MATHEMATICS 502 — SPRING 2016

Theory of functions of a real variable II

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NOTES ON FOURIER TRANSFORMS

Contents

1 Gaussian Integrals

Theorem 1. *If $\alpha \in \mathbb{R}$, $\beta \in \mathbb{C}$ and $\alpha > 0$, then*

$$\boxed{\int_{-\infty}^{\infty} e^{-\alpha x^2+\beta x}dx = \sqrt{\frac{\pi}{\alpha}}\,e^{\frac{\beta^2}{4\alpha}}\,. \qquad (1.1)}$$

Proof. First observe that the integral is convergent, because $\alpha > 0$. (This is trivial, but if you want to see a complete proof you can look at the remark at the end of this subsection.)

Next we observe that, for fixed α, the integral of (1.1) is a holomorphic function of the complex variable β, so to prove (1.1) it suffices, by analytic continuation, to assume that β is real.

Let us make the change of variable

$$\xi = \sqrt{2\alpha}x - \frac{\beta}{\sqrt{2\alpha}}\,,$$

so

$$\xi^2 = 2\alpha x^2 - 2\beta x + \frac{\beta^2}{2\alpha}\,,$$

and then

$$-\frac{\xi^2}{2} = -\alpha x^2 + \beta x - \frac{\beta^2}{4\alpha}\,.$$

Also, $d\xi = \sqrt{2\alpha}\,dx$, so $dx = \frac{d\xi}{\sqrt{2\alpha}}$, and then

$$\int_{-\infty}^{\infty} e^{-\alpha x^2+\beta x} dx = \frac{1}{\sqrt{2\alpha}} e^{\frac{\beta^2}{4\alpha}} \int_{-\infty}^{\infty} e^{-\frac{\xi^2}{2}}\,d\xi\,.$$

If we let

$$I = \int_{-\infty}^{\infty} e^{-\frac{\xi^2}{2}}\,d\xi\,,$$

then

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{\xi^2+\eta^2}{2}}\,d\xi\,d\eta\,,$$

and the last integral can be done in polar coordinates:

$$\begin{aligned} I^2 &= \iint_{\mathbf{R}^2} e^{-\frac{r^2}{2}}\,r\,dr\,d\theta \\ &= \int_0^{\infty}\Big(\int_0^{2\pi} d\theta\Big) r e^{-\frac{r^2}{2}}\,dr \\ &= 2\pi\int_0^{\infty} r e^{-\frac{r^2}{2}}\,dr \\ &= 2\pi\int_0^{\infty}\Big(-\frac{d}{dr}e^{-\frac{r^2}{2}}\Big)dr \\ &= 2\pi\,. \end{aligned}$$

It follows that $I = \sqrt{2\pi}$, and then

$$\int_{-\infty}^{\infty} e^{-\alpha x^2+\beta x} dx = \frac{1}{\sqrt{2\alpha}} \times \sqrt{2\pi} e^{\frac{\beta^2}{4\alpha}}$$

so

$$\int_{-\infty}^{\infty} e^{-\alpha x^2+\beta x} dx = \sqrt{\frac{\pi}{\alpha}} e^{\frac{\beta^2}{4\alpha}}\,,$$

as desired. **Q.E.D.**

Remark 1. Let us prove the convergence of the integral in (1.1). First, we have the inequality

$$|\beta x| \le \frac{1}{2}\Big(\alpha x^2 + \frac{|\beta|^2}{\alpha}\Big)\,,$$

using the inequality $ab \leq \frac{a^2+b^2}{2}$ with $a = \sqrt{\alpha}|x|$, $b = \frac{|\beta|}{\sqrt{\alpha}}$, so that $ab = |\beta x|$. Then

$$\begin{aligned} -\alpha x^2 + |\beta x| &\leq -\alpha x^2 + \frac{1}{2}\Big(\alpha x^2 + \frac{|\beta|^2}{\alpha}\Big) \\ &= -\frac{\alpha x^2}{2} + \frac{|\beta|^2}{\alpha}\,. \end{aligned}$$

