MATHEMATICS 300 — SPRING 2019

Introduction to Mathematical Reasoning

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HINTS FOR PROBLEMS 7, 8 OF HOMEWORK ASSIGNMENT NO. 3.

From the equations

$$ax + by = e\,, \tag{1}$$
$$cx + dy = f, \tag{2}$$

if we multiply the first equation by d and the second equation by b, and subtract the results, then the y-term goes away, and you get

$$\Delta x = ed - fb\,, \tag{3}$$

where $\Delta = ad - bc$.

A similar argument also yields $bc - ad = be - cf$, so

$$\Delta y = fb - ea\,, \tag{4}$$

Then we get the formulas

$$x = \frac{ed - fb}{\Delta} \qquad \text{and} \qquad y = \frac{fb - ea}{\Delta}\ .$$

So far, this is pure symbol manipulation and proves nothing.

Now: if $\Delta \neq 0$, then the quotients that appears in these formulas make sense, and we can plug in the values of x and y given by these formulas and verify that (1) and (2) hold. This proves rigorously that if $\Delta \neq 0$ then the system (1), (2) has a real solution for all real numbers e, f.

In the integer case: if $\Delta = 1$ or $\Delta = -1$, then the quotients that appears in the formulas are integers (when a, b, c, d, e, f are integers) and we can plug in the values of x and y given by these formulas and verify that (1) and (2) hold. This proves rigorously that if $\Delta = 1 \vee \Delta = -1$ then the system (1), (2) has an integer solution for all integers e, f.

Now we look at the implicatiiona in the other direction. So we assume, in the real case, that

(I) the system (1), (2) has a real solution for all real numbers e, f.

and, in the integer case, that

(II) the system (1), (2) has an integer solution for all integers e, f.

In the real case, we want to prove that $\Delta \neq 0$. Assume $\Delta = 0$. Then for every pair e, f of real numbers we can find a solution (x, y) of our system, and the formula $\Delta x = ed - fb$ tells us, since $\Delta = 0$, that $ed - fb = 0$ for all e, f. In particular, we can choose $e = 1$, $f = 0$ and conclude that $d = 0$, and we can chose $e = 0$, $f = 1$ and conclude that $b = 0$. A similar argument using the other equation tells us that $a = 0$ and $b = 0$. So $a = b = c = d = 0$. But if $a = b = c = d = 0$ then (I) clearly, cannot be true, and we got our contradiction.

In the integer case, we want to prove that $|\Delta| = 1$. The the formula $\Delta x = ed - fb$ tells us, that $ed - fb$ is divisible by Δ for all integers e, f. In particular, we can choose $e = 1$, $f = 0$ and conclude that $\Delta | d$, and we can choose $e = 0$, $f = 1$ and conclude that $\Delta | b$. A similar argument using the other equation tells us that Δ divides a and c. So a, b, c, and d are all divisible by Δ. This implies that, if x, y are arbitrary integes, the numbers $ax + by$ and $cx + dy$ are divisible by Δ. In view of (II), if e, f are arbitrary integers, then e and f must be divisible by Δ. So in particular 1 is divisible by Δ. Hence $\Delta = 1$ or $\Delta = -1$.