text{composite}\,. \tag{2}$$

Sentence (1) is an implication, so to prove it we assume the premiss[4] and prove the conclusion.

Assume that n is composite[5].

From now on, the proof is all yours. Obviously, the next thing you have to do is translate the observation that n is composite into something

[3]Notice the extra indentation here! That's because we are starting a subproof (i.e., a "proof within a proof") by introducing the new fact that we have this arbitrary natural number n. You should think of n as a natural number that has been given to you but you don't know what it is, so you can say truthfully certain things about it (for example, "$n + n^2$ is even") but there are lots of things you cannot say because you cannot be sure they are true (for example, you cannot say "n is odd").

[4]The book uses "antecedent" and "consequent" rather than "premiss" and "conclusion". I prefer "premiss" and "conclusion", but you are welcome to use the book's terminology if you like it better.

[5]Once again, notice the extra indentation. That's because we have introduced an assumption.

not involving the word "composite". And the definition in the book tells you how to do that. Please do not write "since n is composite, it is the product of two natural numbers smaller than n". That's precisely what you are trying to prove, but at this point you don't know it's true, because the only things you know are (a) that n is composite, and (b) that this means, according to the book, that n is neither 1 nor prime.

(5) Book, pages 25-26-27: Problems 5, 6 (non-starred items), 8 (non-starred items), 10 (non-starred items).