Hence

$$\begin{aligned} |e^{-\alpha x^2+\beta x}| &= e^{-\alpha x^2}|e^{\beta x}| \\ &\leq e^{-\alpha x^2}e^{|\beta x|} \\ &\leq e^{-\alpha x^2+|\beta x|} \\ &\leq e^{-\frac{\alpha x^2}{2}+\frac{|\beta|^2}{\alpha}} \\ &= e^{-\frac{\alpha x^2}{2}}e^{\frac{|\beta|^2}{\alpha}}\,. \end{aligned}$$

And $e^{\frac{\alpha x^2}{2}} \geq 1 + \frac{\alpha x^2}{4}$, because $e^u \geq 1 + u$ for every nonnegative u, so

$$e^{-\frac{\alpha x^2}{2}} \leq \frac{1}{1+\frac{\alpha x^2}{4}}\,,$$

so the function $x \mapsto e^{-\frac{\alpha x^2}{2}}$ is integrable.

2 Fourier Trnasforms

In this section, we define

a. the <u>Fourier transform</u> $\hat{f}$,

and

b. the <u>inverse Fourier transform</u> $\check{f}$,

of a function $f \in L^1(\mathbb{R};\mathbb{C})$. We do this by letting

$$\begin{aligned} \hat{f}(u) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)e^{-iuv}dv\,, \\ \check{f}(u) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)e^{iuv}dv\,. \end{aligned}$$

With the above definitions, it is clear that

Theorem 2. *If $f \in L^1(\mathbb{R};\mathbb{C})$, then $\hat{f}$ and $\check{f}$ are continuous functions on $\mathbb{R}$, and satisfy*

$$\lim_{|u|\to\infty} \hat{f}(u) = \lim_{|u|\to\infty} \check{f}(u) = 0\,,$$

as well as

$$\|\hat{f}\|_{L^\infty} \le \|f\|_{L^1} \quad \text{and} \quad \|\check{f}\|_{L^\infty} \le \|f\|_{L^1}\,.$$

Furthermore, $\check{f}(u) = \hat{f}(-u)$ for all $u \in \mathbb{R}$, so that

$$\check{f} = \mathcal{R}\hat{f}\,,$$

where $\mathcal{R}$ is the reflection operator*, i.e., the map that sends each funtion f on $\mathbb{R}$ to the function $\mathbb{R} \ni u \mapsto f(-u)$.*

Proof. All these things are very easy to prove, and were proved in class.

1 The Fourier Inversion Formula

We are now ready to prove the *Fourier Inversion Formula* for L^1 functions[1] We define $\Lambda^1(\mathbb{R};\mathbb{C})$ to be the space of all functions $f \in L^1(\mathbb{R};\mathbb{C})$ such that the Fourier transform $\hat{f}$ also belongs to $L^1(\mathbb{R};\mathbb{C})$.

Theorem 3. *Let f be a function belonging to $\Lambda^1(\mathbb{R};\mathbb{C})$. Then*

$$\boxed{f = \check{\hat{f}}\,. \qquad (2.2)}$$

Proof. First of all. the facts that f and $\hat{f}$ belong to L^1 imply that the integrals in the right-hand sides of the formulas

$$\begin{aligned}
\hat{f}(u) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v)e^{-iuv}dv\,,\\
\check{\hat{f}}(u) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(v)e^{-iuv}dv\,,
\end{aligned}$$

[1]As will become clear soon, there are versions of the Fourier Inversion Formula for L^2 functions, and for tempered distributions.

exist for each u, and are bounded continuous functions of u.

Furthermore, if $u \in \mathbb{R}$, then

$$\check{\hat{f}}(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(v) e^{iuv} dv \,.$$

Let

$$g_\varepsilon(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\varepsilon v^2} \hat{f}(v) e^{iuv} dv \,. \tag{2.3}$$

Then it is clear that

$$\lim_{\varepsilon \downarrow 0} g_\varepsilon(u) = \check{\hat{f}}(u) \,, \tag{2.4}$$

because the functions $\mathbb{R} \ni v \mapsto e^{-\varepsilon v^2} \hat{f}(v) e^{iuv}$ converge pointwise to the function $\mathbb{R} \ni v \mapsto \hat{f}(v) e^{iuv}$ and are uniformly dominated by the integrable function $|\hat{f}|$.

It follows from (2.3) that

$$\begin{aligned} g_\varepsilon(u) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Big(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(w) e^{-ivw} \, dw \Big) e^{-\varepsilon v^2} e^{iuv} dv \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\varepsilon v^2} e^{iv(u-w)} f(w) \, dw \, dv \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(w) \Big(\int_{-\infty}^{\infty} e^{-\varepsilon v^2} e^{iv(u-w)} \, dv \Big) dw \,, \end{aligned}$$

where the changes of the orders of integration are justified because the absolute value of the function of two variables

$$\mathbb{R}^2 \ni (v, w) \mapsto e^{-\varepsilon v^2} e^{iv(u-w)} f(w)$$

is $e^{-\varepsilon v^2} |f(w)|$, which is an integrable function on $\mathbb{R}^2$.

The integral

$$J(u, w) = \int_{-\infty}^{\infty} e^{-\varepsilon v^2} e^{iv(u-w)} \, dv$$

can be computed using Formula (1.1) (with $\alpha = \varepsilon$ and $\beta = i(u - w)$), and we get

$$J(u, w) = \sqrt{\frac{\pi}{\varepsilon}} e^{-\frac{(u-w)^2}{4\varepsilon}} \,.$$

It follows that

$$g_\varepsilon(u) = \sqrt{\frac{\pi}{\varepsilon}} \times \frac{1}{2\pi} \int_{-\infty}^{\infty} f(w) e^{-\frac{(u-w)^2}{4\varepsilon}} dw \,,$$

so

$$g_\varepsilon(u) = \frac{1}{2\sqrt{\pi\varepsilon}} \int_{-\infty}^{\infty} f(w) e^{-\frac{(u-w)^2}{4\varepsilon}} dw \,,$$

and then, making the change of variables

$$\xi = \frac{u-w}{2\sqrt{\varepsilon}} \,,$$

so that

$$d\xi = -\frac{dw}{2\sqrt{\varepsilon}} \,,$$

$$dw = -2\sqrt{\varepsilon} d\xi \,,$$

$$w = u - 2\sqrt{\varepsilon}\xi \,,$$

and

$$\frac{(u-w)^2}{4\varepsilon} = \xi^2 \,,$$

we find

$$g_\varepsilon(u) = \frac{1}{2\sqrt{\pi\varepsilon}} \times 2\sqrt{\varepsilon} \int_{-\infty}^{\infty} f(u - 2\sqrt{\varepsilon}\xi) e^{-\xi^2} d\xi \,,$$

so

$$g_\varepsilon(u) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(u - 2\sqrt{\varepsilon}\xi) e^{-\xi^2} d\xi \,.$$

We can compute the integral $\int_{-\infty}^{\infty} e^{-\xi^2} d\xi$ using (1.1), and get

$$\int_{-\infty}^{\infty} e^{-\xi^2} d\xi = \sqrt{\pi} \,,$$

so

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2} d\xi = 1 \,.$$

Therefore,

$$g_\varepsilon(u) - f(u) = \frac{1}{\sqrt{\pi}} = \int_{-\infty}^{\infty} \Big(f(u - 2\sqrt{\varepsilon}\xi) - f(u) \Big) e^{-\xi^2} d\xi \,.$$

Hence

$$|g_\varepsilon(u) - f(u)| \le \frac{1}{\sqrt{\pi}} = \int_{-\infty}^{\infty} \Big| f(u - 2\sqrt{\varepsilon}\xi) - f(u)\Big| e^{-\xi^2} d\xi \,.$$

If we now integrate this with respect to u, we get

$$\int_{-\infty}^{\infty} |g_\varepsilon(u) - f(u)| du \le \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Big| f(u - 2\sqrt{\varepsilon}\xi) - f(u)\Big| e^{-\xi^2} d\xi\, du\,. \quad (2.5)$$

The double integral in the above inequality makes sense because the integrand is positive, and satisfies the inequality

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Big| f(u - 2\sqrt{\varepsilon}\xi) - f(u)\Big| e^{-\xi^2} d\xi\, du \le \int_{-\infty}^{\infty} \theta(2\sqrt{\varepsilon}\xi) e^{-\xi^2} d\xi \,,$$

where we define

$$\theta(h) = \int_{-\infty}^{\infty} |f(u+h) - f(u)| du \,.$$

Clearly, $\theta(h) = \|\tau_h(f) - f\|_{L^1}$, where $\tau_h(f)$ is the h-translate of f, i.e., the function $\mathbb{R} \ni u \mapsto f(u+h)$.

Inequality (2.5) says that

$$\|g_\varepsilon - f\|_{L^1} \le \int_{-\infty}^{\infty} \theta(2\sqrt{\varepsilon}\xi) e^{-\xi^2} d\xi \,. \quad (2.6)$$

It is clear that $\theta(h) \le 2\|f\|_{L^1}$ for every h. Therefore the functions

$$\mathbb{R} \ni \xi \mapsto \theta(2\sqrt{\varepsilon}\xi) e^{-\xi^2} \quad (2.7)$$

are uniformly dominated by the integrable function

$$\mathbb{R} \ni \xi \mapsto 2\|f\|_{L^1} e^{-\xi^2} \,.$$

We now use the fact that θ is continuous (proved below, as Lemma (1)) to conclude that the functions (2.7) converge pointwise to $\theta(0)e^{-\xi^2}$ as $\varepsilon \downarrow 0$.

Since $\theta(0) = 0$, the functions (2.7) converge pointwise to zero. It the follows from the Lenesgue dominated convergence theorem that

$$\lim_{\varepsilon\downarrow 0} \int_{-\infty}^{\infty} \theta(2\sqrt{\varepsilon}\xi) e^{-\xi^2} d\xi = 0 \,.$$

Thefeore (2.6) implies that

$$\lim_{\varepsilon \downarrow 0} \|g_\varepsilon - f\|_{L^1} = 0\,.$$

So the functions g_ε converge to f in L^1. But Equation (2.4) says that the g_ε converge pointwise to $\check{\hat{f}}$. Hence $f = \check{\hat{f}}$, and our conclusion is proved. **Q.E.D.**

Lemma 1. *If* $f \in L^1(\mathbb{R};\mathbb{C})$*, then the translations* $\tau_h(f)$ *depend continuously on* h*. That is, the function*

$$\mathbb{R} \ni h \mapsto \tau_h(f) \in L^1(\mathbb{R};\mathbb{C}) \tag{2.8}$$

is continous. Furthermore, the function (2.8) is actually uniformly continuous.

In particular, if we let $\theta(h) = \|\tau_h(f) - f\|_{L^1}$*, then* θ *is a continuous function.*

Proof. First assume that f is a continuous compactly supported function. Then f is uniformly continuous. Therefore, if ε' is an arbitrary positive number, there exists a positive δ such that

1. $\delta \le 1$,
2. $|f(x+h) - f(x)| < \varepsilon'$ whenever $x, h \in \mathbb{R}$ and $|h| < \delta$.

If $|h| < \delta$, and L is such that the support of f is contained in the interval $[-L, L]$, then

$$\|\tau_h(f) - f\|_{L^1} = \int_{-\infty}^{\infty} |f(x+h) - f(x)| dx \le 2(L+1)\varepsilon'\,,$$

because the integrand is always bounded by ε', and vanishes whenever $|x| > L + 1$.

Therefore, if $\varepsilon > 0$, and we choose ε' such that $2(L+1)\varepsilon' \le \varepsilon$, we find that

$$\|\tau_h(f) - f\|_{L^1} \le \varepsilon \qquad \text{whenever} \quad |h| < \delta\,.$$

It follows that

$$\|\tau_{h_1}(f) - \tau_{h_2}(f)\|_{L^1} \le \varepsilon \qquad \text{whenever} \quad |h_1 - h_2| < \delta\,,$$

because

$$\|\tau_{h_1}(f) - \tau_{h_2}(f)\|_{L^1} = \Big\|\tau_{h_2}\Big(\tau_{h_1-h_2}(f) - f\Big)\Big\|_{L^1} = \|\tau_{h_1-h_2}(f) - f\|_{L^1} \,,$$

in view of the trabslation-invariance of the L^1 norm.

Now, if f is a general L^1 function, and $\varepsilon > 0$, we can find a continuous compactly supported function g such that $\|f-g\|_{L^1} < \frac{\varepsilon}{3}$. Then

$$\|\tau_h(f) - \tau_h(g)\|_{L^1} < \frac{\varepsilon}{3} \qquad \text{for every } h\,.$$

We then find a positive δ such that

$$\|\tau_{h_1}(g) - \tau_{h_2}(g)\|_{L^1} \leq \frac{\varepsilon}{3} \qquad \text{whenever} \quad |h_1 - h_2| < \delta\,,$$

It then follows that, if $|h_1 - h_2| < \delta$, the inequality

$$\|\tau_{h_1}(f) - \tau_{h_2}(f)\|_{L^1} \leq \varepsilon$$

holds, because

$$\begin{aligned} \|\tau_{h_1}(f) - \tau_{h_2}(f)\|_{L^1} \;\leq\; & \|\tau_{h_1}(f) - \tau_{h_1}(g)\|_{L^1} + \|\tau_{h_1}(g) - \tau_{h_2}(g)\|_{L^1} \\ & + \|\tau_{h_2}(g) - \tau_{h_2}(f)\|_{L^1}\,. \end{aligned}$$

Hence the function (2.8) is uniformly continuous, and our proof is complete. **Q.E.D.**

Corollary 1. *If* $f \in \Lambda^1(\mathbb{R};\mathbb{C})$ *then*

1. *Both* f *and* $\hat{f}$ *are continuous functions on* $\mathbb{R}$ *that go to zero at infinity.*
2. $\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$ *and* $\|f\|_{L^\infty} \leq \|\hat{f}\|_{L^1}$.
3. *Both* f *and* $\hat{f}$ *belong to* $L^2(\mathbb{R};\mathbb{C})$.

Proof. We already know that $\hat{f}$ and $\check{\hat{f}}$ are continuous functions that go to zero at infinity, and that $\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$ and $\|\check{\hat{f}}\|_{L^\infty} \leq \|\hat{f}\|_{L^1}$. But now we know in addition that $\check{\hat{f}} = f$. Hence f is a continuous function that goes to zero at infinity, and $\|f\|_{L^\infty} \leq \|\hat{f}\|_{L^1}$.

Finally, the fact that f and $\hat{f}$ belong to L^2 follows by interpolation from the fact that both functions belong to $L^1 \cap L^\infty$. (That is, $\int |f|^2 \leq \|f\|_{L^1}\|f\|_{L^\infty}$, so $\int |f|^2 < \infty$, and similarly for $\hat{f}$.) **Q.E.D.**

2 Plancherel's Theorem

Now that we know that the functions f and $\hat{f}$ belong to L^2 whenever $f \in \Lambda^1$, we can go one step further and prove the very important ***Plancherel theorem***:

Theorem 4. *If* $f \in \Lambda^1(\mathbb{R};\mathbb{C})$ *then*

$$\|\hat{f}\|_{L^2} = \|f\|_{L^2}\,. \quad (2.9)$$

Proof. First observe that, if f and g belong to Λ^1, then

$$\int_{-\infty}^{\infty} f(x)\overline{\check{g}(x)}dx = \int_{-\infty}^{\infty} \hat{f}(x)\overline{g(x)}dx\,, \tag{2.10}$$

To see this, we compute

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\overline{\check{g}(x)}dx &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\overline{\int_{-\infty}^{\infty} g(y)e^{ixy}dy}dx \\
&= \frac{1}{\sqrt{2\pi}}\int\!\!\int)_{\mathbf{R}^2} f(x)\overline{g(y)e^{ixy}}dydx \\
&= \frac{1}{\sqrt{2\pi}}\int\!\!\int)_{\mathbf{R}^2} f(x)e^{-ixy}\overline{g(y)}dydx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} f(x)e^{-ixy}dx\Big)\overline{g(y)}dy \\
&= \int_{-\infty}^{\infty} \hat{f}(y)\overline{g(y)}dy\,.
\end{aligned}$$

This proves (2.10). If we then apply (2.10) with $g = \hat{f}$, we get

$$\int_{-\infty}^{\infty} f(x)\overline{\check{\hat{f}}(x)}dx = \int_{-\infty}^{\infty} \hat{f}(x)\overline{\hat{f}(x)}dx\,.$$

Since $\check{\hat{f}} = f$, we may conclude that

$$\int_{-\infty}^{\infty} f(x)\overline{f(x)}dx = \int_{-\infty}^{\infty} \hat{f}(x)\overline{\hat{f}(x)}dx\,,$$

that is,

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(x)|^2 dx \,,$$

which is the desired identity. **Q.E.D.**

3 Fourier trnasforms of functions in L^2

We have shown that Λ^1 is a subspace of L^2 and the Fourier transform map $\Lambda^1 \ni f \mapsto \hat{f}$ maos Λ^1 to Λ^1 and satisfies $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$ for all $f \in \Lambda^1$. It is easy to see that Λ^1 is a dense subspace of L^2. (For example, the space $\mathcal{S}$ od rapidly decreasing C^∞ functions is conatined in Λ^1 and is dense in L^2.) Hence the Fourier transform map can be extended to a map $\mathcal{F} : L^2 \mapsto L^2$, and this map also satisfies $\|\mathcal{F}(f)\|_{l^2} = \|f\|_{L^2}$. In other words, $\mathcal{F}$ is an isometric map from L^2 to L^2.

We will now go back to our initial notation, and write $\hat{f}$ for $\mathcal{F}(f)$ and $\check{f}$ for $\mathcal{R}(\mathcal{F}(f))$. (Recall that $\mathcal{R}$ os the reflection map, that sends a function f to the function $x \mapsto f(-x)$.)

The formulas

$$\check{\hat{f}} = f \,, \tag{2.11}$$

and

$$\|\hat{f}\|_{L^2} = \|f\|_{L^2} \,, \tag{2.12}$$

that were proved for $f \in \Lambda^1$, extend by contimnuity to all of L^2. The Fourier inversion formula (2.11) says that

$$\mathcal{F} \circ \mathcal{F} = \mathcal{R} \,.$$

Since $\mathcal{R}^2 =$ identity, it follows that

$$\mathcal{F}^4 = \text{identity} \,. \tag{2.13}$$

It is important to note thet. for a general function in L^2, the formula

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-icy} dy$$

is no longer valid as written, because f need not be integrable, so the intefral need not exist.

What is true, however, is that $\hat{f}$ is the L^2-limit of $\hat{f}_L$ as $L \to \infty$, where $\hat{f}_L$ is the Fourier transform of $\chi_{[-L,L]}f$, because $\chi_{[-L,L]}f \to f$ in L^2 as $L \to \infty$. Furthermnore, the functions $\chi_{[-L,L]}f$ are in L^1, so their Fourier transforms are given by the integral fornula. And the same is true for rthe inverse Fourier transform. So we get the following ***Fourier transform formulas***:

$$\hat{f} = L^2 - \lim_{L\to\infty} \hat{f}_L \,, \qquad (2.14)$$

$$\hat{f}_L(x) = \frac{1}{\sqrt{2\pi}} \int_{-L}^{L} f(y) e^{-ixy} dy \,, \qquad (2.15)$$

$$\check{f} = L^2 - \lim_{L\to\infty} \check{f}_L \,, \qquad (2.16)$$

$$\check{f}_L(x) = \frac{1}{\sqrt{2\pi}} \int_{-L}^{L} f(y) e^{ixy} dy \,, \qquad (2.17)$$

$$\check{\hat{f}} = f \,, \qquad (2.18)$$

$$\|\hat{f}\|_{L^2} = \|f\|_{L^2} \,, \qquad (2.19)$$

valid for $f \in L^2(\mathbb{R};\mathbb{C})$.

Formulas (2.14) and (2.16) aare sometimes written in the form

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \lim_{L\to\infty} \int_{-L}^{L} f(y) e^{-ixy} dy \,,$$

and

$$\check{f}(x) = \frac{1}{\sqrt{2\pi}} \lim_{L\to\infty} \int_{-L}^{L} f(y) e^{ixy} dy \,,$$

with the understanding that ***the limits are not pointwise limits, for every*** x***, but limits in*** L^2 ***of functions of*** x/